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CONSTRAINED QUADRILATERAL NONCONFORMING ROTATED $\mathcal{Q}_1$ ELEMENT *1)

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Abstract

In this paper, we define a new nonconforming quadrilateral finite element based on the nonconforming rotated $\mathcal{Q}_1$ element by enforcing a constraint on each element, which has only three degrees of freedom. We investigate the consistency, approximation, superclose property, discrete Green's function and superconvergence of this element. Moreover, we propose a new postprocessing technique and apply it to this element. It is proved that the postprocessed discrete solution is superconvergent under a mild assumption on the mesh.

Mathematics subject classification: 65N30.
Key words: Constrained, Nonconforming Rotated $\mathcal{Q}_1$ element, Superconvergence, Postprocess.

1. Introduction

There are some lower order quadrilateral finite elements, e.g., the conforming isoparametric Q_1 element, the nonconforming rotated $\mathcal{Q}_1$ element and the nonconforming Wilson element. All these finite elements need at least four degrees of freedom. Recently, Park and Sheen have proposed a nonconforming quadrilateral P_1 element, which has only three degrees of freedom [10]. One of the key ideas of the P_1 element is that a linear function on a quadrilateral satisfies a constraint that the summation of values at the midpoints of one pair of opposite edges equals to the summation of values at the midpoints of the other pair of opposite edges.

In this paper, we define a new nonconforming quadrilateral finite element based on the nonconforming rotated $\mathcal{Q}_1$ element (NR $\mathcal{Q}_1$ hereafter)[9] by imposing a similar constraint on each element, the resulting element has only three degrees of freedom, too. We call this element constrained nonconforming rotated $\mathcal{Q}_1$ element(CNR $\mathcal{Q}_1$ for short). The CNR $\mathcal{Q}_1$ element and the P_1 element are equivalent on a rectangle, however, they are different on a general quadrilateral. We investigate some properties of this new element. A new postprocess technique is proposed to obtain a superconvergent discrete postprocessed solution.

The outline of the paper is as follows. In Section 2 and Section 3, we define the CNR $\mathcal{Q}_1$ element and apply it to the second order elliptic problem. In section 4, we define regular derivative Green function of nonconforming finite elements and investigate its properties. Section 5 is devoted to the analysis of the supperclose property and superconvergence of the CNR $\mathcal{Q}_1$ element. In Section 6 , we discuss the postprocessing technique which admits a superconvergent discrete postprocessed solution. This paper ends with numerical examples in Section 7.

We end this section with some notations. Let Ω be a convex polygon with the boundary $\partial\Omega$. We use the standard notation and definition for the Sobolev spaces $H^s(\Omega)$ for $s \geq 0$ [1], the associated inner product is denoted by $(\cdot,\cdot)_s$, and the norm by $\|\cdot\|_s$ with the seminorm $|\cdot|_s$. $H^0(\Omega) = L^2(\Omega)$, in this case, the norm and inner product are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$

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respectively. As usual, $H_0^s(\Omega)$ is the subspace of $H^s(\Omega)$ with vanishing trace on $\partial\Omega$. Define $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$ equipped with the norm $\|\cdot\|_{-1}$, and $<\cdot,\cdot>$ denotes the dual pair between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. We shall also use the Sobolev spaces $W^{s,p}$ for $s \geq 0$ and $p \geq 1$, equipped with the norm $\|\cdot\|_{s,p,\Omega}$ with the seminorm $|\cdot|_{s,p,\Omega}$. If $p = 2$ we have $W^{s,p} = H^s(\Omega)$.

We use the standard gradient operator:

$$\nabla r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \quad \widehat{\nabla} r = \begin{pmatrix} \partial r/\partial \xi \\ \partial r/\partial \eta \end{pmatrix}.$$

Throughout this paper, C denotes a generic constant, which is not necessarily the same at different places, but independent of the mesh size h.

2. Constrained Nonconforming Rotated $\mathcal{Q}_1$ Element

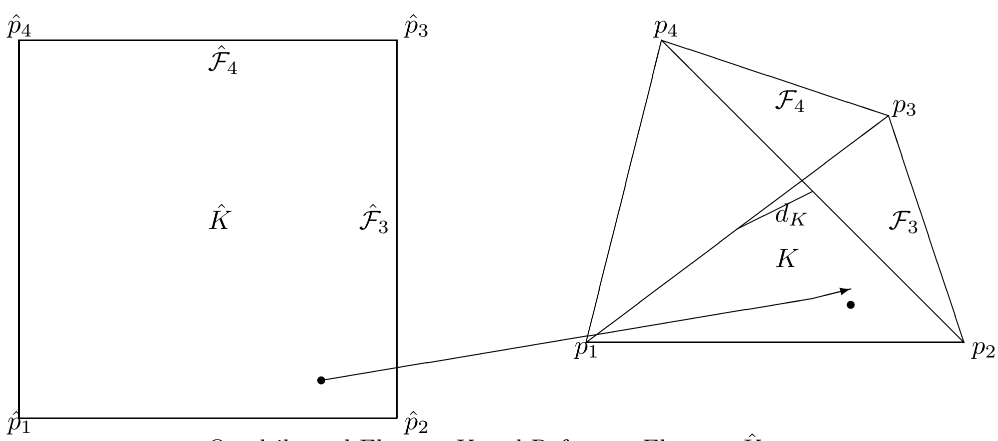
In this section, we introduce some notations and define a new nonconforming finite element method, namely, CNR $\mathcal{Q}_1$ element.

2.1 Quadrilateral Mesh

Let $J^h = \{K_i, i = 1, \cdots, Ne\}$ be a quasi-uniform quadrilateral partition of Ω with $\mathrm{diam}(K_i) \leq h$. Let N^V and N^E denote the numbers of nodes and elements of the partition, respectively, N_i^V and N_B^S denote the numbers of interior nodes and boundary edges, respectively.

We shall frequently use the following assumption on the partition J^h.

Assumption 2.1. *The distance d_K between the midpoints of two diagonals is of order $\mathcal{O}(h^{1+\alpha})$ with $1 \geq \alpha > 0$ when h tends to zero. If $\alpha = 1$, we obtain the usual Bi-section condition [11].*



Quadrilateral Element K and Reference Element $\hat{K}$

For a given element $K \in J^h$, its four nodes are denoted by $p_i(x_i, y_i), i = 1, \cdots, 4$ in the counterclockwise order. Let $\hat{K} = [-1, 1]^2$ denote the reference element with nodes $\hat{p}_i(\xi_i, \eta_i), i = 1, \cdots, 4$. Define the bilinear transformation $\mathcal{F}_K : \hat{K} \to K$ by

$$x = \sum_{i=1}^{4} x_i N_i(\xi, \eta), \quad y = \sum_{i=1}^{4} y_i N_i(\xi, \eta), \quad (\xi, \eta) \in \hat{K},$$

where $N_i(\xi,\eta), i=1,2,3,4$ are the bilinear basis functions, which can be written as

$$\begin{aligned} N_1(\xi,\eta) &= \frac{1}{4}(1-\xi)(1-\eta), \quad N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta),\\ N_3(\xi,\eta) &= \frac{1}{4}(1+\xi)(1+\eta), \quad N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta). \end{aligned}$$

Define

$$\begin{pmatrix} c_0 & d_0\\ c_1 & d_1\\ c_2 & d_2\\ c_{12} & d_{12} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1\\ -1 & 1 & 1 & -1\\ -1 & -1 & 1 & 1\\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\\ x_4 & y_4 \end{pmatrix},$$

then the Jacobian matrix of the bilinear transformation $\mathcal{F}_K$ can be expressed as

$$\mathcal{J}_K=\begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta}\\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} c_1+c_{12}\eta & c_2+c_{12}\xi\\ d_1+d_{12}\eta & d_2+d_{12}\xi \end{pmatrix}$$

with the determinant $J_K(\xi,\eta)=J_0+J_1\xi+J_2\eta$, where $J_0=c_1d_2-c_2d_1,\ J_1=c_1d_{12}-c_{12}d_1$, $J_2=c_{12}d_2-c_2d_{12}$, and its inverse is

$$\mathcal{J}_K^{-1}=\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y}\\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \frac{1}{J_K(\xi,\eta)}\begin{pmatrix} d_2+d_{12}\xi & -c_2-c_{12}\xi\\ -d_1-d_{12}\eta & c_1+c_{12}\eta \end{pmatrix}.$$

In terms of the aforementioned mesh parameters, Assumption2.1 implies

$$| c_{12} | + | d_{12} | \le Ch^{1+\alpha}, \tag{2.1}$$
$$| J_1 | + | J_2 | \le Ch^{2+\alpha}, \tag{2.2}$$
$$| \hat{\nabla}\mathcal{J}_K | \le Ch^{1+\alpha}. \tag{2.3}$$

2.2 The constrained nonconforming rotated $\mathcal{Q}_1$ element

For $S\subset R^2$, denote $P_l(S)$ the space of polynomials of degrees $\le l$ defined on S, $Q_{m,n}(S)$ the space of polynomials of degrees $\le m$ for the first variable and $\le n$ for the second variable. For brevity, $Q_m(S)=Q_{m,m}(S)$.

Before defining the constrained nonconforming rotated $\mathcal{Q}_1$ element, we briefly describe the nonconforming rotated $\mathcal{Q}_1$ element [9]. Define

$$\mathcal{Q}_1(\hat{K})=\mathrm{Span}\langle 1,\xi,\eta,\xi^2-\eta^2\rangle.$$

For any edge $\mathcal{F}\subset\partial K$, the edge functional $\pi_0^{\mathcal{F}}$ is defined as

$$\pi_0^{\mathcal{F}}(v)=\frac{1}{|\mathcal{F}|}\int_{\mathcal{F}} v ds, \quad \forall v\in L^2(K).$$

A local interpolation operator π_K is generated by $\pi_0^{\mathcal{F}}$ with $\pi_{K|\mathcal{F}}=\pi_0^{\mathcal{F}}$ for all $\mathcal{F}\subset\partial K$. The NR $\mathcal{Q}_1$ element space [9] is defined as

$$R^h=\{\, v\in L^2(\Omega)\mid v_{|K}=\hat{v}\circ\mathcal{F}_K^{-1}, \hat{v}\in\mathcal{Q}_1(\hat{K}), v\,\text{is continuous regarding}\,\pi_0^{\mathcal{F}}\,\}.$$

The corresponding homogeneous space is

$$R_0^h=\{\, v\in R^h\mid \pi_0^{\mathcal{F}}(v)=0, \quad \text{for}\,\mathcal{F}\subset\partial\Omega\,\}.$$

On the space R^h, we define the following norm and seminorm, respectively,

$$\|v_h\|_{1,h}=\left\{\sum_{K\in J^h}\|v_h\|_{1,K}^2\right\}^{1/2}, \quad | v_h |_{1,h}=\left\{\sum_{K\in J^h}| v_h |_{1,K}^2\right\}^{1/2}, \quad \forall v_h\in R^h. \tag{2.4}$$

Obviously, $|\cdot|_{1,h}$ is a norm on the space R_0^h [9].

We are in the position to give our element. For a given element $K \in J^h$, let $\mathcal{F}_i, i = 1, 2, 3, 4$ denote its four edges in the counterclockwise order. The constrained nonconforming rotated $\mathcal{Q}_1$ element space CR^h and its homogenous space CR_0^h read

$$CR^h = \{v \in R^h, \int_{\mathcal{F}_1} vds + \int_{\mathcal{F}_3} vds = \int_{\mathcal{F}_2} vds + \int_{\mathcal{F}_4} vds, \forall K\},$$

$$CR_0^h = \{v \in R_0^h, \int_{\mathcal{F}_1} vds + \int_{\mathcal{F}_3} vds = \int_{\mathcal{F}_2} vds + \int_{\mathcal{F}_4} vds, \forall K\}.$$

Remark 2.1. As mentioned in the introduction, the constraint used in the P_1 quadrilateral nonconforming element [10] is that: For a given element $K \in J^h$, let $m_j, 1 \le j \le 4$ denote its four midpoints of the edges in the counterclockwise order. Let $u \in P_1(K)$ be a linear function on K, then $u(m_1) + u(m_3) = u(m_2) + u(m_4)$. Conversely, if $u_j, j = 1, 2, 3, 4$, are given at $m_j, j = 1, 2, 3, 4$, respectively, and $u_1 + u_3 = u_2 + u_4$, then there exists a unique $u \in P_1(K)$ such that, $u(m_j) = u_j, j = 1, 2, 3, 4$. In our element, we use a similar, however different constraint.

For the CNR $\mathcal{Q}_1$ element, there is an equivalent definition. Define

$$\mathcal{CQ}_1(\hat{K}) = \{\, q \in \mathcal{Q}_1(\hat{K}), \int_{\hat{\mathcal{F}}_1} qds + \int_{\hat{\mathcal{F}}_3} qds = \int_{\hat{\mathcal{F}}_2} qds + \int_{\hat{\mathcal{F}}_4} qds \,\},$$

then, we have the following result,

Lemma 2.1.

$$\mathcal{CQ}_1(\hat{K}) = P_1(\hat{K}).$$

Proof. Obviously, we have

$$P_1(\hat{K}) \subset \mathcal{CQ}_1(\hat{K}).$$

We now show the converse relation of $P_1(\hat{K})$ and $\mathcal{CQ}_1(\hat{K})$. Let $q \in \mathcal{CQ}_1(\hat{K})$, which can be expressed as

$$q = a_0 + a_1\xi + a_2\eta + a_3(\xi^2 - \eta^2),$$

we assert that $a_3 = 0$, otherwise

$$\int_{\hat{\mathcal{F}}_1} qds + \int_{\hat{\mathcal{F}}_3} qds \neq \int_{\hat{\mathcal{F}}_2} qds + \int_{\hat{\mathcal{F}}_4} qds,$$

thus

$$\mathcal{CQ}_1(\hat{K}) \subset P_1(\hat{K}),$$

which ends the proof.

With this Lemma at hand, the CNR $\mathcal{Q}_1$ element space CR^h and its homogenous space CR_0^h read

$$CR^h = \{\, v \in L^2(\Omega) \mid v_{|_K} = \hat{v} \circ \mathcal{F}_K^{-1}, \hat{v} \in P_1(\hat{K}), v \text{ is continuous regarding } \pi_0^{\mathcal{F}} \,\}.$$

$$CR_0^h = \{\, v \in CR^h \mid \pi_0^{\mathcal{F}}(v) = 0, \quad \text{for } \mathcal{F} \subset \partial\Omega \,\}.$$

For this new element, several remarks are in order.

Remark 2.2. The CNR element can also be constructed from Cai-Douglas-Ye element [5] by imposing the same constraint. Since Cai-Douglas-Ye element is a constrained Han element [6], the CNR element can also be derived from the Han element by enforcing two constraints on each element.

Remark 2.3. On a rectangle K, the CNR $\mathcal{Q}_1$ element is equivalent to the P_1 element. In fact, in this case, the bilinear transformation $\mathcal{F}_K$ degenerates to a linear one, both the P_1 element [10] and the CNR $\mathcal{Q}_1$ element are linear, morover, $v(m) = \pi_0^{\mathcal{F}} v$ if v is a linear function with $v(m)$ the value of v at the midpoint m of the edge $\mathcal{F}$ of K, therefore the constraint (see Remark 2.1 for the P_1 element) on each element and the continuity(see Ref.[10] for the P_1 element) are the same. On a general quadrilateral, they are different, because the latter is still a linear function in the case, while the former is not a polynomial any more.

Remark 2.4. Compared to the P_1 element, the CNR $\mathcal{Q}_1$ element is defined on the reference element through the bilinear transformation, therefore its implementation is standard; while the P_1 element is directly defined on the physical element, its implementation is not standard.

We now evaluate the dimensions of CR^h and CR_0^h. Let N^V and N^E denote the numbers of nodes and elements of the partition, N_i^V and N_B^S denote the numbers of interior nodes and boundary edges, respectively. From the Euler relation of a quadrilateral partition, we have

$$\mathrm{dim(CR^h) = N^V - 1.}$$

The CR_0^h is a subset of R^h, which satisfies

$$\int_{\mathcal{F}_{1,K}} vds + \int_{\mathcal{F}_{3,K}} vds - \int_{\mathcal{F}_{2,K}} vds - \int_{\mathcal{F}_{4,K}} vds = 0, \forall K \in J^h, \forall v \in CNR_0^h,$$

$$\int_{\mathcal{F}_j} vds = 0, \forall \mathcal{F}_j, \forall v \in CNR_0^h,$$

where $\mathcal{F}_j, j = 1, \cdots, N_B^S$ are boundary edges of the partition. It is easy to find that, among these linear constraints, only $N^E + N_B^S - 1$ of them are linearly independent, which implies

$$\mathrm{dim(CR_0^h) = N_i^V.} \tag{2.5}$$

Now we look for a basis for CR_0^h. From (2.5), we need to choose N_i^V linearly independent functions from CR_0^h. On the reference element $\hat{K}$, define

$$\hat{\phi}_1 = \frac{1}{4}(1 - \xi - \eta), \quad \hat{\phi}_2 = \frac{1}{4}(1 + \xi - \eta),$$

$$\hat{\phi}_3 = \frac{1}{4}(1 + \xi + \eta), \quad \hat{\phi}_4 = \frac{1}{4}(1 - \xi + \eta),$$

which are associated to nodes $\hat{p}_i, i = 1, 2, 3, 4$ of $\hat{K}$, respectively. In particular, it holds that

$$\int_{\hat{\mathcal{F}}_1} \hat{\phi}_i d\hat{s} + \int_{\hat{\mathcal{F}}_3} \hat{\phi}_i d\hat{s} = \int_{\hat{\mathcal{F}}_2} \hat{\phi}_i d\hat{s} + \int_{\hat{\mathcal{F}}_4} \hat{\phi}_i d\hat{s}, i = 1, 2, 3, 4.$$

For each node p_j, let $E(j)$ denote the set of elements with the node p_j as one of their vertexes, define

$$\phi_j(p) = \begin{cases} \hat{\phi}_i(\mathcal{F}_K^{-1}(p)), p \in K \in E(j), \\ 0, p \in K \in J^h \backslash E(j), \end{cases} \tag{2.6}$$

where the subscript i is determined by $p_j = p_{i,K} = \mathcal{F}_K(\hat{p}_i)$ with $p_{i,K}, i = 1, 2, 3, 4$ four nodes of element K. Let $j = 1, \cdots, N_i^V$ denote the interior nodes, it is easy to see $\phi_j, j = 1, \cdots, N_i^V$ are linearly independent and that

$$span\{\phi_1, \cdots, \phi_{N_i^V}\} \subset CR_0^h,$$

therefore, $\{\phi_j\}_{j=1}^{N_i^V}$ is a basis of CR_0^h.

Remark 2.5. *Let Q_1 denote the conforming bilinear finite element space, we can see from (2.6) and the definition of π_h that $CR_0^h = \pi_h Q_1$.*

3. Application to the Second Order Elliptic Problem

We consider the following second order elliptic problem in its weak formulation:

Problem 3.1. *Find $u \in H_0^1(\Omega)$, such that*

$$a(u, v) = < f, v >, \quad \forall v \in H_0^1(\Omega), \tag{3.1}$$

where

$$a(u, v) = \int_\Omega \left[\sum_{i,j=1}^{2} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{j=1}^{2} b_j \frac{\partial u}{\partial x_j} v + b_0 uv \right] dx_1 dx_2.$$

Let CR_0^h approximate $H_0^1(\Omega)$, we get the discrete problem:

Problem 3.2. *Find $u_h \in CR_0^h$, such that*

$$a_h(u_h, v) = < f, v_h >, \quad \forall v_h \in CR_0^h. \tag{3.2}$$

where

$$a(u, v) = \sum_{K \in J^h} \int_K \left[\sum_{i,j=1}^{2} a_{i,j} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{j=1}^{2} b_j \frac{\partial u}{\partial x_j} v + b_0 uv \right] dx_1 dx_2.$$

Before considering the convergence of the discrete problem, we investigate the approximation and consistency properties of the CNR $\mathcal{Q}_1$ element.

Because CR_0^h is a subset of R_0^h, we have the following consistency error estimate [9]:

$$| \sum_K \int_{\partial K} v \Psi \cdot n ds | \leq Ch \|v\|_{1,h} \|\Psi\|_1, \quad \forall v \in R_0^h, \forall \Psi \in (H^1(\Omega))^2. \tag{3.3}$$

In order to study the approximation of the CNR $\mathcal{Q}_1$ element, we first summarize some interpolation results for $\pi_0^{\mathcal{F}}$ and its global version defined as $\pi_{h|_K} = \pi_K$.

Lemma 3.1. *[9] The foregoing defined interpolation operator $\pi_0^{\mathcal{F}}$ and π_h admit the following estimates*

$$\|v - \pi_0^{\mathcal{F}}(v)\|_{0,\mathcal{F}} \leq C h_K^{1/2} | v |_{1,K} \quad \forall v \in H^1(K). \tag{3.4}$$

Moreover, if the $(1+\alpha)-$Section Condition *holds, then*

$$\|v - \pi_h v\|_0 + h \|v - \pi_h v\|_{1,h} \leq C h^{1+\alpha} \|v\|_2 \quad \forall v \in H_0^1 \cap H^2. \tag{3.5}$$

For any $v \in H^2(\Omega)$, let $\Pi_1^h v$ denote its conforming bilinear interpolation, we have

$$| v - \Pi_1^h v |_{m,K} \leq C h^{2-m} \|v\|_{2,K}, m = 0, 1, 2. \tag{3.6}$$

Set $\Pi = \pi_h \Pi_1^h$. Because $\xi\eta$ is a bubble function for the operator π_h, we have

$$\Pi v = \pi_h \Pi_1^h v \in CR^h.$$

From Lemma 3.1 and (3.6), we deduce

$$\begin{aligned} & h \mid v - \Pi v \mid_{1,K} + \|v - \Pi v\|_{0,K} \\ & \leq h \mid v - \Pi_1^h v \mid_{1,K} + h \mid \Pi_1^h v - \Pi v \mid_{1,K} \\ & \quad + \|v - \Pi_1^h v\|_{0,K} + \|\Pi_1^h v - \Pi v\|_{0,K} \\ & \leq C h^{1+\alpha} \|v\|_{2,K}. \end{aligned} \tag{3.7}$$

For the interpolant Πv, there exists a simple expression in terms of $\phi_i = \hat{\phi}_i \circ \mathcal{F}_K^{-1}, i = 1, 2, 3, 4$ for any $v \in H^2(K)$. In fact, let $v_i, i = 1, 2, 3, 4$ be the values of v at vertexes $p_i, i = 1, 2, 3, 4$, a direct calculation gives

$$\Pi v = \sum_{i=1}^{4} v_i \phi_i.$$

Remark 3.1. From (3.7), we note that the CNR $\mathcal{Q}_1$ element shares the same approximation property as the NR $\mathcal{Q}_1$ element. This is partly because the term $\xi^2 - \eta^2$ is added to the NR $\mathcal{Q}_1$ element to satisfy the requirement of degrees of freedom, which has no contribution to the approximation.

Remark 3.2. Compared to the P_1 element, the interpolation error estimate of the CNR $\mathcal{Q}_1$ element depends on the mesh distortion parameter α; while the interpolation error estimate of the P_1 element is independent of α. However, when $\alpha = 1$, they converge at the same rate.

From (3.3) and (3.7), proceeding along the standard line of nonconforming finite element methods, we obtain the following error estimates,

Theorem 3.1. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of Problem 3.1 and $u_h \in CR_0^h$ be the solution of Problem 3.2, then*

$$\|u - u_h\|_{1,h} \leq Ch^{\alpha} \|u\|_2, \tag{3.8}$$

$$\|u - u_h\|_0 \leq Ch^{2\alpha} \|u\|_2. \tag{3.9}$$

4. Regular Green's Function

In order to get the superconvergence in the L^{∞} norm, we introduce the following regular delta function $\delta_{\varepsilon}(X, Z)$ for the point $Z \in K_Z$, which is defined such that

1. $(v, -\partial_{x,h}\delta_{\varepsilon}) = (\partial_{x,h}v, \delta_{\varepsilon}) = \partial_{x,h}v(Z), \quad \forall v \in CR_0^h$, where the differential operator $\frac{\partial}{\partial x}$ is defined element by element and denoted by $\partial_{x,h}$,
2. $| \delta_{\varepsilon} |_{s,\Omega,\infty} \leq C\varepsilon^{-2-s}, s = 0, 1,$
3. $\delta_{\varepsilon} \in C^0(K_Z) \cap H_0^1(K_Z)$ with $\varepsilon \geq C_1 h$.

For example, let $Z = (x_{0,K_Z}, y_{0,K_Z})$ be the center of element K_Z, let K' be a subrectangle of K_Z with (x_{0,K_Z}, y_{0,K_Z}) the center and $h_{x,K'}$ and $h_{y,K'}$ the meshsizes in the x and y direction. The regular delta function $\delta_{\varepsilon}(X, Z)$ can be defined as

$$\delta_{\varepsilon}(X, Z) = \frac{9}{4 \mid K' \mid}(1 - 4(\frac{x - x_{0,K}}{h_{x,K'}})^2)(1 - 4(\frac{y - y_{0,K}}{h_{y,K'}})^2).$$

For the general quadrilateral, $\delta_{\varepsilon}(X, Z)$ can be defined in a similar way.

The regular derivative Green's function $G(X, Z)$ is defined by: Find $G(X, Z) \in H_0^1(\Omega)$ such that

$$a(v, G) = (v, -\frac{\partial \delta_{\varepsilon}}{\partial x}) = (\frac{\partial v}{\partial x}, \delta_{\varepsilon}), \quad \forall v \in H_0^1(\Omega). \tag{4.1}$$

Its discrete problem is: Find $G_h(X, Z) \in CR_0^h$ such that

$$a_h(v, G_h) = (\partial_{x,h}v, \delta_{\varepsilon}) = \partial_{x,h}v(Z), \quad \forall v \in CR_0^h. \tag{4.2}$$

Introduce the weight function ρ defined by

$$\rho(X, Z) = (\|X - Z\|^2 + h^2)^{\frac{1}{2}} \tag{4.3}$$

It is well known that $\rho(X,Z)$ satisfies the following properties

$$\max_{X\in K}\rho(X,Z)\leq C\min_{X\in K}\rho(X,Z),\quad \forall K\in J^h, \tag{4.4}$$

$$\int_\Omega \rho^{-s}(X,Z)dxdy\leq \left\{\begin{array}{ll} Ch^{2-s}, & s>2,\\ C\mid \ln h\mid, & s=2.\end{array}\right. \tag{4.5}$$

For $s\in R$ and $k\in N$, the weighted norms and weighted seminorms are defined by

$$\|v\|_{k,\rho^s,\Omega}=(\int_\Omega \rho^s\sum_{|\beta|\leq k}\mid D^\beta v\mid^2 dxdy)^{\frac{1}{2}},$$

$$\mid v\mid_{k,\rho^s,\Omega}=(\int_\Omega \rho^s\sum_{|\beta|=k}\mid D^\beta v\mid^2 dxdy)^{\frac{1}{2}}.$$

In the analysis in the sequel, we use the following Sobolev inequality(see.Ref.[3] and the references therein for the details),

Lemma 4.1. *Let $f\in H^1(\Omega)$ and $p\gg 1$, we have the following estimate*

$$\|f\|_{0,p,\Omega}\leq Cp^{\frac{1}{2}}\|f\|_{1,\Omega}.$$

With these preparations at hand, we have

Lemma 4.2.

$$\|G\|^2_{1,\rho^2,\Omega}+\|G\|_{1,1,\Omega}+\|G\|_0^2\leq C\mid \ln h\mid .$$

Proof. For the simplicity, we only consider the case where

$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v dxdy.$$

We shall use the duality method to bound $\|G\|_0$. To this end, we introduce the following auxiliary problem: Find $u_G\in H_0^1(\Omega)$ such that

$$a(u_G,v)=(G,v). \tag{4.6}$$

Assume this problem admits the following regularity,

$$\|u_G\|_2\leq C\|G\|_0.$$

Take $v=G$ in (4.6), by the definition of the derivative Green function, we proceed as

$$\begin{aligned}\|G\|_0^2 &= a(u_G,G)=(\frac{\partial u_G}{\partial x},\delta_\varepsilon)=(\frac{\partial u_G}{\partial x},\delta_\varepsilon)_{K_Z}\\ &\leq \|\frac{\partial u_G}{\partial x}\|_{0,p,\Omega}\|\delta_\varepsilon\|_{0,q,K_Z}\quad (\text{with } \tfrac{1}{p}+\tfrac{1}{q}=1)\\ &\leq Ch^{-\frac{2}{p}}p^{\frac{1}{2}}\|\frac{\partial u_G}{\partial x}\|_1\quad (\text{ by Lemma 4.1})\\ &\leq Ch^{-\frac{2}{p}}p^{\frac{1}{2}}\|G\|_0\quad (\text{by the regularity}).\end{aligned}$$

Let $p=\mid\ln h\mid$ in the above inequality, we obtain

$$\|G\|_0\leq C\mid \ln h\mid^{\frac{1}{2}}.$$

Take $v = \rho^2 G$ in (4.1) and integrate by parts, we derive

$$\begin{aligned}\|\rho\nabla G\|_0^2 &= (\frac{\partial \rho^2 G}{\partial x}, \delta_\varepsilon) - 2(\rho\nabla\rho, G\nabla G)\\ &= (\rho^2\frac{\partial G}{\partial x}, \delta_\varepsilon) + 2(\rho G\frac{\partial \rho}{\partial x}, \delta_\varepsilon) - 2(\rho\nabla\rho, G\nabla G)\\ &\le C(\|\rho\nabla G\|_0 + \|G\|_0 + \|\rho\nabla G\|_0\|G\|_0).\end{aligned}$$

An application of Yong's inequality gives

$$\|\rho\nabla G\|_0 \le C \mid \ln h \mid^{\frac{1}{2}} .$$

We now turn to $\|G\|_{1,1,\Omega}$, which can be bounded as

$$\begin{aligned}\|G\|_{1,1,\Omega} &= \sum_{|\beta|\le 1}\int_\Omega \mid D^\beta G \mid dxdy\\ &\le \sum_{|\beta|\le 1}(\int_\Omega \rho^{-2}dxdy)^{\frac{1}{2}}(\int_\Omega \rho^2 \mid D^\beta G\mid^2 dxdy)^{\frac{1}{2}}\\ &\le C \mid \ln h \mid,\end{aligned}$$

which completes the proof.

Lemma 4.3. *There exist a positive constant C independent of h the mesh size and α the mesh distortion parameter such that*

$$\|G_h\|_0^2 \le Ch^{2\alpha-2} + C \mid \ln h \mid .$$

Proof. First it is easy to see

$$\|G_h\|_{1,h} \le Ch^{-1}.$$

We use the duality method again to bound $\|G_h\|_0$, thus introduce the auxiliary problem: Find $u_{G_h} \in H_0^1(\Omega)$ such that

$$a(u_{G_h}, v) = (G_h, v), \forall v \in H_0^1(\Omega). \tag{4.7}$$

Let $u_{G_h,h}$ be the CNR $\mathcal{Q}_1$ finite element solution to the above problem, from the consistency error estimate of CNR $\mathcal{Q}_1$ element and the regularity of u_{G_h}, we derive that

$$\begin{aligned}(G_h, G_h) &= (G_h, G_h) - a_h(u_{G_h}, G_h) + a_h(u_{G_h}, G_h)\\ &\le Ch\|u_{G_h}\|_2\|G_h\|_{1,h} + a_h(u_{G_h} - u_{G_h,h}, G_h) + a_h(u_{G_h,h}, G_h)\\ &\le Ch\|u_{G_h}\|_2\|G_h\|_{1,h} + (\partial_{x,h}u_{G_h,h}, \delta_\varepsilon)\\ &\le Ch^\alpha\|u_{G_h}\|_2\|G_h\|_{1,h} + (\partial_{x,h}(u_{G_h,h} - u_{G_h}), \delta_\varepsilon) + (\frac{\partial u_{G_h}}{\partial x}, \delta_\varepsilon)\\ &\le Ch^{\alpha-1}\|G_h\|_0 + (\frac{\partial u_{G_h}}{\partial x}, \delta_\varepsilon) \quad \text{(using Lemma 4.1 again)}\\ &\le C(h^{\alpha-1} + \mid \ln h \mid^{\frac{1}{2}})\|G_h\|_0.\end{aligned}$$

which ends the proof.

Lemma 4.4. *Let J^h be a rectangle partition of Ω, it holds*

$$\|G_h\|_{1,\rho^2,h}^2 + \|G_h\|_{1,1,h} \le C \mid \ln h \mid .$$

Proof.

$$\begin{aligned}\|\rho\nabla_h G_h\|_0^2 &= \sum_{K\in J^h}\int_K \rho^2\nabla G_h\cdot\nabla G_h dxdy\\ &= \sum_{K\in J^h}\int_K \nabla(\rho^2 G_h)\cdot\nabla G_h - 2\rho G_h\nabla\rho\cdot\nabla G_h dxdy\\ &= I_1+I_2.\end{aligned}$$

We first estimate the first term in the above equation. Let $Z\in K_Z$, for a given element K with the center $(x_{0,k}, y_{0,K})$ and the meshsizes $2h_x$ and $2h_y$ in the x and y direction respectively, define

$$d_x = -(x_Z - x_{0,K}),\quad d_y = -(y_Z - y_{0,K}).$$

It follows that

$$\rho^2(X,Z) = (h_x^2\xi^2 + 2h_x d_x\xi + h_y^2\eta + 2h_y d_y\eta + d_x^2 + d_y^2 + h^2).$$

By virtue of (4.4), we have

$$d_x^2 + d_y^2 \le C\rho^2. \tag{4.8}$$

Denote $p_i, i = 1,2,3,4$ the nodes of K numbered counterclockwise. Suppose G_h can be expressed as

$$G_h\mid_K = \sum_{i=1}^4 g_i\phi_i \text{ (see Section 2 for the definition of } \phi_i).$$

Let $w_h\in CR_0^h$ be a formal interpolant of $\rho^2 G_h$ defined by

$$w_h\mid_K = \sum_{i=1}^4 \rho_i^2 g_i\phi_i,$$

with ρ_i the values of ρ at nodes p_i, then we have

$$\begin{aligned}w_h\mid_K = &(h_x^2 - 2h_x d_x + h_y^2 - 2h_y d_y + d_x^2 + d_y^2 + h^2)g_1\phi_1\\ &+(h_x^2 + 2h_x d_x + h_y^2 - 2h_y d_y + d_x^2 + d_y^2 + h^2)g_2\phi_2\\ &+(h_x^2 + 2h_x d_x + h_y^2 + 2h_y d_y + d_x^2 + d_y^2 + h^2)g_3\phi_3\\ &+(h_x^2 - 2h_x d_x + h_y^2 + 2h_y d_y + d_x^2 + d_y^2 + h^2)g_4\phi_4.\end{aligned}$$

Consequently,

$$\begin{aligned}\rho^2 G_h - w_h = &(h_x^2(\xi^2-1) + 2h_x d_x(\xi+1) + h_y^2(\eta^2-1) + 2h_y d_y(\eta+1))g_1\phi_1\\ &+(h_x^2(\xi^2-1) + 2h_x d_x(\xi-1) + h_y^2(\eta^2-1) + 2h_y d_y(\eta+1))g_2\phi_2\\ &+(h_x^2(\xi^2-1) + 2h_x d_x(\xi-1) + h_y^2(\eta^2-1) + 2h_y d_y(\eta-1))g_3\phi_3\\ &+(h_x^2(\xi^2-1) + 2h_x d_x(\xi+1) + h_y^2(\eta^2-1) + 2h_y d_y(\eta+1))g_4\phi_4\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial\rho^2 G_h - w_h}{\partial x}\\
&= (2h_x\xi + 2d_x)g_1\phi_1 - \frac{g_1}{4h_x}(h_x^2(\xi^2-1) + 2h_xd_x(\xi+1) + h_y^2(\eta^2-1) + 2h_yd_y(\eta+1))\\
&\quad +(2h_x\xi + 2d_x)g_2\phi_2 + \frac{g_2}{4h_x}(h_x^2(\xi^2-1) + 2h_xd_x(\xi-1) + h_y^2(\eta^2-1) + 2h_yd_y(\eta+1))\\
&\quad +(2h_x\xi + 2d_x)g_3\phi_3 + \frac{g_3}{4h_x}(h_x^2(\xi^2-1) + 2h_xd_x(\xi-1) + h_y^2(\eta^2-1) + 2h_yd_y(\eta-1))\\
&\quad +(2h_x\xi + 2d_x)g_4\phi_4 - \frac{g_4}{4h_x}(h_x^2(\xi^2-1) + 2h_xd_x(\xi+1) + h_y^2(\eta^2-1) + 2h_yd_y(\eta+1)).
\end{aligned}$$

Which implies

$$\begin{aligned}
\int_K \frac{\partial(\rho^2 G_h - w_h)}{\partial x}\frac{\partial G_h}{\partial x}dxdy &= \int_{\hat{K}} 3(-g_1+g_2+g_3-g_4)\xi^2 bJ_K d\xi d\eta\\
&\quad + \int_{\hat{K}} (g_1-g_2-g_3+g_4)bJ_K d\xi d\eta\\
&\quad + \int_{\hat{K}} (-g_1+g_2+g_3-g_4)\frac{h_y^2}{h_x^2}(\eta^2-1)bJ_K d\xi d\eta\\
&\quad + \int_{\hat{K}} 2(-g_1+g_2+g_3-g_4)d_y\frac{h_y}{h_x^2}bJ_K d\xi d\eta\\
&\leq Ch^2\|\nabla G_h\|_{0,K}^2 + Ch \mid d_y \mid \|\nabla G_h\|_{0,K}^2\\
&\leq Ch^2\|\nabla G_h\|_{0,K}^2 + Ch\|\nabla G_h\|_{0,K}\|\rho\nabla G_h\|_{0,K},
\end{aligned}$$

where$b = \frac{1}{16}(-g_1+g_2+g_3-g_4)$. In the same way, we can prove

$$\int_K \frac{\partial(\rho^2 G_h - w_h)}{\partial y}\frac{\partial G_h}{\partial y}dxdy \leq Ch^2\|\nabla G_h\|_{0,K}^2 + Ch\|\nabla G_h\|_{0,K}\|\rho\nabla G_h\|_{0,K}.$$

Since

$$\begin{aligned}
\frac{\partial w_h}{\partial x}\mid_K &= \frac{1}{2}(g_1+g_2+g_3+g_4)d_x + 2(g_1-g_2+g_3-g_4)\frac{h_yd_y}{4h_x}\\
&\quad +(h_x^2+h_y^2+d_x^2+d_y^2+h^2)(-g_1+g_2+g_3-g_4)\frac{1}{4h_x},
\end{aligned}$$

we proceed as

$$\begin{aligned}
\mid \sum_{K\in J^h} \nabla w_h\cdot\nabla G_h \mid &= \mid (\frac{\partial w_h}{\partial x}, \delta_\varepsilon)_{K_Z} \mid\\
&\leq C\|G_h\|_{0,K_Z} + Ch\|G_h\|_{0,\infty,K_Z} + Ch\|\nabla G_h\|_{0,K_Z}\\
&\leq C \mid \ln h \mid^{\frac{1}{2}} .
\end{aligned}$$

Therefore

$$\begin{aligned}
\mid I_1 \mid &= \mid \sum_{K\in J^h} \nabla(\rho^2 G_h - w_h)\cdot\nabla G_h + \nabla w_h\cdot\nabla G_h dxdy \mid\\
&\leq C \mid \ln h \mid^{\frac{1}{2}} + C\|\rho\nabla_h G_h\|_0
\end{aligned}$$

Obviously, we have

$$\mid I_2 \mid \leq C \mid \ln h \mid^{\frac{1}{2}} \|\rho\nabla_h G_h\|_0.$$

Owing to the two estimates, using Young's inequality, we come to

$$\|\rho\nabla_h G_h\|_0^2 \leq C \mid \ln h \mid .$$

By the usual way, we have

$$\|G_h\|_{1,1,h} \leq C \mid \ln h \mid,$$

which completes the proof.

Lemma 4.5. *For a general quadrilateral mesh J^h, we can prove*

$$\|G_h\|_{1,\rho^2,h}^2 + \|G_h\|_{1,1,h} \leq Ch^{2\alpha-2} + C \mid \ln h \mid .$$

Proof. The proof is similar but lengthy expressions, for the brevity we omit it here.

5. Superconvergence of the CNR $\mathcal{Q}_1$ Element

In this section, we discuss the superconvergence of the CNR $\mathcal{Q}_1$ element. We shall get three kinds of superconvergence points.

We consider the rectangular mesh with constant coefficients in the first subsection and the rectangular mesh with variable coefficients in the second subsection. The general case is studied in the third subsection.

5.1 The case of constant coefficients

Let $J^h = \cup K$ be a regular rectangle partition of the domain Ω with $2h_{x,K}$ and $2h_{y,K}$ the mesh size of the element K in the x and y direction respectively, $h = \max\limits_{K\in J^h} \max(h_{x,K}, h_{y,K})$.

For a given element $K \in J^h$, let $(x_{0,K}, y_{0,K})$ be its center. Setting $R = u - \Pi_1^h u$, expanding R at the point (x, y), we have

$$R = \phi(x)u_{xx} + \psi(y)u_{yy} + r_3,$$

where

$$\phi(x) = \frac{(x - x_{0,K})^2}{2} - \frac{h_{x,K}^2}{2}, \qquad \psi(y) = \frac{(y - y_{0,K})^2}{2} - \frac{h_{y,K}^2}{2},$$

$$\|r_3\|_{s,p.K} \leq Ch^{3-s}\|u\|_{3,p,K}, s = 0, 1, 1 \leq p \leq \infty.$$

For any $v \in CR_0^h$, we have the following decomposition

$$\begin{aligned} a_h(u_h - \Pi u, v) &= a_h(u_h - \Pi_1^h u, v) + a_h(\Pi_1^h u - \Pi u, v) \\ &= a_h(u_h - u, v) + a_h(R, v) + a_h(\Pi_1^h u - \Pi u, v) \\ &= s_1 + s_2 + s_3. \end{aligned} \tag{5.1}$$

In order to show the superconvergence of the CNR $\mathcal{Q}_1$ element, we have to prove the superconvergence of the consistency error term s_1, the interpolation error term s_2 and s_3.

For s_1, we have

Theorem 5.1. *Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ be the solution of Problem 3.1, u_h be the solution of Problem 3.2, then*

$$\mid s_1 \mid \leq Ch^2\|u\|_{3,\Omega}\|v\|_{1,h}, \forall v \in CR_0^h. \tag{5.2}$$

Proof. Let $\nu = (\nu_1, \nu_2)$ denote the unit outward normal, we have

$$\begin{aligned} s_1 &= a_h(u_h - u, v) = (f, v) - a_h(u, v) \\ &= -\sum_{K\in J^h} \int_{\partial K} \sum_{i,j=1}^{2} a_{i,j} \frac{\partial u}{\partial x_i} \nu_j v ds \\ &= -\sum_{K\in J^h F\subset \partial k} \int_F \sum_{i,j=1}^{2} a_{i,j} \Big(\frac{\partial u}{\partial x_i} - \pi_0^F \frac{\partial u}{\partial x_i}\Big)(v - \pi_0^F v)\nu_j ds \\ &= -\sum_{K\in J^h} \sum_{i,j=1}^{2} \sum_{m=1}^{4} I_{i,j}^m, \end{aligned}$$

where $\pi_0^F w = \frac{1}{|F|}\int_F w ds$. Now we study the cancellation in the above identity. For a given element K with the center (a, b) and the mesh sizes $2r$ and $2s$ in x and y direction respectively, we only consider the case where $i = 1$ and $j = 1$.

$$\begin{aligned} I_{1,1}^1 + I_{1,1}^3 &= -a_{1,1} \int_{b-s}^{b+s} \Big[\frac{\partial u}{\partial x}(a - r, y)dy - \frac{1}{2s}\int_{b-s}^{b+s} \frac{\partial u}{\partial x}(a - r, t)dt\Big] \\ &\qquad \times\Big[v(a - r, y) - \frac{1}{2s}\int_{b-s}^{b+s} v(a - r, t)dt\Big]dy \\ &\quad +a_{1,1} \int_{b-s}^{b+s} \Big[\frac{\partial u}{\partial x}(a + r, y)dy - \frac{1}{2s}\int_{b-s}^{b+s} \frac{\partial u}{\partial x}(a + r, t)dt\Big] \\ &\qquad \times\Big[v(a + r, y) - \frac{1}{2s}\int_{b-s}^{b+s} v(a + r, t)dt\Big]dy. \end{aligned} \tag{5.3}$$

Note that v is a linear function on K, which implies

$$v(a - r, y) - \frac{1}{2s}\int_{b-s}^{b+s} v(a - r, t)dt = v(a + r, y) - \frac{1}{2s}\int_{b-s}^{b+s} v(a + r, t)dt. \tag{5.4}$$

Substituting (5.4) into (5.3) gives

$$\begin{aligned} I_{1,1}^1 + I_{1,1}^3 &= a_{1,1} \int_{b-s}^{b+s} \Big[\int_{a-r}^{a+r} \frac{\partial^2 u}{\partial x^2}(x, y)dx - \frac{1}{2s}\int_{b-s}^{b+s}\int_{a-r}^{a+r} \frac{\partial^2 u}{\partial x^2}(x, t)dxdt\Big] \\ &\qquad \times\Big[v(a + r, y) - \frac{1}{2s}\int_{b-s}^{b+s} v(a + r, t)dt\Big]dy \\ &= \frac{a_{1,1}}{2s}\int_{b-s}^{b+s} \left\{\int_{a-r}^{a+r}\int_{b-s}^{b+s}\int_t^y \frac{\partial^3 u}{\partial x^2 \partial z} dzdtdx\right\} \\ &\qquad \times \left\{\frac{1}{2s}\int_{b-s}^{b+s}\int_t^y \frac{\partial v}{\partial z}(a + r, z)dzdt\right\} dy. \end{aligned}$$

Because $\frac{\partial v}{\partial z}$ is a constant on K, we get

$$I_{1,1}^1 + I_{1,1}^3 \le Ch^2 \|u\|_{3,K} \mid v \mid_{1,K}.$$

Similarly

$$I_{i,j}^1 + I_{i,j}^3 \le Ch^2 \|u\|_{3,K} \mid v \mid_{1,K}, i, j = 1, 2,$$

and

$$I_{i,j}^2 + I_{i,j}^4 \le Ch^2 \|u\|_{3,K} \mid v \mid_{1,K}, i, j = 1, 2.$$

From these inequalities, we conclude that

$$| s_1 |= \sum_{K\in J^h}\sum_{i,j=1}^{2}\sum_{m=1}^{4} I_{i,j}^m \le Ch^2\|u\|_{3,\Omega} \mid v \mid_{1,h},$$

which completes the proof.

For s_2, we have

Theorem 5.2. *Let $u \in H^3(\Omega)\cap H_0^1(\Omega)$ be the solution of Problem 3.1, then*

$$| s_2 |\le Ch^2\|u\|_{3,\Omega}\|v\|_{1,h}, \forall v \in CR_0^h. \tag{5.5}$$

Proof.

$$\begin{aligned} s_2 &= \sum_{K\in J^h}\int_K \left\{\sum_{i,j=1}^{2} a_{i,j}\frac{\partial R}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{2} b_i\frac{\partial R}{\partial x_i}v + b_0 R v\right\} dxdy \\ &= J_1 + J_2 + J_3. \end{aligned}$$

On any element K, $\frac{\partial v}{\partial x_i}, i = 1, 2$ are constants and

$$\frac{\partial R}{\partial x} = (x - x_{0,K})u_{xx} + r_4, \quad \frac{\partial R}{\partial y} = (y - y_{0,K})u_{yy} + r_5,$$

where

$$\|r_4\|_{0,K} + \|r_5\|_{0,K} \le Ch^2\|u\|_{3,K}.$$

Let Π_0 denote the L^2 projection operator onto the piecewise constant space, we proceed as

$$\begin{aligned} \left| \int_K a_{1,1}\frac{\partial R}{\partial x}\frac{\partial v}{\partial x}dxdy \right| &\le \left| \int_K a_{1,1} r_4 \frac{\partial v}{\partial x}dxdy \right| \\ &\quad + \left| \int_K a_{1,1}(x - x_{0,K})u_{xx}\frac{\partial v}{\partial x}dxdy \right| \\ &= \left| \int_K a_{1,1} r_4 \frac{\partial v}{\partial x}dxdy \right| \\ &\quad + \left| \int_K a_{1,1}(x - x_{0,K})(I - \Pi_0)u_{xx}\frac{\partial v}{\partial x}dxdy \right| \\ &\le Ch^2\|u\|_{3,K}\|v\|_{1,K}. \end{aligned}$$

In the same way, we can obtain

$$\left| \int_K a_{i,j}\frac{\partial R}{\partial x_i}\frac{\partial v}{\partial x_j}dxdy \right| \le Ch^2\|u\|_{3,K}\|v\|_{1,K},$$

therefore,

$$| J_1 |\le Ch^2\|u\|_{3,\Omega}\|v\|_{1,h}.$$

We turn to the second term J_2.

$$\begin{aligned} \int_K \sum_{i=1}^{2} b_i\frac{\partial R}{\partial x_i}v dxdy &= \int_K \sum_{i=1}^{2} b_i\frac{\partial R}{\partial x_i}(v - \Pi_0 v)dxdy + \int_K \sum_{i=1}^{2} b_i\frac{\partial R}{\partial x_i}\Pi_0 v dxdy \\ &\le Ch^2\|u\|_{2,K} \mid v \mid_{1,K} + Ch^2\|u\|_{3,K}\|v\|_{0,K}, \end{aligned}$$

which implies

$$| J_2 |\le Ch^2\|u\|_{3,\Omega}\|v\|_{1,h}.$$

It is easy to see

$$| J_3 | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h},$$

which ends the proof.

We remain to estimate the third term s_3 in (5.1), which is bounded in the following theorem.

Theorem 5.3. *Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ be the solution of Problem 3.1, then*

$$| a_h(\Pi_1^h u - \Pi u, v) | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}, \forall v \in CR_0^h. \tag{5.6}$$

Proof. Denote $\delta u = \Pi_1^h u - \Pi u$, we have

$$\begin{aligned} a_h(\Pi_1^h u - \Pi u, v) &= \sum_{K \in J^h} \int_K \left\{ \sum_{i,j=1}^2 a_{i,j} \frac{\partial \delta u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^2 b_i \frac{\partial \delta u}{\partial x_i} v + b_0 \delta u v \right\} dxdy \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Since

$$\delta u \,|_K = C_K \frac{x - x_{0,K}}{h_{x,K}} \frac{y - y_{0,k}}{h_{y,K}},$$

where C_K is a constant, and $\frac{\partial v}{\partial x_j}$, $j = 1, 2$, are piecewise constants, we have

$$J_1 = 0.$$

By virtue of $| \Pi_1^h u |_{2,K} \leq C \|u\|_{3,K}$, proceeding along the same line of Theorem 5.2, we can show that

$$| J_2 + J_3 | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h},$$

which ends the proof.

Combining Theorem 5.1, Theorem 5.2 and Theorem 5.3, we obtain

Theorem 5.4. *Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ be the solution of Problem 3.1 and u_h be the solution of Problem 3.2, then*

$$| a_h(u_h - \Pi u, v) | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}, \forall v \in CR_0^h, \tag{5.7}$$

and

$$| u_h - \Pi u |_{1,h} \leq Ch^2 \|u\|_{3,\Omega}. \tag{5.8}$$

Theorem 5.5. *A slight modification of the above analysis, we can obtain*

$$| a_h(u_h - \Pi u, v) | \leq Ch^2 \|u\|_{3,\infty,\Omega} \|v\|_{1,1,h}, \forall v \in CR_0^h \tag{5.9}$$

where the broken norm $\| \cdot \|_{1,1,h}$ is similarly defined as in (2.4)

For the analysis, we need the following result concerning the superconvergence at the centrial points of elements for the interpolation operator Π

Theorem 5.6. *Let $u \in W^{3,\infty}(\Omega)$, then it holds that*

$$\max_{K \in J^h} | \nabla(u - \Pi u)(x_{0,K}, y_{0,K}) | \leq Ch^2 \|u\|_{3,\infty,\Omega}. \tag{5.10}$$

Proof. Denote $\hat{\mathcal{Q}}(\hat{u}) = | \hat{\nabla}(\hat{u} - \hat{\Pi}\hat{u})(\hat{O}) |$, where $\hat{O}$ is the center of $\hat{K}$. Note that

$$| \hat{\mathcal{Q}}(\hat{u}) | \leq C \|\hat{u}\|_{3,\infty,\hat{K}},$$

For any $\hat{v} \in P_2(\hat{K})$, it is easy to see

$$\hat{\mathcal{Q}}(\hat{v}) = 0.$$

It follows from Bramble-Hilbert Lemma that

$$\hat{\mathcal{Q}}(\hat{u}) \leq C \mid \hat{u} \mid_{3,\infty,\hat{K}},$$

which, together with the scaling argument, implies the desired result.

From Theorem 5.5, Lemma 4.4 and Theorem5.6, we have the following superconvergence

Theorem 5.7. *Let $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$ be the solution of Problem 3.1, and u_h be the solution of Problem 3.2, $\mathcal{O}_K$ be the center of K, then*

$$\max_{K \in J^h} \mid \nabla(u - u_h)(\mathcal{O}_K) \mid \leq Ch^2 \mid \ln h \mid |u\|_{3,\infty,\Omega}. \tag{5.11}$$

Proof. For a given element K with $Z = (x_{0,K}, y_{0,k})$ the center, we have

$$\begin{aligned}
\partial_{x,h}(u_h - \Pi u)(Z) &= a_h(u_h - \Pi u, G_h) \\
&\leq Ch^2 \|u\|_{3,\infty,\Omega} \|G_h\|_{1,1,h} \\
&\leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega}.
\end{aligned}$$

In the same way, we can prove

$$\partial_{y,h}(u_h - \Pi u)(Z) \leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega}.$$

The above two inequalities imply

$$\max_K \mid \nabla(u_h - \Pi u)(x_{0,K}, y_{0,K}) \mid \leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega}.$$

Owing to Theorem 5.6, we come to

$$\begin{aligned}
\max_K \mid \nabla(u_h - u)(x_{0,K}, y_{0,K}) \mid &\leq \max_K \mid \nabla(u_h - \Pi u)(x_{0,K}, y_{0,K}) \mid \\
&\quad + \max_K \mid \nabla(\Pi u - u)(x_{0,K}, y_{0,K}) \mid \\
&\leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega},
\end{aligned}$$

which is the desired result.

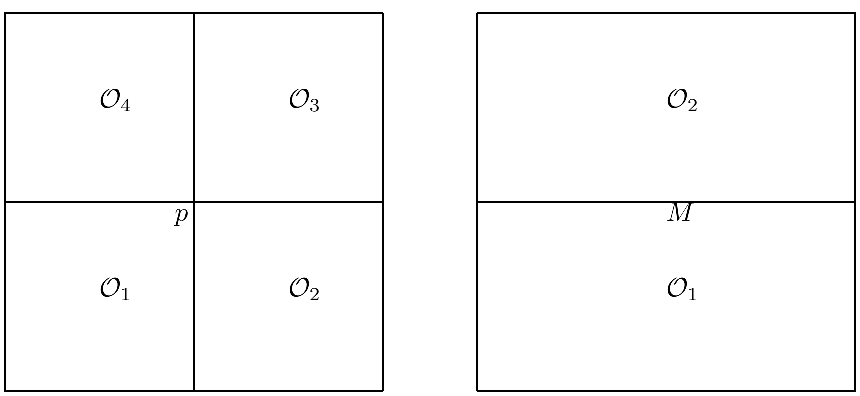


Figure 1: Node and Midpoint

Now we are going to discuss the superconvergence at nodes and midpoints of edges. In this stage, we assume the partition is uniform. As illustrated by Figure 1, let the centrial points near to the node p be denoted by $\mathcal{O}_i, i = 1, \cdots, 4$, assume that $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$.

Define

$$\overline{\partial_i u_h}(p) = \frac{1}{4}\sum_{j=1}^{4}\frac{\partial u_h}{\partial x_i}(\mathcal{O}_j), i = 1, 2.$$

Let $\mathcal{V}_I$ denote the set of interior nodes, then we have

Theorem 5.8. *Let $u \in W^{3,\infty}(\Omega)\cap H_0^1(\Omega)$ be the solution of Problem 3.1, and u_h be the solution of Problem 3.2, then*

$$\max_{p\in\mathcal{V}_i} \mid \partial_i u(p) - \overline{\partial_i u_h}(p) \mid \leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega}.$$

Proof. Denote $\partial_i u = \frac{\partial u}{\partial x_i}, i = 1, 2$, by Taylor expansion, we have

$$\partial_i u(p) = \partial_i u(\mathcal{O}_1) + \nabla\partial_i u(\mathcal{O}_1)(h_x, h_y) + r_1,$$

$$\partial_i u(p) = \partial_i u(\mathcal{O}_2) + \nabla\partial_i u(\mathcal{O}_2)(-h_x, h_y) + r_2,$$

$$\partial_i u(p) = \partial_i u(\mathcal{O}_3) + \nabla\partial_i u(\mathcal{O}_3)(-h_x, -h_y) + r_3,$$

$$\partial_i u(p) = \partial_i u(\mathcal{O}_4) + \nabla\partial_i u(\mathcal{O}_2)(h_x, -h_y) + r_4,$$

which give

$$\partial_i u(p) = \frac{1}{4}\sum_{j=1}^{4}\partial_i u(\mathcal{O}_j) + r_p,$$

$$\mid r_p \mid \leq Ch^2\|u\|_{3,\infty,\Omega}.$$

Then

$$\mid \partial_i u(p) - \overline{\partial_i u_h}(p) \mid \leq \mid \frac{1}{4}\sum_{j=1}^{4}(\partial_i u(\mathcal{O}_j) - \partial_i u_h(\mathcal{O}_j)) \mid + Ch^2\|u\|_{3,\infty,\Omega}. \tag{5.12}$$

Owing to Theorem 5.7, we come to

$$\max_{p\in\mathcal{V}_i} \mid \partial_i u(p) - \overline{\partial_i u_h}(p) \mid \leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega},$$

which is the desired result.

As illustrated by Figure 1, let the centrial points near to the midpoint M be $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively, define

$$\overline{\partial_i u_h}(M) = \frac{1}{2}(\frac{\partial u_h}{\partial x_i}(\mathcal{O}_1) + \frac{\partial u_h}{\partial x_i}(\mathcal{O}_2)), i = 1, 2.$$

Let $\mathcal{M}_I$ be the set of midpoints of interior edges, similarly,

Theorem 5.9. *Let $u \in W^{3,\infty}(\Omega)\cap H_0^1(\Omega)$ be the solution of Problem 3.1, and u_h be the solution of Problem 3.2, then*

$$\max_{p\in\mathcal{M}_I} \mid \partial_i u(p) - \overline{\partial_i u_h}(p) \mid \leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega}.$$

5.2 The case of variable coefficients

In this case, our main task is still to bound the three terms s_1, s_2 and s_3 in the decoposition (5.1), which are estimated in the following Theorem.

Theorem 5.10. *Assume that* $a_{i,j} \in W^{2,\infty}(\Omega), i, j = 1, 2,\ b_i \in W^{1,\infty}(\Omega), i = 1, 2,\ b_0 \in L^\infty(\Omega)$, *then*

$$| s_1 | + | s_2 | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}, \forall v \in CR_0^h, \tag{5.13}$$

$$| a_h(\Pi_1^h u - \Pi u, v) | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}, \forall v \in CR_0^h. \tag{5.14}$$

$$\|u_h - \Pi u\|_{1,h} \leq Ch^2 \|u\|_{3,\Omega}. \tag{5.15}$$

Proof.

$$\begin{aligned} s_1 &= a_h(u_h - u, v) = -\sum_{K\in J^h} \int_{\partial K} \sum_{i,j=1}^{2} a_{i,j} \frac{\partial u}{\partial x_i} \nu_j v ds \\ &= -\sum_{K\in J^h F\subset\partial k} \int_F \sum_{i,j=1}^{2} [a_{i,j} \frac{\partial u}{\partial x_i} - \pi_0^F (a_{i,j} \frac{\partial u}{\partial x_i})] \times [v - \pi_0^F v] \nu_j ds. \end{aligned}$$

By the assumption on $a_{i,j}$, repeating the line of Theorem5.1, it can be shown that

$$| s_1 | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}.$$

$$\begin{aligned} s_2 &= \sum_{K\in J^h} \int_K \left\{ \sum_{i,j=1}^{2} a_{i,j} \frac{\partial R}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial R}{\partial x_i} v + b_0 R v \right\} dxdy \\ &= \sum_{K\in J^h} \int_K \left\{ \sum_{i,j=1}^{2} \Pi_0 a_{i,j} \frac{\partial R}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{2} \Pi_0 b_i \frac{\partial R}{\partial x_i} v + b_0 R v \right\} dxdy \\ &\quad + \sum_{K\in J^h} \int_K \left\{ \sum_{i,j=1}^{2} (a_{i,j} - \Pi_0 a_{i,j}) \frac{\partial R}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{2} (b_i - \Pi_0 b_i) \frac{\partial R}{\partial x_i} v \right\} dxdy \\ &= J_1 + J_2. \end{aligned}$$

The first term J_1 can be bounded in the same way as in Theorem 5.2, which reads

$$| J_1 | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}.$$

It is easy to see that

$$| J_2 | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}.$$

Similarly,

$$| a_h(\Pi_1^h u - \Pi u, v) | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h},$$

which completes the proof.

Similarly, we have

$$| a_h(u_h - \Pi u, v) | \leq Ch^2 \|u\|_{3,\infty,\Omega} \|v\|_{1,1,h}, \quad \forall v \in CR_0^h. \tag{5.16}$$

Owing to (5.16), the same procedures of Theorem 5.7, Theroem 5.8 and Theorem 5.9 yield

Theorem 5.11. *Let* $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$ *be the solution of Problem 3.1, and* u_h *be the solution of Problem 3.2. In addition, assume that* $a_{i,j} \in W^{2,\infty}(\Omega), i, j = 1, 2,\ b_i \in W^{1,\infty}(\Omega), i = 1, 2, b_0 \in L^\infty(\Omega)$, *we have*

$$\max_{K\in J^h} | \nabla(u - u_h)(x_{0,K}, y_{0,K}) | \leq Ch^2 | \ln h | \|u\|_{3,\infty,\Omega}, \tag{5.17}$$

$$\max_{p\in\mathcal{V}_I} \mid \partial_i u(p) - \overline{\partial_i u_h}(p) \mid \leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega},$$

$$\max_{p\in\mathcal{M}_I} \mid \partial_i u(p) - \overline{\partial_i u_h}(p) \mid \leq Ch^2 \mid \ln h \mid \|u\|_{3,\infty,\Omega}.$$

In the last two estimates, we assume that J^h is a uniform partition of the domain Ω.

5.3 The General Case

In this subsection, we give the analysis of the superconvergence on a general quadrilateral mesh. We assume the partition satisfied the $1+\alpha$ condition defined by Assumption 2.1.

From the above discussion, in order to analyze the superconvergence, the key is to bound the three term s_1, s_2 and s_3 in the decomposition (5.1). We first deal with the consistency error term s_1. As usual, we only consider the case where $i=1$, $j=1$ with constant coefficients. We shall apply the technique of [11]. For a given element K,

$$\begin{aligned} I_{1,1}^1 + I_{1,1}^3 &= a_{1,1}\int_{F_1}(\frac{\partial u}{\partial x} - \pi_0^{F_1}\frac{\partial u}{\partial x}) \times (v - \pi_0^{F_1}v)\nu_{1,1}ds \\ &\quad +a_{1,1}\int_{F_3}(\frac{\partial u}{\partial x} - \pi_0^{F_3}\frac{\partial u}{\partial x}) \times (v - \pi_0^{F_3}v)\nu_{1,3}ds \\ &= a_{1,1}\frac{\mid F_1 \mid}{\mid \hat{F}_1 \mid} \mid \hat{F}_1 \mid \int_{-1}^1 (\widehat{\frac{\partial u}{\partial x}}(-1,\eta) - \widehat{\pi_0^{F_1}\frac{\partial u}{\partial x}}) \\ &\quad \times(\hat{v}(-1,\eta) - \widehat{\pi_0^{F_1}v})\frac{d_{12}-d_2}{\mid F_1 \mid}d\eta \\ &\quad +a_{1,1}\frac{\mid F_3 \mid}{\mid \hat{F}_3 \mid} \mid \hat{F}_3 \mid \int_{-1}^1 (\widehat{\frac{\partial u}{\partial x}}(1,\eta) - \widehat{\pi_0^{F_3}\frac{\partial u}{\partial x}}) \\ &\quad \times(\hat{v}(1,\eta) - \widehat{\pi_0^{F_3}v})\frac{d_{12}+d_2}{\mid F_3 \mid}d\eta. \end{aligned}$$

Since $\mid \hat{F}_1 \mid = \mid \hat{F}_3 \mid = 2$, it follows that

$$\pi_0^{F_1}\frac{\partial u}{\partial x} = \frac{1}{\mid \hat{F}_1 \mid}\int_{-1}^1 \widehat{\frac{\partial u}{\partial x}}(-1,t)dt, \quad \pi_0^{F_3}\frac{\partial u}{\partial x} = \frac{1}{\mid \hat{F}_3 \mid}\int_{-1}^1 \widehat{\frac{\partial u}{\partial x}}(1,t)dt.$$

Note that $\hat{v}$ is a linear function with respect to ξ and η on $\hat{K}$, we have

$$\hat{v}(-1,\eta) - \widehat{\pi_0^{F_1}v} = \hat{v}(1,\eta) - \widehat{\pi_0^{F_3}v}.$$

Using these equations, we derive as

$$\begin{aligned} I_{1,1}^1+I_{1,1}^3 &= \frac{a_{1,1}}{4}d_2\int_{-1}^1\left\{\int_{-1}^1\int_t^\eta\int_{-1}^1\frac{\partial^2}{\partial s\partial z}\widehat{\frac{\partial u}{\partial x}}(s,z)dsdzdt\right\} \\ &\qquad\times\left\{\int_{-1}^1\int_t^\eta\frac{\partial}{\partial z}\hat{v}(1,z)dzdt\right\}d\eta \\ &\quad+\frac{a_{1,1}}{4}d_{12}\int_{-1}^1\left\{\int_{-1}^1\int_t^\eta[\frac{\partial}{\partial z}\widehat{\frac{\partial u}{\partial x}}(-1,z)+\frac{\partial}{\partial z}\widehat{\frac{\partial u}{\partial x}}(1,z)](s,z)dzdt\right\} \\ &\qquad\times\left\{\int_{-1}^1\int_t^\eta\frac{\partial}{\partial z}\hat{v}(1,z)dzdt\right\}d\eta \\ &\leq Ch\mid\widehat{\frac{\partial u}{\partial x}}\mid_{2,\hat{K}}\mid\hat{v}\mid_{1,\hat{K}}+Ch^{1+\alpha}\mid\widehat{\frac{\partial u}{\partial x}}\mid_{1,\hat{K}}\mid\hat{v}\mid_{1,\hat{K}} \\ &\leq Ch^{1+\alpha}\|u\|_{3,K}\|v\|_{1,K}. \end{aligned}$$

Similarly,

$$\mid I_{i,j}^1+I_{i,j}^3\mid\leq Ch^{1+\alpha}\|u\|_{3,K}\|v\|_{1,K},\, i,j=1,2,$$

and

$$\mid I_{i,j}^2+I_{i,j}^4\mid\leq Ch^{1+\alpha}\|u\|_{3,K}\|v\|_{1,K},\, i,j=1,2.$$

Therefore,

$$\mid s_1\mid=\mid\sum_{K\in J^h}\sum_{i,j=1}^2\sum_{l=1}^4 I_{i,j}^l\mid\leq Ch^{1+\alpha}\|u\|_{3,\Omega}\|v\|_{1,h}. \tag{5.18}$$

We now estimate the second s_2 of (5.1). On the reference element $\hat{K}$, we have

$$\hat{R}(\xi,\eta)=\frac{(\xi^2-1)}{2}u_{\xi\xi}+\frac{(\xi^2-1)}{2}u_{\eta\eta}+r_3, \tag{5.19}$$

where r_3 only consists of the third order derivatives of u with respect to ξ and η.

$$\begin{aligned} s_2 &= \sum_{K\in J^h}\int_K\left\{\sum_{i,j=1}^2 a_{i,j}\frac{\partial R}{\partial x_i}\frac{\partial v}{\partial x_j}+\sum_{i=1}^2 b_i\frac{\partial R}{\partial x_i}v+b_0Rv\right\}dxdy \\ &= J_1+J_2+J_3. \end{aligned}$$

It is easy to find

$$\mid J_3\mid\leq Ch^2\|u\|_{3,\Omega}\|v\|_{1,h}.$$

The key is to bound J_1 and J_2. For a given element K, we consider the case where $i=1$ and $j=1$. In this case,

$$\begin{aligned} \int_K a_{1,1}\frac{\partial R}{\partial x}\frac{\partial v}{\partial x}dxdy &= a_{1,1}\int_{\hat{K}}\left[\frac{\partial\hat{R}}{\partial\xi}(d_2+d_{12}\xi)+\frac{\partial\hat{R}}{\partial\eta}(-d_1-d_{12}\eta)\right] \\ &\qquad\times\left[\frac{\partial\hat{v}}{\partial\xi}\frac{(d_2+d_{12}\xi)}{J_K}+\frac{\partial\hat{v}}{\partial\eta}\frac{(-d_1-d_{12}\eta)}{J_K}\right]d\xi d\eta. \end{aligned}$$

Assume that $\alpha>0$. From Assumption 2.1, when h is small enough, it holds

$$\frac{1}{J_K}=\frac{1}{J_0}[1-\frac{J_1\xi+J_2\eta}{J_0}+r)],$$

with $\mid r\mid\leq Ch^{2\alpha}$.

Taking into account the expansion of $\hat{R}$ and the fact that $\frac{\partial \hat{v}}{\partial \xi}$ and $\frac{\partial \hat{v}}{\partial \eta}$ are constants on $\hat{K}$, we can prove

$$| \int_K a_{1,1} \frac{\partial R}{\partial x} \frac{\partial v}{\partial x} dxdy | \leq Ch^{2\alpha} \|u\|_{3,K} \|v\|_{1,K},$$

which implies

$$| J_1 | \leq Ch^{2\alpha} \|u\|_{3,\Omega} \|v\|_{1,h}.$$

We now turn to J_2, which can be decomposed as

$$\begin{aligned} J_2 &= \sum_{K \in J^h} \int_K \sum_{i=1}^{2} \frac{\partial R}{\partial x_i} v dxdy \\ &= \sum_{K \in J^h} \int_K \sum_{i=1}^{2} \frac{\partial R}{\partial x_i} (v - \Pi_0 v) + \frac{\partial R}{\partial x_i} \Pi_0 v dxdy \\ &= I_1 + I_2. \end{aligned}$$

Obviously,

$$| I_1 | \leq Ch^{2\alpha} \|u\|_{3,\Omega} \|v\|_{1,h}.$$

Repeating the line of estimating J_1, we obtain

$$| I_2 | \leq Ch^{2\alpha} \|u\|_{3,\Omega} \|v\|_{1,h}.$$

Therefore,

$$| s_2 | \leq Ch^{2\alpha} \|u\|_{3,\Omega} \|v\|_{1,h}. \tag{5.20}$$

From (5.18) and (5.20), we obtain

$$| a_h(u_h - \Pi_1^h u, v) | \leq Ch^{2\alpha} \|u\|_{3,\Omega} \|v\|_{1,h}, \forall v \in CR_0^h. \tag{5.21}$$

Because $\widehat{\delta u} =$ only consists of the intersected term $\xi\eta$ and $\hat{v}$ is a linear function on $\hat{K}$, taking into account the approximation of the operator π_h (see Lemma 3.1) and (2.1)-(2.3), a similar procedure of estimating s_2 yields

$$| s_3 | = | a_h(\Pi_1^h u - \Pi u, v) | \leq Ch^2 \|u\|_{3,\Omega} \|v\|_{1,h}, \forall v \in CR_0^h. \tag{5.22}$$

Combining these estimates, we obtain

$$\|u_h - \Pi u\|_{1,h} \leq Ch^{2\alpha} \|u\|_{3,\Omega}. \tag{5.23}$$

Repeating the same line, we can prove

$$| a_h(u_h - \Pi u, v) | \leq Ch^{2\alpha} \|u\|_{3,\infty,\Omega} \|v\|_{1,1,h}, \quad \forall v \in CR_0^h. \tag{5.24}$$

Applying (2.1)-(2.3) again, using Bramble-Hilbert Lemma and the scaling argument as in Theorem 5.6, we get

$$\max_{K \in J^h} | \nabla(u - \Pi u)(x_{0,K}, y_{0,K}) | \leq Ch^{2\alpha} \|u\|_{3,\infty,\Omega}. \tag{5.25}$$

From (5.24) and (5.25), applying Lemma 4.5, we obtain

Theorem 5.12. *Let $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$ be the solution of Problem 3.1 and u_h be the solution of Problem 3.2, assume that $a_{i,j} \in W^{2,\infty}(\Omega), i,j = 1,2$, $b_i \in W^{1,\infty}(\Omega), i = 1,2, b_0 \in L^\infty(\Omega)$, and the partition satisfies $1+\alpha$ condition, then*

$$\max_{K \in J^h} | \nabla(u - u_h)(\mathcal{O}_K) | \leq Ch^{4\alpha - 2} | \ln h | \|u\|_{3,\infty,\Omega}. \tag{5.26}$$

Remark 5.1. For the general quarilateral mesh, there are no similar superconvergences at the nodes of the partition and the midpoints of edges.

5.4 Summary

In this section, we analyze the superconvergence of the CNR $\mathcal{Q}_1$ element. Because the CNR $\mathcal{Q}_1$ element is equivalent to the P_1 element on a rectangle, the analysis and results hold equally for the the P_1 element when a rectangular mesh is uased. When a general quadrilateral mesh is used, we don't know whether there are similar results. The technique in this section is inapplicable for the NR $\mathcal{Q}_1$ element, because in this case, Theorem 5.2 doesn't hold any more.

6. Postprocessing

In this section, we shall propose a new postprocessing technique which admits a superconvergence postprocessed discrete solution.

Let J^h be obtained from a coarse quadrilateral mesh J^{2h} by bi-sectioning each quadrilateral M, $p_i, i = 1, \cdots, 9$ be the nodes on M(see Figure 2 for an example).

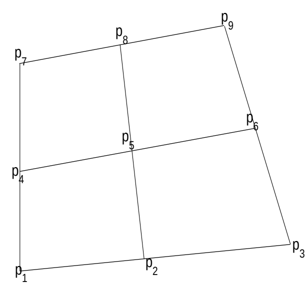


Figure 2: Macroelement

For any $v_h \in CR_0^h$ in the form

$$v_h \mid_M = \sum_{i=1}^{9} v_i \phi_i,$$

we define an interpolant $\Pi_2 v_h \in Q_2(M)$ by

$$\Pi_2 v_h = \sum_{j=1}^{9} v_i \Phi_i,$$

where $\Phi_i, i = 1, \cdots, 9$ are the basis functions of the space $Q_2(M)$. For any $w \in H^2(\Omega) \cap H_0^1(\Omega)$, let $\Pi_2' w$ its piecewise biquadratic interpolant with respect to the coarse partition J^{2h} defined by

$$\Pi_2' w \mid_M = \sum_{i}^{9} w_i \Phi_i,$$

where w_i are the values of w on the nodes p_i. Obviously, we have

Lemma 6.1. *For any $w \in H^2(\Omega) \cap H_0^1(\Omega)$, it holds that*

$$\Pi_2' w = \Pi_2 \Pi w.$$

We now prove that Π_2 is a bounded operator with repspect to the norm $\|\cdot\|_{1,h}$,

Lemma 6.2. *For any $v_h \in CR_0^h$, it holds that*

$$\mid \Pi_2 v_h \mid_{1,M} \leq C \mid v_h \mid_{1,M,h}, \tag{6.1}$$

$$\mid \Pi_2 v_h \mid_1 \leq C \mid v_h \mid_{1,h} . \tag{6.2}$$

Proof. In order to verify this inequality we consider the reference macroelement $\widehat{M}$ consisting of four subcells $F_M^{-1}(K_j)$, $j = 1, 2, 3, 4$. Let $\widehat{W}$ denote the nonconforming finite element space of piecewise linear functions(spanned by $1, \xi, \eta$) with zero meanvalue of the jumps over four inner edges of the subcells. Define the following seminorm

$$\mid \widehat{w} \mid_{1,\widehat{M},h} = \left\{ \sum_j^4 \mid \widehat{w} \mid^2_{1,F_M^{-1}(K_j)} \right\}^{\frac{1}{2}}, \forall \widehat{w} \in \widehat{W}.$$

By virtue of the definition, for any $\widehat{w} \in \widehat{W}$, we have,

$$\mid \widehat{\Pi_2 w} \mid_{1,F_M^{-1}(K_j)} = \mid \widehat{\Pi}_2 \widehat{w} \mid_{1,F_M^{-1}(K_j)} \leq C\|\widehat{w}\|_{0,\infty,F_M^{-1}(K_j)} \leq C\|\widehat{w}\|_{1,F_M^{-1}(K_j)},$$

here $w = \widehat{w} \circ F_M^{-1}$. Thus,

$$\mid \widehat{\Pi_2 w} \mid_{1,\hat{M}} \leq C \left\{ \sum_j^4 \|\widehat{w}\|^2_{1,F_M^{-1}(K_j)} \right\}^{\frac{1}{2}}$$

Moreover, $\mid \cdot \mid_{1,\widehat{M},h}$ is also a norm on the factor space $\widehat{W}/R$, which implies

$$\begin{aligned} \mid \widehat{\Pi_2 w} \mid_{1,\hat{M}} &\leq C \left\{ \sum_j^4 \|\widehat{w}\|^2_{1,F_M^{-1}(K_j)} \right\}^{\frac{1}{2}} \\ &\leq C \left\{ \sum_j^4 \mid \widehat{w} \mid^2_{1,F_M^{-1}(K_j)} \right\}^{\frac{1}{2}}, \quad \forall \widehat{w} \in \widehat{W}/R. \end{aligned}$$

Further, both sides of the inequality vanish for a constant funcition $\widehat{w} \in \widehat{W}$. Thus,

$$\mid \widehat{\Pi_2 w} \mid_{1,\hat{M}} \leq C \left\{ \sum_j^4 \mid \widehat{w} \mid^2_{1,F_M^{-1}(K_j)} \right\}^{\frac{1}{2}}, \quad \forall \widehat{w} \in \widehat{W},$$

which prove (6.1). It is easy to see that $\Pi_2 v_h \in H_0^1(\Omega)$, and then (6.2) is the direct consequence of (6.1).

For the postprocessed solution $\Pi_2 u_h$, we have the following superconvergence,

Theorem 6.1. *Let $u \in H^3(\Omega) \cap H_0^1(\Omega)$ be the solution of Problem 3.1 and u_h be the solution of Problem 3.2, assume that $a_{i,j} \in W^{2,\infty}(\Omega), i, j = 1, 2,$ $b_i \in W^{1,\infty}(\Omega), i = 1, 2, b_0 \in L^\infty(\Omega)$, and the partition satisfies $1 + \alpha$ condition, then*

$$\mid u - \Pi_2 u_h \mid_1 \leq Ch^{2\alpha}\|u\|_3. \tag{6.3}$$

Proof. Owing to estimate (5.23), Lemma 6.1 and Lemma 6.2, we derive

$$\begin{aligned} | u - \Pi_2 u_h |_1 &\leq | u - \Pi_2' u |_1 + | \Pi_2' u - \Pi_2 \Pi u |_1 + | \Pi_2 \Pi u - \Pi_2 u_h |_1 \\ &\leq Ch^{2\alpha} \|u\|_3 + C \mid \Pi u - u_h |_{1,h} \\ &\leq Ch^{2\alpha} \|u\|_3. \end{aligned}$$

7. Numerical Examples

In this section, we consider numerical experiments of second order elliptic problems with Dirichlet boundary condition, which reads as

$$\begin{cases} -\triangle u = f, & \Omega = [0,1]^2, \\ u = 0, & \partial\Omega. \end{cases}$$

We choose f such that the exact solution is $u(x,y) = \sin 2\pi x \sin 2\pi y$. The example meshes are illustrated by Figures (3)-(5). In numerical examples, we compare the CNR $\mathcal{Q}_1$ element with the nonconforming quadrilateral P_1 element from [10]. The numerical results are listed in Table 1-Table 3.

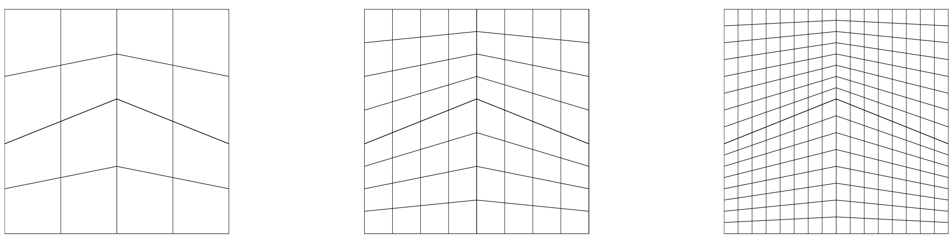


Figure 3: Example Mesh $\alpha = 1.0$

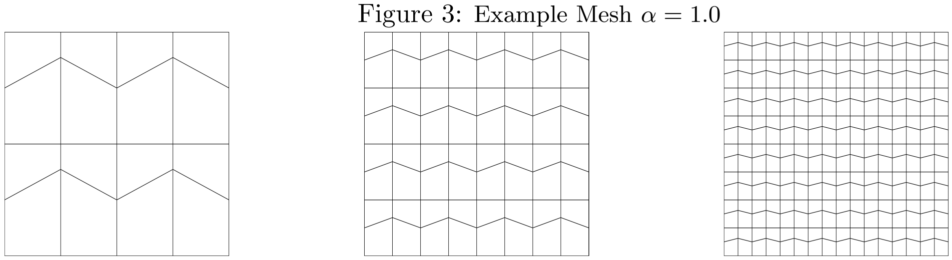


Figure 4: Example Mesh $\alpha = 0.5$

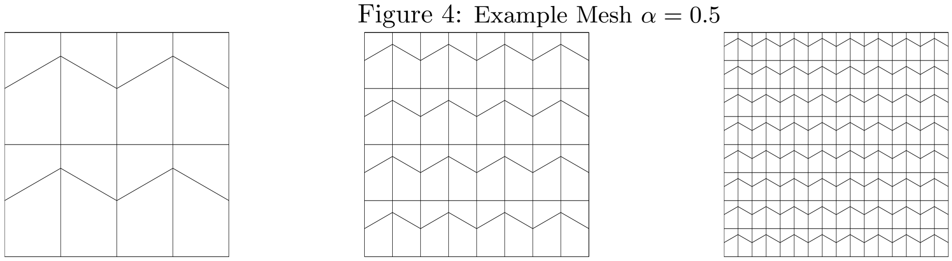


Figure 5: Example Mesh $\alpha = 0$

Table 1. Error of P_1, CNR $\mathcal{Q}_1$ and NR $\mathcal{Q}_1$ element $\alpha = 1$

Elements	$\|u-u_h\|_0$			$\|\nabla_h(u-u_h)\|_0$		
	CNR	NR	P_1	CNR	NR	P_1
8×8	0.034054	0.031303	0.034251	1.442686	1.422368	1.448856
16×16	0.008501	0.007715	0.008538	0.727732	0.715461	0.730244
32×32	0.002124	0.001921	0.002132	0.365815	0.358225	0.364638
64×64	0.000530	0.000479	0.000533	0.182415	0.179173	0.182993
128×128	0.000132	0.000120	0.000133	0.091219	0.089594	0.091507
256×256	0.000033	0.000030	0.000034	0.045123	0.044766	0.046180

Table 2. Error of P_1, CNR $\mathcal{Q}_1$ and NR $\mathcal{Q}_1$ element $\alpha=0.5$

Elements	$\|u-u_h\|_0$			$\|\nabla_h(u-u_h)\|_0$		
	CNR	NR	P_1	CNR	NR	P_1
8×8	0.039254	0.038142	0.036890	1.541388	1.521077	1.476560
16×16	0.009854	0.009708	0.008447	0.796036	0.791288	0.727414
32×32	0.002834	0.002811	0.002006	0.426718	0.425625	0.360132
64×64	0.000972	0.000968	0.000488	0.241019	0.240780	0.179091
128×128	0.000394	0.000393	0.000120	0.144607	0.144558	0.089292
256×256	0.000178	0.000178	2.99E-05	0.091841	0.091837	0.044582

Table 3. Error of P_1, CNR $\mathcal{Q}_1$ and NR $\mathcal{Q}_1$ element $\alpha=0$

Elements	$\|u-u_h\|_0$			$\|\nabla_h(u-u_h)\|_0$		
	CNR	NR	P_1	CNR	NR	P_1
8×8	0.041618	0.0410213	0.038437	1.57476	1.48928	1.49358
16×16	0.014419	0.0141182	0.009707	0.92364	0.91044	0.754329
32×32	0.008793	0.0085398	0.002429	0.66085	0.66491	0.378065
64×64	0.007793	0.0076641	0.000607	0.57625	0.56245	0.189144
128×128	0.007589	0.0074404	0.000152	0.55305	0.55130	0.094586
256×256	0.007542	0.0074012	3.79E-05	0.54710	0.54848	0.047294

Table 1 indicates that if the Bisection condition holds, namely $\alpha=1$, the CNR $\mathcal{Q}_1$ element and the P_1 element converge at the same rate. Otherwise, as demonstrated by Table 2-3, the convergence rate of the CNR $\mathcal{Q}_1$ element deteriorates when α tends to zero, and the convergence rate of the P_1 element is independent of the mesh distortion parameter α. These numerical results coincide with the theoretical result.

In the next example, we test the superconvergence at three kinds of points. In this case, we select f such that the exact solution is $u(x,y)=x(x-1)y(y-1)$. The numerical result is reported in Table.4, where uniform rectangle meshes are used.

Table 4. Derivative Error of CNR $\mathcal{Q}_1$ at three kinds of points

Elements	CNR		
	Center	Node	Midpoint
4×4	0.0059193	0.0069444	0.0095486
8×8	0.0015681	0.0036764	0.0040020
16×16	0.0003971	0.0012943	0.0013351
32×32	9.961E-05	0.0003880	0.0003931
64×64	2.491E-05	0.000107	0.0001079
128×128	6.232E-06	2.839E-05	2.847E-05
256×256	1.55E-06	7.328E-06	7.338E-06

In the last example, we examine the superconvergence of the postprocessed sulotion $\Pi_2 u_h$. The exact solution is still $u(x,y)=x(x-1)y(y-1)$, the numerical results is reported in Table.5.

Table 5. Superconvergence of postprocessed solution $\Pi_2 u_h$

Elements	rectangle	$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0$
4×4	0.0124225999	0.013832899	0.0158845238	0.0132328551
8×8	0.0028412121	0.00303176098	0.0036921097	0.0033364464
16×16	0.000695237312	0.000724880704	0.00093215325	0.000988615941
32×32	0.000172902193	0.000180134184	0.000272992359	0.000396499858
64×64	4.3169582E-05	4.50965764E-05	8.81364918E-05	0.000205713438
128×128	1.07889121E-05	1.12934922E-05	3.0224819E-05	0.000131823701
256×256	2.69701051E-06	2.82645534E-06	1.0850847E-05	0.000105685113

References

[1] R. A. Adams, Sobolev Space, New York, Academic Press, 1978.

[2] D. N. Arnold, D. Boffi and R. S. Falk, Approximation by quadrilateral finite elements, Report No. AM220, 2000.

[3] C. M. Chen and Y. Q. Huang, High Accuracy Theory of Finite Element Methods, Hunan Science and Technology Press, 1995.

[4] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North Holland, Amsterdam, 1978.

[5] J. Douglas Jr, J. E. Santos, D. W. Sheen and X. Ye, Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems, *Math. Modelling and Numerical Analysis*, **33** (1999), 747-770.

[6] H. Han, Nonconforming elements in the mixed finite element method, *J.Comp.Math*, **2** (1984), 223-233.

[7] Q. Lin, L. Tobiska and A. Zhou, Superconvergence and Extrapolation of nonconforming lower order finite elements applied to the Poisson equation, To appear.

[8] P. B. Ming and Z.C Shi, *Quadrilateral mesh*, Chinese Annals of Mathematics, **23B** (2002), 1-18.

[9] R. Rannacher and S. Turek, Simple nonconforming quadrilateral Stokes element, *Numer. Meth. Part. Diff. Equations.*, **8** (1992), 97-111.

[10] C. Park and D.W.Sheen, P_1 nonconforming quadrilateral finite element methods for second-order elliptic problems, *SIAM. J.Numer.Anal.*,**41** (2003), 624-640.

[11] Z. C. Shi, A convergence condition for quadrilateral Wilson element, *Numer.Math.*, **44** (1984), 349-361.

[12] Z. C. Shi, B. Jiang and W. M. Xue, A new superconvergence property of Wilson nonconforming finite element, *Numer.Math.*, **78** (1997), 259-268.