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SPECTRAL APPROXIMATION ORDERS OF MULTIDIMENSIONAL NONSTATIONARY BIORTHOGONAL SEMI-MULTIRESOLUTION ANALYSIS IN SOBOLEV SPACE *1)

Wen-sheng Chen
(*Department of Mathematics, Shenzhen University, Shenzhen 518060, China; Key Laboratory of Mathematics Mechanization, CAS, Beijing 100080, China*)

Chen Xu
(*Department of Mathematics, Shenzhen University, Shenzhen 518060, China*)

Wei Lin
(*Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, China*)

Abstract

Subdivision algorithm (Stationary or Non-stationary) is one of the most active and exciting research topics in wavelet analysis and applied mathematical theory. In multidimensional non-stationary situation, its limit functions are both compactly supported and infinitely differentiable. Also, these limit functions can serve as the scaling functions to generate the multidimensional non-stationary orthogonal or biorthogonal semi-multiresolution analysis (Semi-MRAs). The spectral approximation property of multidimensional non-stationary biorthogonal Semi-MRAs is considered in this paper. Based on nonstationary subdivision scheme and its limit scaling functions, it is shown that the multidimensional nonstationary biorthogonal Semi-MRAs have spectral approximation order r in Sobolev space $H^s(\mathbb{R}^d)$, for all $r \geq s \geq 0$.

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1. Introduction

Subdivision algorithm, resulting from several fields of applied mathematics and signal processing, is an iterative method to generate smooth curves and surfaces. For example, to construct planar curves, such a scheme begins with the initial control points $f_0(k)$ defined on the integer lattice $\mathbb{Z}$, and then expends the control points to the fine lattice $\mathbb{Z}/2 := \{j/2|j \in \mathbb{Z}\}$ via a specified mask $h_{j,k} = \{h_{j,k}(l)\}_{l\in\mathbb{Z}}$. Usually, we assume that the mask $h_{j,k}$ is a finite sequence, i.e. for every $j \geq 0$ and each $k \in \mathbb{Z}$, the set $\{l| \in \mathbb{Z}, h_{j,k}(l) \neq 0\}$ only contains finite elements. After j iterative steps, it derives a new sequence $f_j(2^{-j}k)$. The iterative procedure satisfies the following linear rule:

$$f_j(2^{-j}k) = 2 \sum_{n\in k+2\mathbb{Z}} h_{j,k}(n) f_{j-1}(2^{-j}(k-n))\,. \tag{1.1}$$

If mask $h_{j,k}$ is independent of both scale j and position k, namely $h_{j,k}(l) = h_l$, then this subdivision scheme is said to be stationary, otherwise to be nonstationary. In the case of stationary subdivision algorithm, (1.1) can be rewritten as:

$$f_j(2^{-j}k) = 2\sum_n h_{k-2n} f_{j-1}(2^{-j+1}n)\,.$$

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The convergence of above stationary subdivision scheme is closely connected with the existence of the solution to the refinement equation as follows.

$$f(x) = 2\sum_{n\in\mathbb{Z}} h_n f(2x-n)\,.$$

Thereby, the stationary subdivision schemes play an important role in the wavelet theory [7, 10, 11, 12, 13].

However, stationary multiresolution analysis based on a compactly supported refinable function is limited to generators (scaling functions) with a finite degree of smoothness. So, one cannot build a C^∞ refinable function which is also compactly supported in stationary case.

More recently, attention has been given to nonstationary subdivision schemes [3, 4, 5, 6]. Since the masks may vary from different scale j or different position k, it is possible to construct a nonstationary Semi-MRA which is generated by C^∞ compactly supported scaling functions. In fact, by virtue of Rvachev[8] up-function method, N. Dyn and A. Ron[4] constructed a compactly supported scaling function in C^∞ and the corresponding nonstationary Semi-MRA $\{V_j\}_{j\geq 0}$. The constructed scaling function $\varphi_j(x)$ is defined in the Fourier domain by

$$\hat{\varphi}_j(\omega) = \prod_{k=1}^{+\infty}\left(\frac{1+e^{-i2^{-k}\omega}}{2}\right)^{k+j}, \quad j\geq 0, \tag{1.2}$$

The length of its support is $L_j = \sum_{k\geq 1}(k+j)2^{-k} = j+2$. The scaling space is defined as:

$$V_j := \mathrm{Span}\{\varphi_j(2^j x - k)\}_{k\in\mathbb{Z}}$$

From equation (1.2), it yields that

$$\hat{\varphi}_j(\omega) = m_{j+1}(\omega/2)\hat{\varphi}_{j+1}(\omega/2)\,, \tag{1.3}$$

where

$$m_{j+1}(\omega/2) = \left(\frac{1+e^{-i\omega}}{2}\right)^{j+1}.$$

It also concludes from (1.3) that the spaces V_j are embeded, namely,

$$V_j \subset V_{j+1}, \quad \text{for all } j\geq 0.$$

In addition, the investigation of the spectral approximation order in L^2 or Sobolev space is also gaining considerable attention because of its powerful theoretical analysis for approximation theory. Encouraging results have been reported in some literatures [4, 5], [14]-[17]. More details, the paper [4] showed that its constructed nonstationary Semi-MRA $\{V_j\}_{j\geq 0}$ has spectral approximation property in $L^2(\mathbb{R})$, i.e., for all $r\geq 0$ and $f(x)\in H^r(\mathbb{R})$, $\lim\limits_{j\to+\infty} 2^{jr}\|P_j f - f\|_0 = 0$. Cohen and Dyn [5] exploited a technique introduced in [14] to generalize these results to some nonstationary subdivision schemes in one dimensional case. de Boor, DeVore and Ron [14] are concerned with approximation in the L^2 norm from shift-invariant spaces. Cohen and Dyn [5] adapted their technique to the derivation of density orders in Sobolev norms. Approximation orders in Sobolev norms by shift-invariant spaces are studied in paper [15] and [16]. Yoon [17] considered the spectral approximation orders in Sobolev space using radial basis function interpolation.

In paper [18], we previously obtained some results on the convergence of multidimensional nonstationary subdivision algorithm and properties of its limit functions. We also exploited these results to generate multidimensional nonstationary biorthogonal Semi-MRAs [19]. The goal of this paper is to prove that the multidimensional nonstationary biorthogonal Semi-MRAs constructed in [19] have spectral approximation order r in Sobolev space $H^s(\mathbb{R}^d)$.

To this end, some multi-index notations are given as follows:

- Multi-index $m = (m_1,\cdots,m_d)\in\mathbb{N}_0^d$, $|m| := m_1+\cdots+m_d$;

- $x\,, y \in \mathbb{R}^d\,,\ x \cdot y := \sum_{i=1}^{d} x_i y_i\ \|x\| := (x \cdot x)^{1/2},\ x^m := \prod_{i=1}^{d} x_i^{m_i};$
- $C_0^\infty(\mathbb{R}^d)$ denotes the space of all functions which are both infinity differentiable and compactly supported in space $\mathbb{R}^d$;
- Multi-derivative $D^m := (\partial^{m_1}/\partial x_1^{m_1}) \cdots (\partial^{m_d}/\partial x_d^{m_d})$;
- $\mathrm{sinc}(x) := \frac{\sin x}{x} := \prod_{i=1}^{d} \frac{\sin x_i}{x_i} := \prod_{i=1}^{d} \mathrm{sinc}(x_i)\,, \quad x \in \mathbb{R}^d$;
- $r_k := \prod_{i=1}^{d}[-r_k(i), r_k(i)] \bigcap \mathbb{Z}^d$, where $k > 0$, $r_k(i) \in Z_+$, $0 \le i \le d$;
- $T^d := [-\pi\,, \pi]^d$, $E_d := \{2^d$ vertices of square box $[0,1]^d\}$.

The rest of this paper is organized as follows. In section 2, some properties on limit function of nonstationary subdivision algorithm are proposed. Nonstationary biorthogonal Semi-MRAs are reported in Section 3. Finally, Section 4 shows the main theorem on spectral approximation in Sobolev space.

2. Multidimensional Nonstationary Subdivision Algorithm

In this section, we briefly introduce some results on multidimensional nonstationary subdivision algorithm. Details can be found in paper [18].

Let $\{h_k\}_{k>0}$ be a finite mask, the corresponding filter function $m_k(\omega)(k > 0)$ are defined by

$$m_k(\omega) := \sum_{l \in r_k} h_k(l) e^{-il\cdot\omega} = \sum_{l_1=-r_k(1)}^{r_k(1)} \cdots \sum_{l_d=-r_k(d)}^{r_k(d)} h_k(l) e^{-il\cdot\omega}\,, \quad \omega \in \mathbb{R}^d\,.$$

The nonstationary algorithm associated with this mask is

$$f_j(2^{-j}k) = 2^d \sum_{l \in \mathbb{Z}^d} h_j(k-2l) f_{j-1}(2^{-j+1}l)\,, \quad k \in \mathbb{Z}^d\,,\ j \ge 1\,. \tag{2.1}$$

It shows in [18] that if the input data is a Dirac sequence $f_0(k) = \delta_{k,0}$ in the nonstationary subdivision algorithm (2.1), then after n times iterative procedure, the generated sequence data on the lattice $2^{-n}\mathbb{Z}^d$ can be interpolated by a function $\varphi^{[n]}(x)$, where $\varphi^{[n]}(x)$ is a band-limited function defined by $\hat{\varphi}^{[n]}(\omega) = \prod_{k=1}^{n} m_k(2^{-k}\omega) \cdot \chi_{T^d}(2^{-n}\omega)$.

Theorem 2.1 *If $\{m_k(\omega)\}_{k>0}$ are uniformly bounded(assuming the bound $M \ge 1$), $\{u_k := |m_k(0) - 1|\}_{k>0}$ is l_1 sequence, and for all $1 \le i \le d\,, r_k(i) = \mathcal{O}(k)$, then $\hat{\varphi}^{[n]}(\omega)$ converges uniformly on any compact set to $\hat{\varphi}(\omega)$ and $\varphi^{[n]}(x)$ converges to $\varphi(x)$ in the sense of tempered distributions with*

$$\mathrm{supp}\,\varphi(x) \subseteq \prod_{i=1}^{d}[-L_i, L_i],\ L_i = \sum_{k>0} 2^{-k} r_k(i),\ i = 1, \cdots, d.$$

Theorem 2.2 *Assume that the hypotheses of theorem 2.1 are satisfied and $|m_k(\omega)| \le (1 + a_k) \cdot |m(\omega)|^k$ with*

$$\sum_{k>0} |a_k| < \infty,\ m(\omega) := \prod_{i=1}^{d} \left(\cos\frac{\omega_i}{2}\right)^{\beta_i} \cdot \widetilde{m}(\omega),\ \text{for some } \beta_i \in R_+,$$

where $\widetilde{m}(\omega)$ satisfies the following conditions:

- *$\widetilde{m}(\omega)$ is bounded and $\widetilde{m}(0) = 1$;*
- *$\widetilde{m}(\omega)$ is Hölder continuous at the origin;*
- *For some fixed $\lambda > 0$, $\sigma_\lambda := \sup_\omega \prod_{k=1}^{\lambda} \left|\widetilde{m}(2^{-k}\omega)\right| < 2^{\lambda L}$, $L := \min_{1 \le i \le d}\{\beta_i\}$. Then $\varphi(x) \in C_0^\infty(\mathbb{R}^d)$ and for all $m \in Z_+^d$, $D^m\varphi^{[n]}(x)$ converges uniformly to $D^m\varphi(x)$.*

3. Nonstationary Biorthogonal Semi-MRAs

By virtue of the results stated in section 2, we formerly constructed the nonstationary biorthogonal Semi-MRAs [19]. Details can be found in paper [19]. Let $\{h_k\}_{k>0}$ and $\{\widetilde{h}_k\}_{k>0}$ be two group finite masks, their associated filter functions are $\{m_k(\omega)\}_{k>0}$ and $\{\widetilde{m}_k(\omega)\}_{k>0}$ respectively, which all satisfy the conditions stated in theorem 2.2. Then we can define two sequences of scaling functions $\varphi_j(x)$ and $\widetilde{\varphi}_j(x) \in C_0^\infty(\mathbb{R}^d)$, their Fourier transformations are given respectively as follows:

$$\hat{\varphi}_j(\omega) = \prod_{k=1}^{\infty} m_{k+j}(2^{-k}\omega)\,, \quad \hat{\widetilde{\varphi}}_j(\omega) = \prod_{k=1}^{\infty} \widetilde{m}_{k+j}(2^{-k}\omega)\,, \tag{3.1}$$

where $j \geq 0,\ \omega \in \mathbb{R}^d$. Hence from (3.1), we have

$$\hat{\varphi}_j(\omega) = m_{j+1}(\omega/2)\hat{\varphi}_{j+1}(\omega/2)\,, \quad \hat{\widetilde{\varphi}}_j(\omega) = \widetilde{m}_{j+1}(\omega/2)\hat{\widetilde{\varphi}}_{j+1}(\omega/2). \tag{3.2}$$

We know from (3.2) that $\varphi_j(x)$ and $\widetilde{\varphi}_j(x)$ satisfy a series of recursive refinement equations respectively as follows:

$$\varphi_j(x) = 2^d \sum_{n \in r_{j+1}} h_{j+1}(n)\varphi_{j+1}(2x-n). \tag{3.3}$$

$$\widetilde{\varphi}_j(x) = 2^d \sum_{n \in \widetilde{r}_{j+1}} \widetilde{h}_{j+1}(n)\widetilde{\varphi}_{j+1}(2x-n). \tag{3.4}$$

It is thus natural to define two semi-MRAs $\{V_j\}_{j\geq 0}$ and $\{\widetilde{V}_j\}_{j\geq 0}$ respectively by

$$V_j := \mathrm{span}\{2^{jd/2}\varphi_j(2^j x - k)\}\,, \quad \widetilde{V}_j := \mathrm{span}\{2^{jd/2}\widetilde{\varphi}_j(2^j x - k)\} \tag{3.5}$$

By (3.3) and (3.4), it is easy to verify that

$$V_j \subset V_{j+1}, \qquad \widetilde{V}_j \subset \widetilde{V}_{j+1}, \qquad (j \geq 0)\,.$$

Theorem 3.1 *Assume conditions stated in theorem 2.2 are satisfied, then $\{V_j\}_{j\geq 0}$ and $\{\widetilde{V}_j\}_{j\geq 0}$ are biorthogonal Semi-MRAs if and only if for all $\omega \in \mathbb{R}^d$, $\forall\, j \geq 1$, it holds that*

$$\sum_{\nu \in E_d} \overline{m}_j(\omega+\nu\pi)\widetilde{m}_j(\omega+\nu\pi) = 1, \qquad \text{a.e.} \tag{3.6}$$

4. Main Theorem on Spectral Approximation

This section will prove the main theorem that multidimensional nonstationary biorthogonal semi-MRAs $\{V_j\}_{j\geq 0}$ and $\{\widetilde{V}_j\}_{j\geq 0}$ have spectral approximation properties in Sobolev space $H^s(\mathbb{R}^d)$. We first give some definitions as follows.

Definition 4.1 *For $0 \leq s \leq r$ and $f \in H^r(\mathbb{R}^d)$, distance $d(f,V_j)_s$ is defined by $d(f,V_j)_s := \inf_{g\in V_j} \|f-g\|_s$, where $\|\cdot\|_s$ is the norm of Sobolev space $H^s(\mathbb{R}^d)$, i.e., $\|f\|_s^2 := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\hat{f}(\omega)|^2(1+\|\omega\|^2)^s d\omega$.*

Definition 4.2 *For $0 \leq s \leq r$ and $f \in H^r(\mathbb{R}^d)$, if $2^{(r-s)j}d(f,V_j)_s$ is bounded as $j \to +\infty$, then we say that V_j has approximation order r in $H^s(\mathbb{R}^d)$; If $2^{(r-s)j}d(f,V_j)_s \to 0$ as $j \to +\infty$, then it is to say that V_j has spectral approximation order r in $H^s(\mathbb{R}^d)$.*

Definition 4.3 *$Q_j := 2^j[-t,t]^d$, $Q_j^c := \mathbb{R}^d - Q_j$, where $t \in (0,\pi)$ is assigned to the value such that statements (i) and (ii) in lemma 4.1 hold simultaneously.*

Definition 4.4 *$\mathcal{P}_j : L^2 \to V_j$ is projection operator, i.e., for all $f(x) \in L^2(\mathbb{R}^d)$*

$$\mathcal{P}_j f(x) = \sum_{k\in\mathbb{Z}^d} \langle f(\cdot)|\widetilde{\varphi}_{jk}(\cdot)\rangle \varphi_{jk}(x) = 2^{dj}\sum_{k\in\mathbb{Z}^d} \langle f(\cdot)|\widetilde{\varphi}_j(2^j\cdot - k)\rangle \varphi_j(2^j x - k)\,.$$

Definition 4.5 *The operator* $\mathcal{S}_j$ *is defined by* $(\mathcal{F}\mathcal{S}_j f)(\omega) = \hat{f}(\omega)\cdot\chi_{Q_j}(\omega)$, $\widetilde{\mathcal{P}}_j$ *and* $\widetilde{\mathcal{S}}_j$ *are defined respectively by*

$$\widetilde{\mathcal{P}}_j := I - \mathcal{P}_j, \quad \widetilde{\mathcal{S}}_j := I - \mathcal{S}_j\,,$$

where I is identity operator.

Lemma 4.1 *Assume $m_k(\omega)$ and $\widetilde{m}_k(\omega)$ satisfy all the conditions stated in theorem 2.2, then there exists $t \in (0,\pi)$ such that for all $r,\ v \geq 0$, the following two statements hold simultaneously.*

$$\text{(i)}\ \sup_{\omega\in[-t,t]^d} \|\omega\|^{-2r}\left|\sum_{n\neq 0}\overline{\hat{\varphi}}_j(\omega+2n\pi)\hat{\widetilde{\varphi}}_j(\omega+2n\pi)\right|^2 \to 0\,,\quad (j\to+\infty)\,;$$

$$\text{(ii)}\ \sup_{\omega\in[-t,t]^d} \|\omega\|^{-2r}\left|\hat{\widetilde{\varphi}}_j(\omega)\right|^2 \sum_{n\neq 0}\|\omega+2n\pi\|^{2v}|\hat{\varphi}_j(\omega+2n\pi)|^2 \to 0\,,\quad (j\to+\infty)\,.$$

Proof. First, we define two functions $g(\omega)$ and $h(\omega)$ respectively by

$$g(\omega) := \prod_{k=1}^{\infty}|m(2^{-k}\omega)|\,,\qquad h(\omega) := \prod_{k=1}^{\infty}|m(2^{-k}\omega)|^k\,, \tag{4.1}$$

then

$$|\hat{\varphi}_j(\omega)| \leq \alpha h(\omega)[g(\omega)]^j\,,\qquad \alpha = \prod_{k=1}^{\infty}(1+|\alpha_k|)\,. \tag{4.2}$$

It yields from (4.1) that

$$g(\omega) = \prod_{i=1}^{d}\operatorname{sinc}^{\beta_i}(\omega_i/2)g_0(\omega)\,,\qquad g_0(\omega) = \prod_{k=1}^{\infty}|\widetilde{m}(2^{-k}\omega)| \tag{4.3}$$

From the hypotheses stated in theorem 2.2, we obtain that for all $l \geq 0$ and $2^l \leq \|\omega\| \leq 2^{l+1}$,

$$\begin{aligned} g_0(\omega) &= g_0(2^{-l}\omega)\prod_{k=1}^{l}|\widetilde{m}(2^{-k}\omega)| \\ &\leq \sup_{\|\omega\|\leq 2}[g_0(\omega)]\cdot(\sup_{\omega}|\widetilde{m}|)^{\lambda-1}\cdot(\sigma_\lambda)^{[l/\lambda]} \\ &\leq K_1\|\omega\|^{\frac{\log_2\sigma_\lambda}{\lambda}} \leq K_1\|\omega\|^L\,. \end{aligned} \tag{4.4}$$

Also, for all $\omega\in\mathbb{R}^d$, we have the following estimate:

$$\begin{aligned} \prod_{i=1}^{d}|\operatorname{sinc}(\omega_i/2)|^{\beta_i} &\leq \prod_{i=1}^{d}\left|\frac{2\sin(\omega_i/2)}{\omega_i}\right|^L \\ &\leq \left[\frac{|2\sin(\omega_{i_0}/2)|}{|\omega_{i_0}|}\right]^L,\quad (|\omega_{i_0}| = \max\{|\omega_i|, 1\leq i\leq d\}) \\ &\leq \left[\frac{|2\sqrt{d}\sin(\omega_{i_0}/2)|+1}{\sqrt{d}|\omega_{i_0}|+1}\right]^L \leq \left[\frac{2\sqrt{d}+1}{\|\omega\|+1}\right]^L \\ &= (2\sqrt{d}+1)^L\cdot(\|\omega\|+1)^{-L} = D_1\cdot(\|\omega\|+1)^{-L}. \end{aligned}$$

Thereby for all $\omega\in\mathbb{R}^d$, it yields that

$$g(\omega) \leq K_2(1+\|\omega\|)^{-\varepsilon}\,,\quad \text{where } \varepsilon = L - \frac{\log_2\sigma_\lambda}{\lambda} > 0\,. \tag{4.5}$$

As $n\in\mathbb{Z}^d-\{0\},\ \omega\in[-\pi,\pi]^d$, we have

$$\begin{aligned} \prod_{i=1}^{d}|\operatorname{sinc}(n_i\pi+\omega_i/2)|^{\beta_i} &\leq \prod_{i=1}^{d}\left|\frac{\sin(n_i\pi+\omega_i/2)}{n_i\pi+\omega_i/2}\right|^L \leq \left|\frac{\sin(n_i\pi+\omega_i/2)}{\max_i|n_i\pi+\omega_i/2|}\right|^L \\ &\leq \frac{|\omega_i|^L}{\max_i|2n_i\pi+\omega_i|^L} \leq \frac{C\cdot\|\omega\|^L}{\|2n\pi+\omega\|^L}. \end{aligned}$$

Hence,

$$g(\omega+2n\pi)\le K_3\|\omega\|^L\,,\quad n\in\mathbb{Z}^d-\{0\}\,,\quad \omega\in[-\pi,\pi]^d\,. \tag{4.6}$$

Moreover $m(\omega)$ is Hölder continuous at the origin, thereby $g(\omega)$ is also Hölder continuous at origin with $g(0)=1$ and $g(\omega)$ has the same Hölder index as $m(\omega)$. It implies that $h(\omega)$ can be expressed as

$$h(\omega)=\prod_{k=0}^{\infty}g(2^{-k}\omega).$$

From (4.5), we know that for any $\eta\in(0,\,1)$, there exists $\omega_\eta>0$, such that $g(\omega)<\eta$ as $\|\omega\|>\omega_\eta$.
So, for all $l\ge 0$ and $2^l\omega_\eta\le\|\omega\|\le 2^{l+1}\omega_\eta$, we have:

$$\begin{aligned} h(\omega) &= h(2^{-l-1}\omega)\prod_{k=0}^{l}g(2^{-k}\omega)\\ &\le \left(\sup_{\|\omega\|\le\omega_\eta}h(\omega)\right)\cdot\eta^{l+1}\le\left(\sup_{\|\omega\|\le\omega_\eta}h(\omega)\right)\cdot 2^{(l+1)\log_2\eta}\\ &\le D(\eta)\cdot\|\omega\|^{\log_2\eta} \end{aligned}\tag{4.7}$$

Since $\eta\in(0,\,1)$ is arbitrary, (4.7) shows that $h(\omega)$ has rapid decay at infinity. Hence for any given $v\ge 0$,

$$A(v):=\sup_{\omega\in T^d}\sum_{n\in\mathbb{Z}^d}\|\omega+2n\pi\|^{2v}|h(\omega+2n\pi)|^2<+\infty \tag{4.8}$$

Apparently, $\{\widetilde{m}_k(\omega)\}_{k>0}$ has the above similar results.

Therefore, as $\omega\in T^d$, it derives from (4.2) and (4.6) and (4.8) that:

$$\begin{aligned} \sum_{n\ne0}|\hat{\varphi}_j(\omega+2n\pi)|^2 &\le \alpha^2\sum_{n\ne0}|h(\omega+2n\pi)|^2|g(\omega+2n\pi)|^{2j}\\ &\le \alpha^2(K_3\|\omega\|^L)^{2j}\sum_{n\ne0}|h(\omega+2n\pi)|^2\le\alpha^2A(0)(K_3\|\omega\|^L)^{2j}\\ &= K_4(K_3\|\omega\|^L)^{2j}\,, \end{aligned}$$

Then, for any given $r\ge 0$ and for all $\omega\in T^d$, we have:

$$\begin{aligned} I_1(j) &:= \|\omega\|^{-2r}\left|\sum_{n\ne0}\overline{\hat{\varphi}}_j(\omega+2n\pi)\hat{\tilde{\varphi}}_j(\omega+2n\pi)\right|^2\\ &\le \|\omega\|^{-2r}\sum_{n\ne0}|\hat{\varphi}_j(\omega+2n\pi)|^2\sum_{n\ne0}\left|\hat{\tilde{\varphi}}_j(\omega+2n\pi)\right|^2\\ &\le \|\omega\|^{-2r}K_4(K_3\|\omega\|^L)^{2j}\cdot\widetilde{K}_4(\widetilde{K}_3\|\omega\|^{\widetilde{L}})^{2j}\\ &= C_1\left(C_2\|\omega\|^{L+\widetilde{L}-r/j}\right)^{2j}\,. \end{aligned}$$

If $\omega\in[-t,t]^d$, then $\|\omega\|\le\sqrt{d}\,t$, it yields that $\|\omega\|<1$ as $0<t<1/\sqrt{d}$. In addition, $L+\widetilde{L}-r/j\ge(L+\widetilde{L})/2$ as j is sufficiently large. So,

$$I_1(j)\le C_1\left(C_2\|\omega\|^{(L+\widetilde{L})/2}\right)^{2j}\le C_1\left(C_3\cdot t^{(L+\widetilde{L})/2}\right)^{2j}\,.$$

From the above estimate, we set $t=t_1$ such that $0<t_1<1/\sqrt{d}$ and $C_3\cdot t_1^{(L+\widetilde{L})/2}<1$, then

$$\lim_{j\to+\infty}I_1(j)=0.$$

It shows from (4.2) and (4.5) and (4.7) and (4.8) that as $\omega \in T^d$, we have

$$\left|\hat{\widetilde{\varphi}}_j(\omega)\right|^2 \leq \widetilde{\alpha}^2|\widetilde{h}(\omega)|^2 \cdot [\widetilde{g}(\omega)]^{2j} \leq \widetilde{\alpha}^2 \widetilde{D}(1/2)\|\omega\|^{-2} \cdot \widetilde{K}_2^{2j}(1+\|\omega\|)^{-2j\varepsilon} \leq C_4\widetilde{K}_2^{2j}\|\omega\|^{-2}.$$

By (4.2) and (4.6), we have the following estimate:

$$\begin{aligned}\sum_{n\neq 0}\|\omega+2n\pi\|^{2v}\,|\hat{\varphi}_j(\omega+2n\pi)|^2 &\leq \sum_{n\neq 0}\|\omega+2n\pi\|^{2v}|h(\omega+2n\pi)|^2|g(\omega+2n\pi)|^{2j}\\ &\leq (K_3\|\omega\|^L)^{2j}\sum_{n\neq 0}\|\omega+2n\pi\|^{2v}|h(\omega+2n\pi)|^2\\ &\leq A(v)(K_3\|\omega\|^L)^{2j}.\end{aligned}$$

Consequently, for any $r \geq 0$ and $\forall\,\omega \in \mathbb{R}^d$, we obtain:

$$\begin{aligned}I_2(j) &:= \|\omega\|^{-2r}\left|\hat{\widetilde{\varphi}}_j(\omega)\right|^2 \cdot \sum_{n\neq 0}\|\omega+2n\pi\|^{2v}|\hat{\varphi}_j(\omega+2n\pi)|^2\\ &\leq C_4A(v)\|\omega\|^{-2(r+1)}(K_3\widetilde{K}_2\|\omega\|^L)^{2j}\\ &= D_2(D_3\|\omega\|^{L-(r+1)/j})^{2j}.\end{aligned}$$

Similarly, as $\omega \in [-t,t]^d$ and $0 < t < 1/\sqrt{d}$, then $\|\omega\| < 1$. Let j be sufficiently large such that $L-(r+1)/j \geq L/2$. This yields that

$$I_2(j) \leq D_2\left(D_3\|\omega\|^{L/2}\right)^{2j} \leq D_2\left(D_4\cdot t^{L/2}\right)^{2j}.$$

So if choose $t = t_2$ such that $0 < t_2 < 1/\sqrt{d}$ and $D_4 \cdot t_2^{L/2} < 1$, then it demonstrates that

$$\lim_{j\to+\infty} I_2(j) = 0.$$

Based on the above analysis, we set $t = \min\{t_1,\, t_2\}$, then as $\omega \in [-t,\, t]^d$, the statements (i) and (ii) hold simultaneously.

Having above lemma 4.1, we can prove the following main theorem on spectral approximation orders of multidimensional nonstationary biorthogonal Semi-MRAs $\{V_j\}_{j\geq 0}$ and $\{\widetilde{V}_j\}_{j\geq 0}$ in Sobolev space.

Theorem 4.1 *Assume $\{m_k(\omega)\}_{k>0}$ and $\{\widetilde{m}_k(\omega)\}_{k>0}$ satisfy the conditions stated in theorem 2.2 and equation (3.6), $\{\varphi_j\}_{j\geq 0}$ and $\{\widetilde{\varphi}_j\}_{j\geq 0}$ are scaling functions defined by (3.1), then the nonstationary biorthogonal Semi-MRAs $\{V_j\}_{j\geq 0}$ and $\{\widetilde{V}_j\}_{j\geq 0}$ (see(3.5)) generated by these two group scaling functions have property of spectral approximation, namely for all $r \geq s \geq 0$, $\{V_j\}_{j\geq 0}$ and $\{\widetilde{V}_j\}_{j\geq 0}$ have spectral approximation order r in Sobolev space $H^s(\mathbb{R}^d)$.*

Proof. It is sufficient to show that for all $f(x) \in H^r(\mathbb{R}^d)$,

$$d(f,V_j)_s \leq \|\mathcal{P}_j\mathcal{S}_jf - f\|_s \leq C\cdot 2^{j(s-r)}\|f\|_r\cdot\varepsilon(f,j),$$

and $\lim_{j\to+\infty}\varepsilon(f,j) = 0$.

For the approximation error $\|\mathcal{P}_j\mathcal{S}_jf - f\|_s$, we have estimate as follows:

$$\begin{aligned}\|\mathcal{P}_j\mathcal{S}_jf - f\|_s &\leq \|\widetilde{\mathcal{S}}_jf\|_s + \|\mathcal{P}_j\mathcal{S}_jf - \mathcal{S}_jf\|_s\\ &\leq \|\widetilde{\mathcal{S}}_jf\|_s + \|\widetilde{\mathcal{S}}_j\mathcal{P}_j\mathcal{S}_jf + \mathcal{S}_j\mathcal{P}_j\mathcal{S}_jf - \mathcal{S}_j^2f\|_s\\ &\leq \|\widetilde{\mathcal{S}}_jf\|_s + \|\mathcal{S}_j\widetilde{\mathcal{P}}_j\mathcal{S}_jf\|_s + \|\widetilde{\mathcal{S}}_j\mathcal{P}_j\mathcal{S}_jf\|_s.\end{aligned}$$

Hence, we need to estimate these three terms on the right side of above inequality separately.

For the first item, we have:

$$\begin{aligned}\|\widetilde{\mathcal{S}}_j f\|_s^2 &= (2\pi)^{-d}\int_{Q_j^c}|\hat{f}(\omega)|^2(1+\|\omega\|^2)^s d\omega\\ &= (2\pi)^{-d}\int_{Q_j^c}|\hat{f}(\omega)|^2(1+\|\omega\|^2)^r(1+\|\omega\|^2)^{s-r} d\omega\\ &\le (2\pi)^{-d}(2\sqrt{d}\,2^j t)^{2(s-r)}\int_{Q_j^c}|\hat{f}(\omega)|^2(1+\|\omega\|^2)^r d\omega\\ &\le C\cdot 2^{2j(s-r)}\|f\|_r^2\cdot\varepsilon_1(f,j)\,.\end{aligned}$$

where $\varepsilon_1(f,j)=\dfrac{\int_{Q_j^c}|\hat{f}(\omega)|^2(1+\|\omega\|^2)^r d\omega}{\|f\|_r^2}\to 0 \qquad (j\to+\infty)\,.$

For the second term, we have:

$$\begin{aligned}\|\mathcal{S}_j\widetilde{\mathcal{P}}_j\mathcal{S}_j f\|_s^2 &= (2\pi)^{-d}\int_{Q_j}|\mathcal{F}\widetilde{\mathcal{P}}_j\mathcal{S}_j f(\omega)|^2(1+\|\omega\|^2)^s d\omega\\ &\le (2\pi)^{-d}\left[1+(\sqrt{d}\,2^j t)^2\right]^s\int_{Q_j}|\mathcal{F}\widetilde{\mathcal{P}}_j\mathcal{S}_j f(\omega)|^2 d\omega\\ &\le C\cdot 2^{2js}\|\mathcal{S}_j\widetilde{\mathcal{P}}_j\mathcal{S}_j f(\omega)\|_0^2\,.\end{aligned}$$

Before estimating the term $\|\mathcal{S}_j\widetilde{\mathcal{P}}_j\mathcal{S}_j f(\omega)\|_0^2$, it notices that

$$\begin{aligned}\mathcal{F}\mathcal{P}_j\mathcal{S}_j f(\omega) &= \hat{\varphi}_j(2^j\omega)\cdot\sum_{k\in\mathbb{Z}^d}\langle\mathcal{S}_j f(\cdot)|\widetilde{\varphi}_j(2^j\cdot-k)\rangle e^{-i2^{-j}k\cdot\omega}\\ &= \hat{\varphi}_j(2^j\omega)\cdot(2^{j+1}\pi)^{-d}\sum_{k\in\mathbb{Z}^d}\langle\mathcal{F}\mathcal{S}_j f(\cdot)|\hat{\widetilde{\varphi}}_j(2^{-j}\cdot)e^{-i2^{-j}k\cdot}\rangle e^{-i2^{-j}k\cdot\omega}\,.\end{aligned} \tag{4.9}$$

The above sum defines a $2^{j+1}\pi\mathbb{Z}^d$-periodic function, which coincides on square box 2^jT^d with

$$\hat{f}(\omega)\cdot\overline{\hat{\widetilde{\varphi}}_j(2^{-j}\omega)}\cdot\chi_{Q_j}(\omega).$$

So as $\omega\in 2^jT^d$, we have

$$\mathcal{F}\mathcal{P}_j\mathcal{S}_j f(\omega)=\hat{\varphi}_j(2^j\omega)\cdot\overline{\hat{\widetilde{\varphi}}_j(2^{-j}\omega)}\hat{f}(\omega)\cdot\chi_{Q_j}(\omega).$$

From above equation, it derives that:

$$\begin{aligned}\|\mathcal{S}_j\widetilde{\mathcal{P}}_j\mathcal{S}_j f(\omega)\|_0^2 &= (2\pi)^{-d}\int_{Q_j}|\hat{f}(\omega)-\mathcal{F}\mathcal{P}_j\mathcal{S}_j f(\omega)|^2 d\omega\\ &= (2\pi)^{-d}\int_{Q_j}|\hat{f}(\omega)|^2\left|1-\overline{\hat{\varphi}}_j(2^j\omega)\cdot\hat{\widetilde{\varphi}}_j(2^{-j}\omega)\right|^2 d\omega\\ &\le (2\pi)^{-d}\int_{Q_j}\frac{\left|1-\overline{\hat{\varphi}}_j(2^j\omega)\cdot\hat{\widetilde{\varphi}}_j(2^{-j}\omega)\right|^2}{\|\omega\|^{2r}}\cdot|\hat{f}(\omega)|^2\|\omega\|^{2r}d\omega\\ &\le (2\pi)^{-d}\cdot 2^{-2jr}\sup_{\omega\in[-t,t]^d}\frac{\left|1-\overline{\hat{\varphi}}_j(\omega)\cdot\hat{\widetilde{\varphi}}_j(\omega)\right|^2}{\|\omega\|^{2r}}\cdot\|f\|_r^2\\ &= C\cdot 2^{-2jr}\|f\|_r^2\cdot\varepsilon_2(j)\,.\end{aligned}$$

Thereby,

$$\|\mathcal{S}_j\widetilde{\mathcal{P}}_j\mathcal{S}_j f\|_s^2\le C\cdot 2^{2j(s-r)}\|f\|_r^2\cdot\varepsilon_2(j)\,.$$

Combining (3.6) with statement (i) in lemma 4.1, we obtain

$$\begin{aligned}\varepsilon_2(j) &= \sup_{\omega\in[-t,t]^d} \frac{\left|1-\overline{\hat{\varphi}}_j(\omega)\cdot\hat{\tilde{\varphi}}_j(\omega)\right|^2}{\|\omega\|^{2r}} \\ &= \sup_{\omega\in[-t,t]^d} \|\omega\|^{-2r}\left|\sum_{n\neq 0}\overline{\hat{\varphi}}_j(\omega+2n\pi)\hat{\tilde{\varphi}}_j(\omega+2n\pi)\right|^2 \to 0 \quad (j\to+\infty)\,.\end{aligned}$$

Finally, for the last term $\|\widetilde{\mathcal{S}}_j\mathcal{P}_j\mathcal{S}_jf\|_s$, we notice from (4.9) that as $\omega\in Q_j+2^{j+1}n\pi$, $n\neq 0$, we have

$$\mathcal{F}\mathcal{P}_j\mathcal{S}_jf(\omega)=\hat{\varphi}_j(2^j\omega)\cdot\overline{\hat{\tilde{\varphi}}_j(2^{-j}\omega-2n\pi)}\hat{f}(\omega-2^{j+1}n\pi)\,,$$

and $\mathcal{F}\mathcal{P}_j\mathcal{S}_jf(\omega)=0$ for $\omega\in 2^jT^d-Q_j$. Hence

$$\begin{aligned}&\|\widetilde{\mathcal{S}}_j\mathcal{P}_j\mathcal{S}_jf\|_s^2=(2\pi)^{-d}\int_{\omega\in Q_j^c}|\mathcal{F}\mathcal{P}_j\mathcal{S}_jf(\omega)|^2(1+\|\omega\|^2)^s d\omega \\ &=(2\pi)^{-d}\sum_{n\neq 0}\int_{Q_j+2^{j+1}n\pi}|\mathcal{F}\mathcal{P}_j\mathcal{S}_jf(\omega)|^2(1+\|\omega\|^2)^s d\omega \\ &\le C\sum_{n\neq 0}\int_{Q_j}|\mathcal{F}\mathcal{P}_j\mathcal{S}_jf(\omega+2^{j+1}n\pi)|^2\|\omega+2^{j+1}n\pi\|^{2s}d\omega \\ &=C\sum_{n\neq 0}\int_{Q_j}\left|\hat{\varphi}_j(2^{-j}\omega+2n\pi)\overline{\hat{\tilde{\varphi}}_j(2^{-j}\omega)}\hat{f}(\omega)\right|^2\|\omega+2^{j+1}n\pi\|^{2s}d\omega \\ &=C\int_{Q_j}|\hat{f}(\omega)|^2\left|\hat{\tilde{\varphi}}_j(2^{-j}\omega)\right|^2\sum_{n\neq 0}\|\omega+2^{j+1}n\pi\|^{2s}|\hat{\varphi}_j(2^{-j}\omega+2n\pi)|^2 d\omega \\ &\le C\sup_{Q_j}\left(\|\omega\|^{-2r}\left|\hat{\tilde{\varphi}}_j(2^{-j}\omega)\right|^2\sum_{n\neq 0}\|\omega+2^{j+1}n\pi\|^{2s}|\hat{\varphi}_j(2^{-j}\omega+2n\pi)|^2\right)\|f\|_r^2 \\ &\le C\cdot 2^{2(s-r)j}\sup_{[-t,t]^d}\left(\|\omega\|^{-2r}\left|\hat{\tilde{\varphi}}_j(\omega)\right|^2\sum_{n\neq 0}\|\omega+2n\pi\|^{2s}|\hat{\varphi}_j(\omega+2n\pi)|^2\right)\|f\|_r^2 \\ &=C\cdot 2^{2(s-r)j}\|f\|_r^2\cdot\varepsilon_3(j)\,.\end{aligned}$$

Combining above estimate with statement (ii) of lemma 4.1 in the case of $v=s$, we have

$$\varepsilon_3(j)=\sup_{[-t,t]^d}\left(\|\omega\|^{-2r}\left|\hat{\tilde{\varphi}}_j(\omega)\right|^2\sum_{n\neq 0}\|\omega+2n\pi\|^{2s}|\hat{\varphi}_j(\omega+2n\pi)|^2\right)\to 0 \quad (j\to+\infty).$$

The above three estimates show that

$$d(f,V_j)_s\le\|\mathcal{P}_j\mathcal{S}_jf-f\|_s\le C\cdot 2^{j(s-r)}\|f\|_r\cdot\varepsilon(f,j),\quad \text{where } \lim_{j\to\infty}\varepsilon(f,j)=0.$$

It indicates that $\{V_j\}_{j\ge 0}$ has spectral approximation order r in Sobolev space $H^s(\mathbb{R}^d)$. Similarly, we can also show that $\{\widetilde{V}_j\}_{j\ge 0}$ has property of spectral approximation for all Sobolev norms. This concludes the proof of the theorem.

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