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REMARKS ON ERROR ESTIMATES FOR THE TRUNC PLATE ELEMENT *1)

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Abstract

This paper provides a simplified derivation for error estimates of the TRUNC plate element. The error analysis for the problem with mixed boundary conditions is also discussed.

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1. Introduction

The TRUNC element is very effective for the numerical solution of Kirchhoff plates. Applications to some sample problems showed that it converged rapidly [1, 2, 3]. Shi first established the error estimates in [9], and the derivation is rather technical.

This paper intends to revisit error analysis of the element. We will give a simple but very useful identity for the approximate solution. From this identity, we obtain a desired estimate for the term $E_1(u^*, \bar{w}_h)$ in [9] in a simplified way, which is essential in producing optimal error estimates. We also discuss error analysis of the method for corresponding problems with mixed boundary conditions. It deserves to point out that our derivation is different from that in [14], where the deduction of (3.18) is not rigorous (see Remark 1.4.4.7 in [6, p.32]).

2. Error Estimates for Plate Bending Problem with Clamped Conditions

Given a polygonal domain Ω, consider the following plate bending problem with clamped conditions [5]:

$$\begin{cases} -\mathcal{M}_{\alpha\beta,\alpha\beta}(u^*) = \Delta^2 u^* = f \text{ in } \Omega, \\ u^* = \partial_n u^* = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where

$$\mathcal{M}_{\alpha\beta}(u) := (1-\sigma)\mathcal{K}_{\alpha\beta}(u) + \sigma\mathcal{K}_{\mu\mu}(u),\ \mathcal{K}_{\alpha\beta}(u) := -\partial_{\alpha\beta}u,\ 1 \le \alpha,\, \mu,\, \beta \le 2,$$

with $\sigma \in (0, 0.5)$ being the Poisson ratio of the plate and n the unit outward normal to $\partial\Omega$. Throughout this paper we use Einstein's convention for summation, and always assume that

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$u^* \in H^3(\Omega) \cap H_0^2(\Omega)$ in this section. The variational formulation of (2.1) is to find $u^* \in V = H_0^2(\Omega)$ such that

$$a(u^*, v) = f(v) = \int_\Omega f v dx, \quad \forall v \in V,$$

where

$$\begin{aligned} a(u, v) :=& \int_\Omega \mathcal{M}_{\alpha\beta}(u)\mathcal{K}_{\alpha\beta}(v)dx \\ =& \int_\Omega [\Delta u \Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v)]dx. \end{aligned}$$

We next give some useful identities [8] for later uses. Given a polygon G, let v be a function in $H^3(G)$ and w a function in $H^2(G)$. Then

$$\begin{aligned} a_G(v, w) :=& \int_G \mathcal{M}_{\alpha\beta}(v)\mathcal{K}_{\alpha\beta}(w)dx \\ =& \int_G \mathcal{Q}_\alpha(v)\partial_\alpha w dx - \int_{\partial G} \{\mathcal{M}_{nn}(v)\partial_n w + \mathcal{M}_{n\tau}(v)\partial_\tau w\} ds, \end{aligned} \tag{2.2}$$

where

$$\mathcal{M}_{nn}(v) := \mathcal{M}_{\alpha\beta}(v)n_\alpha n_\beta, \ \mathcal{M}_{n\tau}(v) := \mathcal{M}_{\alpha\beta}(v)n_\alpha\tau_\beta, \ \mathcal{Q}_\alpha(v) := \partial_\beta\mathcal{M}_{\alpha\beta}(v),$$

with $n = (n_1, n_2)$ and $\tau = (\tau_1, \tau_2)$ being the unit outward normal and tangent vector to ∂G such that (n, τ) forms a right-hand system. Moreover, we have by (2.1) that

$$\int_G \mathcal{Q}_\alpha(u)\partial_\alpha v dx - f(v) = \int_{\partial G} \mathcal{Q}_n(u) v ds, \ \forall v \in H^1(G), \tag{2.3}$$

where $\mathcal{Q}_n(u) := \mathcal{Q}_\alpha(u)n_\alpha \in H^{-1/2}(\partial G)$. Since the tangent derivative is only the derivative with respect to the arc length parameter s in the boundary ∂G, we also write ∂_s for ∂_τ in what follows.

We divide the region of interest Ω into a regular family of triangular elements K with the diameter $h_K \le h$, $\bar{\Omega} = \cup_{K\in\mathcal{T}_h}\bar{K}$, and define on each triangle K the shape function to be an incomplete cubic polynomial,

$$\begin{aligned} v_h =& a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1 \\ &+ a_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + a_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + a_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \end{aligned} \tag{2.4}$$

with the nodal parameters being the function values and the values of two first order derivatives at vertices of the triangle K, i.e., $v_h(p_i)$, $\partial_1 v_h(p_i)$, $\partial_2 v_h(p_i)$, $1 \le i \le 3$, where $\{p_i\}_{i=1}^3$ denote the three vertices of K. We then obtain the usual Zienkiewicz element space V_h related to V.

For each $v_h \in V_h$, we split the function into two parts,

$$v_h := \bar{v}_h + v_h', \tag{2.5}$$

where

$$\bar{v}_h|_K := a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1 \tag{2.6}$$

and

$$v_h'|_K := a_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + a_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + a_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2). \tag{2.7}$$

Thus, we define a bilinear form on V_h by

$$b_h(u_h, v_h) := a_h(\bar{u}_h, \bar{v}_h) + a_h(u_h', v_h'), \ \forall u_h, \ v_h \in V_h,$$

where

$$a_h(u_h,v_h) := \sum_K a_K(u_h,v_h),\ a_K(u_h,v_h) := \int_K \mathcal{M}_{\alpha\beta}(u_h)\mathcal{K}_{\alpha\beta}(v_h)dx.$$

With these notations, the TRUNC element method is to find $u_h \in V_h$ such that

$$b_h(u_h,v_h) = f(v_h),\ \forall v_h \in V_h. \tag{2.8}$$

Lemmas 2.1-2.5 are due to Shi [9].

Lemma 2.1 *For the split functions determined by (2.4)-(2.7), it holds*

$$|\bar{v}_h|_{2,K} \lesssim |v_h|_{2,K},\ |v_h'|_{2,K} \lesssim h_K|v_h|_{3,K},\ |\partial_n v_h'|_{0,\infty,F} \lesssim h_K|v_h|_{3,K}.$$

Lemma 2.2 *Let r_h be the usual interpolation operator related to the Zienkiewicz element space V_h. Given $u \in H^3(\Omega)$ and the decomposition of the interpolant $r_h u = \overline{r_h u} + (r_h u)'$. Then*

$$|u - r_h u|_{m,K} \lesssim h_K^{3-m}|u|_{3,K},\ |u - \overline{r_h u}|_{m,K} \lesssim h_K^{3-m}|u|_{3,K},\ |(r_h u)'| \lesssim h_K^{3-m}|u|_{3,K},\ 0 \le m \le 2.$$

Lemma 2.3 *The seminorm $|v_h|_{2,h} = \left(\sum_K |v_h|_{2,K}^2\right)^{1/2}$ is a norm over the space V_h and the bilinear form $b_h(\cdot,\cdot)$ is coercive,*

$$|v_h|_{2,h}^2 \lesssim b_h(v_h,v_h),\ \forall v_h \in V_h.$$

Lemma 2.4 *Let u^*, u_h be the solutions of (2.1) and (2.8) respectively. Then*

$$|u^* - u_h|_{2,h} \lesssim |u^* - r_h u^*|_{2,h} + \sup_{w_h \in V_h} \frac{|G_h(u^*, r_h u^*, w_h)|}{|w_h|_{2,h}}, \tag{2.9}$$

where

$$G_h(u^*, r_h u^*, w_h) := E_1(u^*, \bar{w}_h) + E_1(u^* - \overline{r_h u^*}, w_h') - E_1(\bar{w}_h, (r_h u^*)'),$$
$$E_1(u,w) := -\sum_K \int_{\partial K} \mathcal{M}_{nn}(u)\partial_n w ds = \sum_K \int_{\partial K} [\Delta u - (1-\sigma)\partial_{ss}u]\partial_n w ds.$$

Lemma 2.5 *Let $\varphi = u^* - \overline{r_h u^*}$. Then*

$$|E_1(\varphi, w_h')| \lesssim h|u^*|_3|w_h|_{2,h},\ |E_1(\bar{w}_h, (r_h u^*)') \lesssim h|u^*|_3|w_h|_{2,h}.$$

In the above results and henceforth, we always use "$\lesssim \cdots$" to indicate "$\le C\cdots$", where the generic constant C is independent of related parameters (e.g., h_K and h) and the functions under considerations, which may take different values in different appearances. Moreover, we simply write $|\cdot|_k$ (resp. $\|\cdot\|_k$) for $|\cdot|_{k,\Omega}$ (resp. $\|\cdot\|_{k,\Omega}$) where there is no confusion caused.

Lemma 2.6 *For any $v_h \in V_h$, it holds the identity:*

$$\sum_{F \subset \partial K} [\mathcal{M}_{nn}(w)|_F Q^F(\partial_n v_h') + \mathcal{M}_{n\tau}(w)|_F Q^F(\partial_s v_h')] = 0,\ \forall w \in P_2(K),$$

where

$$Q^F(f) := 1/2|F|(f(a) + f(b))$$

for a continuous function f in $F := (a,b)$.

Lemma 2.6 follows from the identities (18)-(19) in [9] combined with the identities [4, p.15]

$$\mathcal{M}_{nn}(u) = -\{\Delta u - (1-\sigma)\partial_{ss}u\},\ \mathcal{M}_{n\tau}(u) = -(1-\sigma)\partial_{ns}u.$$

One of the main results of this paper is to give a new proof of the following result, which simplifies the original derivation [9] essentially.

Lemma 2.7 *We have the estimate*

$$|E_1(u^*,\bar{w}_h)| \lesssim h|u^*|_3|w_h|_{2,h}. \tag{2.10}$$

Proof. Let F be an edge of a triangle $K\in\mathcal{T}_h$. For each function $v\in L^2(F)$, we define

$$P_0^F v := \frac{1}{|F|}\int_F v ds,\ \ R_0^F v := v - P_0^F v.$$

Since P_0^F is an orthogonal projection operator, we can rewrite $E_1(u^*,\bar{w}_h)$ in the form

$$\begin{aligned}
-E_1(u^*,\bar{w}_h) =& \sum_K \int_{\partial K}\mathcal{M}_{nn}(u^*)\partial_n\bar{w}_h ds\\
=&\sum_K\sum_{F\subset\partial K}\int_F R_0^F[\mathcal{M}_{nn}(u^*)]R_0^F(\partial_n\bar{w}_h)ds\\
&+\sum_K\sum_{F\subset\partial K}\int_F P_0^F[\mathcal{M}_{nn}(u^*)]\partial_n\bar{w}_h ds\\
=:& I_1+I_2.
\end{aligned}\tag{2.11}$$

From Lemma 2.1, the estimate for P_0^F [12] and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}
|I_1| &\lesssim \sum_K\sum_{F\subset\partial K}\|R_0^F[\mathcal{M}_{nn}(u^*)]\|_{0,F}\|R_0^F(\partial_n\bar{w}_h)\|_{0,F}\\
&\lesssim \sum_K\sum_{F\subset\partial K} h_K|u^*|_{3,K}|\bar{w}_h|_{2,K}\lesssim h|u^*|_3|w_h|_{2,h}.
\end{aligned}\tag{2.12}$$

The second term in (2.11) is

$$\sum_K\sum_{F\subset\partial K}\int_F P_0^F[\mathcal{M}_{nn}(u^*)]\partial_n\bar{w}_h ds = \sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{nn}(u^*)]\int_F\partial_n\bar{w}_h ds.$$

The integrand $\partial_n\bar{w}_h$ is a linear polynomial in one variable on F, so the trapezoidal rule is exact and yields,

$$\int_F\partial_n\bar{w}_h ds = Q^F(\partial_n\bar{w}_h) = Q^F(\partial_n w_h) - Q^F(\partial_n w_h').$$

Since the first derivatives of w_h are continuous at vertices, and $\partial_n u = 0$ on the vertices of $F\subset\partial\Omega$, we know

$$\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{nn}(u^*)]Q^F(\partial_n w_h)=0.$$

Therefore,

$$\begin{aligned}
I_2 =& -\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{nn}(u^*)]Q^F(\partial_n w_h')\\
=&\Big\{-\sum_K\sum_{F\subset\partial K}\{P_0^F[\mathcal{M}_{nn}(u^*)]Q^F(\partial_n w_h')+P_0^F[\mathcal{M}_{n\tau}(u^*)]Q^F(\partial_s w_h')\}\Big\}\\
&+\Big\{\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{n\tau}(u^*)]Q^F(\partial_s w_h')\Big\}\\
=:&I_{21}+I_{22}.
\end{aligned}\tag{2.13}$$

It follows from Lemma 2.6 that

$$\sum_{F\subset\partial K}\{P_0^F[\mathcal{M}_{nn}(I_K^{(2)}u^*)]Q^F(\partial_n w_h')+P_0^F[\mathcal{M}_{n\tau}(I_K^{(2)}u^*)]Q^F(\partial_s w_h')\}=0,$$

where $I_K^{(2)}$ denotes the usual interpolation operator related the Lagrange element of second order [4]. Hence,

$$\begin{aligned}|I_{21}|=&\left|\sum_{F\subset\partial K}\{P_0^F[\mathcal{M}_{nn}(u^*-I_K^{(2)}u^*)]Q^F(\partial_n w_h')+P_0^F[\mathcal{M}_{n\tau}(u^*-I_K^{(2)}u^*)]Q^F(\partial_s w_h')\}\right|\\ \lesssim& h_K^{1/2}\sum_{F\subset\partial K}\{\|\mathcal{M}_{nn}(u^*-I_K^{(2)}u^*)\|_{0,F}\|\partial_n w_h'\|_{0,\infty,F}\\ &+\|\mathcal{M}_{n\tau}(u^*-I_K^{(2)}u^*)\|_{0,F}\|\partial_s w_h'\|_{0,\infty,F}\}.\end{aligned}\tag{2.14}$$

However, we have by Lemma 2.1, the error estimate for the interpolation operator $I_K^{(2)}$ [4] and the inverse inequality that

$$\|\mathcal{M}_{nn}(u^*-I_K^{(2)}u^*)\|_{0,F}+\|\mathcal{M}_{n\tau}(u^*-I_K^{(2)}u^*)\|_{0,F}\lesssim h_K^{1/2}|u^*-I_K^{(2)}u^*|_{2,\infty,K}\lesssim h_K^{1/2}|u^*|_{3,K},$$
$$\|\partial_n w_h'\|_{0,\infty,F}+\|\partial_s w_h'\|_{0,\infty,F}\lesssim h_K|w_h|_{3,K}\lesssim |w_h|_{2,K}.$$

Plugging these estimates into (2.14) and using the Cauchy-Schwarz inequality then yields

$$|I_{21}|\lesssim\sum_K h_K|u^*|_{3,K}|w_h|_{2,K}\lesssim h|u^*|_3|w_h|_{2,h}.\tag{2.15}$$

Moreover, since w_h' is identically zero at the vertices of any K in $\mathcal{T}_h$, w_h is a continuous function with zero values on $\partial\Omega$, and $\partial_s\bar{w}_h$ is a linear polynomial in one variable on $F\subset\partial K$, we have

$$\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{n\tau}(u^*)]\int_F\partial_s w_h ds=0,$$
$$\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{n\tau}(u^*)]Q^F(\partial_s w_h)=0,$$

and

$$Q^F(\partial_s\bar{w}_h)=\int_F\partial_s\bar{w}_h ds=\int_F\partial_s w_h ds,$$

hence

$$\begin{aligned}I_{22}=&\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{n\tau}(u^*)]Q^F(\partial_s w_h')\\ =&\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{n\tau}(u^*)]Q^F(\partial_s w_h)\\ &-\sum_K\sum_{F\subset\partial K}P_0^F[\mathcal{M}_{n\tau}(u^*)]\int_F\partial_s w_h ds\\ =&0.\end{aligned}\tag{2.16}$$

The estimate (2.10) now follows from (2.11)-(2.13) and (2.15)-(2.16) directly.

The next result is an immediate consequence of Lemmas 2.2-2.7.

Theorem 2.1 *Let $u^*\in H^3(\Omega)\cap H_0^2(\Omega)$ be the solution of problem (2.1) and $u_h\in V_h$ the solution of problem (2.8). Then*

$$|u^*-u_h|_{2,h}\lesssim h|u^*|_3.$$

3. Error Estimates for the Problem with Mixed Boundary Conditions

Suppose that Ω is a polygonal domain with the boundary $\partial\Omega$ consisting of N line segments $\{\Gamma_j\}_{j=1}^N$, which are numbered anticlockwise. Let $\tilde{\Gamma}_0$ be the union of $\{\Gamma_i\}_{i=1}^{N_0}$, $1 \le N_0 < N$, i.e., $\tilde{\Gamma}_0 = (\cup_{i=1}^{N_0}\bar{\Gamma}_i)^0$, while $\tilde{\Gamma}_1$ the other part of $\partial\Omega$. We consider the plate bending problem with the clamped conditions on $\tilde{\Gamma}_0$, i.e.,

$$u^* = \partial_n u^* = 0 \text{ on } \tilde{\Gamma}_0,$$

and the force and moment free conditions on $\tilde{\Gamma}_1$. The related variational formulation is to find $u^* \in V = H_0^2(\Omega;\tilde{\Gamma}_0)$ such that

$$a(u^*, v) = f(v),\ \forall v \in V, \tag{3.1}$$

where $a(u,v) := \int_\Omega \mathcal{M}_{\alpha\beta}(u)\mathcal{K}_{\alpha\beta}(v)dx$.

From now on we always assume that $u^* \in H^3(\Omega) \cap H_0^2(\Omega;\tilde{\Gamma}_0)$. Then we have by the similar argument in [7] that

$$\mathcal{M}_{\alpha\beta,\alpha\beta}(u^*) + f = 0 \text{ in } L^2(\Omega),$$

and

$$\mathcal{M}_{nn}(u^*) = 0 \text{ in } H^{1/2}(\Gamma_i),\ N_0 + 1 \le i \le N. \tag{3.2}$$

We introduce an auxiliary space by

$$\tilde{H}_0^1(\Omega;\tilde{\Gamma}_0) := \{v \in H_0^1(\Omega;\tilde{\Gamma}_0);\ v|_{\Gamma_i} \in H_*^1(\Gamma_i),\ 1 \le i \le N\},$$

where $H_*^1(\Gamma_i)$ consists of all functions in $C^\infty(\bar{\Gamma}_i)$ whose first-order derivatives are identically zero at two endpoints. The next result follows from the same argument for proving Lemma 4.1 in [7].

Lemma 3.1 *$\tilde{H}_0^1(\Omega;\tilde{\Gamma}_0)$ is dense in $H_0^1(\Omega;\tilde{\Gamma}_0)$ in the norm $\|\cdot\|_{1,\partial\Omega}$.*

Lemma 3.2 *For all $v \in H_0^1(\Omega;\tilde{\Gamma}_0)$,*

$$< \mathcal{Q}_n(u^*), v >_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} - \int_{\partial\Omega} \mathcal{M}_{n\tau}(u^*)\partial_s v ds = 0. \tag{3.3}$$

Proof. For each $v \in \tilde{H}_0^1(\Omega;\tilde{\Gamma}_0)$, we have by the trace theorem for polygonal domains [6] that there exists a function $w \in H_0^2(\Omega;\tilde{\Gamma}_0)$ such that

$$\partial_n w = 0 \text{ on } \partial\Omega;\ w = 0 \ \text{ on } \tilde{\Gamma}_0;\ w = v \ \text{ on } \Gamma_i,\ N_0 + 1 \le i \le N.$$

Now it follows (3.1), (3.2), (2.2) and (2.3) that

$$0 = a(u^*, w) - f(w) =< \mathcal{Q}_n(u^*), w >_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} - \int_{\partial\Omega} [\mathcal{M}_{nn}(u^*)\partial_n w + \mathcal{M}_{n\tau}(u^*)\partial_s w]ds$$

$$=< \mathcal{Q}_n(u^*), w >_{H^{-1/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)} - \int_{\partial\Omega} \mathcal{M}_{n\tau}(u^*)\partial_s w ds.$$

This with Lemma 3.1 implies that (3.3) also holds for each $v \in H_0^1(\Omega;\tilde{\Gamma}_0)$ by the density argument.

The TRUNC element method for solving (3.1) is to find $u_h \in V_h$ such that

$$b_h(u_h, v_h) = f(v_h),\ \forall v_h \in V_h, \tag{3.4}$$

where

$$b_h(u_h, v_h) := a_h(\bar{u}_h, \bar{v}_h) + a_h(u_h', v_h'),\ \forall v_h \in V_h. \tag{3.5}$$

In contrast to the method (2.8), the only difference is that for a function v_h in the current admissible space V_h, it only holds that $v_h(p) = \partial_1 v_h(p) = \partial_2 v_h(p) = 0$, for any vertex $p \in \tilde{\Gamma}_0$ (including the endpoints of $\tilde{\Gamma}_0$).

We next check the validity of Lemma 2.4 in the present case. We have by the identities (2.2)-(2.3) that for all $w_h \in V_h$,

$$\begin{aligned} f(w_h) - a_h(u^*, w_h) &= -\sum_K \int_{\partial K} \mathcal{Q}_n(u^*) w_h ds + \sum_K \int_{\partial K} [\mathcal{M}_{nn}(u^*)\partial_n w_h + \mathcal{M}_{n\tau}(u^*)\partial_s w_h] ds \\ &= -\sum_K \int_{\partial K} \mathcal{Q}_n(u^*) w_h ds + \sum_K \int_{\partial K} \mathcal{M}_{n\tau}(u^*)\partial_s w_h ds - E_1(u^*, w_h). \quad (3.6) \end{aligned}$$

Observing that each function in V_h is continuous, by the density argument we get

$$-\sum_K \int_{\partial K} \mathcal{Q}_n(u^*) w_h ds = -\int_{\partial\Omega} \mathcal{Q}_n(u^*) w_h ds.$$

Since $w_h|_{\partial\Omega} \in H_0^1(\Omega; \tilde{\Gamma}_0)$, combining the last equation with (3.3) and (3.6) shows

$$f(w_h) = a_h(u^*, w_h) - E_1(u^*, w_h), \ \forall w_h \in V_h. \quad (3.7)$$

On the other hand, recalling definition (3.5), we can write

$$\begin{aligned} b_h(v_h, w_h) &= a_h(v_h, w_h) - a_h(\bar{v}_h, w_h') - a_h(v_h', \bar{w}_h) \\ &= a_h(v_h - u^*, w_h) - a_h(\bar{v}_h, w_h') - a_h(v_h', \bar{w}_h) + a_h(u^*, w_h), \end{aligned}$$

from which and (3.7) we obtain

$$\begin{aligned} f(w_h) - b_h(v_h, w_h) &= f(w_h) - a_h(u^*, w_h) + a_h(\bar{v}_h, w_h') + a_h(v_h', \bar{w}_h) + a_h(u^* - v_h, w_h) \\ &= -E_1(u^*, w_h) + a_h(\bar{v}_h, w_h') + a_h(v_h', \bar{w}_h) + a_h(u^* - v_h, w_h). \quad (3.8) \end{aligned}$$

Using Lemma 2.4 and (3.8), we find, for each $v_h \in V_h$,

$$\begin{aligned} |u_h - v_h|_{2,h}^2 &\lesssim b_h(u_h - v_h, w_h) = f(w_h) - b_h(v_h, w_h) \\ &= a_h(u^* - v_h, w_h) - E_1(u^*, w_h) + a_h(\bar{v}_h, w_h') + a_h(v_h', \bar{w}_h), \quad (3.9) \end{aligned}$$

where $w_h := u_h - v_h$.

Noting that $\bar{v}_h|_K$ is a quadratic polynomial, and w_h' takes zero values at vertices of triangles, we have by identity (2.2) that

$$a_h(\bar{v}_h, w_h') = E_1(\bar{v}_h, w_h'). \quad (3.10)$$

Similarly,

$$a_h(v_h', \bar{w}_h) = E_1(v_h', \bar{w}_h).$$

This with (3.9) and (3.10) implies

$$|u_h - v_h|_{2,h}^2 \lesssim a_h(u^* - v_h, w_h) - E_1(u^*, \bar{w}_h) - E_1(u^* - \bar{v}_h, w_h') + E_1(\bar{w}_h, v_h'),$$

which leads to the estimate (2.9) by taking $v_h = r_h u^*$. Therefore, Lemma 2.4 still holds true for the method (3.4). It is easy to check that Lemma 2.5 is also valid for this method.

Now the critical step is to estimate $E_1(u^*, \bar{w}_h)$. Employing the similar argument for proving Lemma 2.7, we find

$$\begin{aligned} -E_1(u^*, \bar{w}_h) &= \sum_K \int_{\partial K} \mathcal{M}_{nn}(u^*)\partial_n \bar{w}_h ds \\ &= \sum_K \sum_{F\subset\partial K} \int_F R_0^F[\mathcal{M}_{nn}(u^*)] R_0^F(\partial_n \bar{w}_h) ds + \sum_K \sum_{F\subset\partial K} \int_F P_0^F[\mathcal{M}_{nn}(u^*)]\partial_n \bar{w}_h ds \\ &=: II_1 + II_2, \quad (3.11) \end{aligned}$$

$$
\begin{aligned}
|II_1| &\lesssim \sum_K \sum_{F\subset\partial K} \|R_0^F[\mathcal{M}_{nn}(u^*)]\|_{0,F}\|R_0^F(\partial_n \bar{w}_h)\|_{0,F} \\
&\lesssim \sum_K \sum_{F\subset\partial K} h_K|u^*|_{3,K}|\bar{w}_h|_{2,K} \lesssim h|u^*|_3|w_h|_{2,h},
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
II_2 =& -\sum_K \sum_{F\subset\partial K} P_0^F[\mathcal{M}_{nn}(u^*)]Q^F(\partial_n w_h') \\
=&\Big\{ -\sum_K \sum_{F\subset\partial K} \{P_0^F[\mathcal{M}_{nn}(u^*)]Q^F(\partial_n w_h') + P_0^F[\mathcal{M}_{n\tau}(u^*)]Q^F(\partial_s w_h')\}\Big\} \\
&+\Big\{\sum_K \sum_{F\subset\partial K} P_0^F[\mathcal{M}_{n\tau}(u^*)]Q^F(\partial_s w_h')\Big\} \\
=:& \, II_{21} + II_{22}.
\end{aligned}
\tag{3.13}
$$

II_{21} can be estimated as for the clamped bending problems,

$$
|II_{21}| \lesssim \sum_K h_K|u^*|_{3,K}|w_h|_{2,K} \lesssim h|u^*|_3|w_h|_{2,h}. \tag{3.14}
$$

It remains to bound II_{22}. Applying the similar argument for deriving (2.16) we have

$$
\begin{aligned}
II_{22} =& \sum_{F\subset\partial\Omega} P_0^F[\mathcal{M}_{n\tau}(u^*)]\Big\{Q^F(\partial_s w_h) - \int_F \partial_s w_h ds\Big\} \\
=&\frac{1}{12}\sum_{F\subset\partial\Omega} |F|^2 \int_F M_{n\tau}(u^*)\partial_s^3 w_h' ds,
\end{aligned}
\tag{3.15}
$$

where we have also used the identity

$$
Q^F(\partial_s \bar{w}_h) = \int_F \partial_s \bar{w}_h ds, \ \forall F\subset\partial K,
$$

and the error estimate for numerical integration formula [11]. We remark that in the equation (3.15), $F\subset\partial\Omega$ means that F is a side of some triangle $K\in\mathcal{T}_h$ which also belongs to $\partial\Omega$.

By the Hölder inequality we have for a side F,

$$
\begin{aligned}
\left|\int_F M_{n\tau}(u^*)\partial_s^3 w_h' ds\right| &\lesssim \|M_{n\tau}(u^*)\|_{L^r(F)}\|\partial_s^3 w_h'\|_{L^{r'}(F)} \\
&\lesssim \sum_{\alpha,\beta=1}^{2} \|\partial_{\alpha\beta}u^*\|_{L^r(F)}\|\partial_s^3 w_h'\|_{L^{r'}(F)},
\end{aligned}
\tag{3.16}
$$

where $r>2$ is a real number, $r' = r/(r-1)$.

Recalling definition (2.7), using the scaling argument, estimate (24) in [9] and the inverse inequality for finite elements, we deduce that

$$
\begin{aligned}
\|\partial_s^3 w_h'\|_{L^{r'}(F)} &\lesssim h_{K_F}^{-3+1/r'}(|a_7|+|a_8|+|a_9|) \\
&\lesssim h_{K_F}^{-1+1/r'}|w_h|_{3,K_F} \lesssim h_{K_F}^{-3/2}|w_h|_{2,r',K_F},
\end{aligned}
\tag{3.17}
$$

where K_F is the triangle with F as one side. Hence, by the Hölder inequality, and (3.15)-(3.17) we get

$$
\begin{aligned}
|II_{22}| &\lesssim h^{1/2}\sum_{\alpha,\beta=1}^{2}\Big\{\sum_{F\subset\partial\Omega}\|\partial_{\alpha\beta}u^*\|_{L^r(F)}^r\Big\}^{1/r}\Big\{\sum_{F\subset\partial\Omega}|w_h|_{2,r',K_F}^{r'}\Big\}^{1/r'}\\
&\lesssim h^{1/2}\sum_{\alpha,\beta=1}^{2}\|\partial_{\alpha\beta}u^*\|_{L^r(\partial\Omega)}\Big\{\sum_{F\subset\partial\Omega}|w_h|_{2,r',K_F}^{r'}\Big\}^{1/r'}. \qquad (3.18)
\end{aligned}
$$

Noting that $1 < r' < 2$, using the Hölder inequality again, we obtain

$$
\begin{aligned}
\Big\{\sum_{F\subset\partial\Omega}|w_h|_{2,r',K_F}^{r'}\Big\}^{1/r'} &\lesssim \Big\{\sum_{F\subset\partial\Omega}|w_h|_{2,K_F}^{2}\Big\}^{1/2}\Big\{\sum_{F\subset\partial\Omega}\int_{K_F}1^{\frac{2r-2}{r-2}}dx\Big\}^{\frac{r-2}{2r-2}}\\
&\lesssim |w_h|_{2,h}\mathrm{meas}(\tilde\Omega)^{(r-2)/(2r-2)} \lesssim h^{(r-2)/(2r-2)}|w_h|_{2,h}, \qquad (3.19)
\end{aligned}
$$

where $\tilde\Omega$ stands for the set of all points in Ω with distance of $\partial\Omega$ no more than h.

On the other hand, it follows from [10] and [13, p.68] that, for all $v \in H^{1/2}(\partial\Omega)$,

$$\|v\|_{L^r(\partial\Omega)} \lesssim \sqrt{r}\|v\|_{H^{1/2}(\partial\Omega)},$$

where the generic constant does not depend on r. This with the trace theorem for Sobolev spaces gives

$$\|\partial_{\alpha\beta}u^*\|_{L^r(\partial\Omega)} \lesssim \sqrt{r}\|u^*\|_3,$$

from which and (3.18)-(3.19) it comes that

$$|II_{22}| \lesssim h\sqrt{r}h^{-1/(2r-2)}\|u^*\|_3|w_h|_{2,h}.$$

By taking $r = 2(1+|\ln h|)$ in the above estimate, we have after a simple computation that

$$|II_{22}| \lesssim h(1+|\ln h|)^{1/2}\|u^*\|_3|w_h|_{2,h}. \qquad (3.20)$$

Now it follows from (3.11)-(3.14) and (3.20) that

$$|E_1(u^*,\bar w_h)| \lesssim h(1+|\ln h|)^{1/2}\|u^*\|_3|w_h|_{2,h},$$

which with Lemmas 2.4-2.5 yields

$$|u^*-u_h|_{2,h} \lesssim h(1+|\ln h|)^{1/2}\|u^*\|_3.$$

Theorem 3.1 *Let $u^* \in H^3(\Omega)\cap H_0^2(\Omega;\tilde\Gamma_0)$ be the solution of problem (3.1) and $u_h \in V_h$ the solution of the discrete problem (3.4). Then*

$$|u^*-u_h|_{2,h} \lesssim h(1+|\ln h|)^{1/2}\|u^*\|_3.$$

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