Journal of Computational Mathematics, Vol.24, No.2, 2006, 193–208.

NATURAL BOUNDARY ELEMENT METHOD FOR THREE DIMENSIONAL EXTERIOR HARMONIC PROBLEM WITH AN INNER PROLATE SPHEROID BOUNDARY *1)

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Abstract

In this paper, we study natural boundary reduction for Laplace equation with Dirichlet or Neumann boundary condition in a three-dimensional unbounded domain, which is the outside domain of a prolate spheroid. We express the Poisson integral formula and natural integral operator in a series form explicitly. Thus the original problem is reduced to a boundary integral equation on a prolate spheroid. The variational formula for the reduced problem and its well-posedness are discussed. Boundary element approximation for the variational problem and its error estimates, which have relation to the mesh size and the terms after the series is truncated, are also presented. Two numerical examples are presented to demonstrate the effectiveness and error estimates of this method.

Mathematics subject classification: 65N38, 65N30.
Key words: Natural boundary reduction, Prolate spheroid boundary, Finite element, Exterior harmonic problem.

1. Introduction

Starting from Green's function and Green's formula, natural boundary element method reduces the boundary value problem of partial differential equation into a hypersingular integral equation on the boundary, and then solves the latter numerically [1,15]. Since the variational principle can be conserved after the natural boundary reduction, some useful properties, e.g., self-adjointness and coerciveness, can also be preserved well. Thus the existence, uniqueness and stability of the solution of resulting boundary integral equation can be obtained conveniently. However, it is difficult to obtain Green's functions for most general domains. Therefore the natural boundary element method is very efficient when it is used to solve some exterior boundary value problems and singular problems with a special boundary, such as circle [8,15], ellipse [9,16], and spherical surface [2,6]. But for general cases, only natural boundary element method is not enough, we need the coupling or domain decomposition methods.

The coupling of natural boundary element method and finite element method is applied to solve boundary value problems in general unbounded domains, sometimes for simplicity we still call it as natural boundary element method, or more shortly, DtN method [1,8,10,14,15]. Its basic idea is described as follows. First, the unbounded domain is divided into two subregions, a bounded inner region and an unbounded outer one, by introducing an artificial boundary.

* Received March 22, 2005; Final revised October 12, 2005.
1) This work was subsidized by the National Basic Research Program of China under the grant G19990328, 2005CB321701, and the National Natural Science Foundation of China under the grant 10531080.

Next, the original problem is reduced to an equivalent one in the bounded region. There are many ways to accomplish this reduction. However, the advantages of the natural boundary reduction just as described above ensure that the coupled bilinear form preserves automatically the symmetry and coerciveness of the original bilinear form, so not only the analysis of the discrete problem is simplified but also the optimal error estimates and the numerical stability are restored [10].

In the three-dimensional unbounded domain, a sphere [2,6] is usually selected as the artificial boundary. However, for elongated cigar-shaped or ship-shaped obstacles, a prolate spheroid boundary can enclose the obstacles very efficiently, since it leads to smaller computational domain. Therefore, in this paper, we study natural boundary reduction for Laplace equation with Dirichlet or Neumann boundary condition in a three-dimensional unbounded domain outside a prolate spheroid. On the basis of the given results in this paper, we will further study the coupling of finite element and natural boundary element and domain decomposition algorithm based on natural boundary reduction. By using the method of separation of variables and spherical harmonic functions, we express the Poisson integral formula and natural integral operator in a series form explicitly. Thus the original problem is reduced to a boundary integral equation on a prolate spheroid. In real calculation, we truncate the series in finite terms. The variational formula for the reduced problem, the concerned formula after truncating and their well-posedness are all discussed. Boundary element approximation for the variational problem and the concerned error estimates are also presented. The truncation error is often ignored in lots of previous papers but appears in [12,13]. Our error estimates are not only based on the mesh size but also on the terms N after truncating. Two numerical examples are presented to demonstrate the numerical method and their error estimates. We may apply the similar method to solving the same problem outside an oblate spheroid boundary.

2. Poisson Integral Formula and Natural Integral Equation

Let $\Gamma_0 = \{(x,y,z) : \frac{x^2+y^2}{b^2} + \frac{z^2}{a^2} = 1, a > b > 0\}$ denote a prolate spheroid and Ω^c be an unbounded domain outside the boundary Γ_0. We consider the following exterior Dirichlet problem:

$$\begin{cases} \Delta u = 0, & in\ \Omega^c, \\ u = u_0, & on\ \Gamma_0, \\ \textit{some conditions at infinity,} \end{cases} \tag{2.1}$$

and the exterior Neumann problem:

$$\begin{cases} \Delta u = 0, & in\ \Omega^c, \\ \dfrac{\partial u}{\partial \nu} = g_0, & on\ \Gamma_0, \\ \textit{some conditions at infinity,} \end{cases} \tag{2.2}$$

where ν denotes the unit exterior normal vector on Γ_0(regarded as the inner boundary of Ω^c), u_0 and g_0 are the known function on Γ_0 for corresponding problem, respectively. From [5], we know if $g_0 \in H^{-\frac{1}{2}}(\Gamma_0)$, problem (2.2) is well-posed in $W^1(\Omega^c)$ and if $u_0 \in H^{\frac{1}{2}}(\Gamma_0)$, problem (2.1) is also well-posed in $W^1(\Omega^c)$, here

$$W^1(\Omega^c) = \{v \in \mathcal{D}'(\Omega^c) : \frac{v}{r}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z} \in L^2(\Omega^c)\}, \tag{2.3}$$

where $\mathcal{D}(\Omega^c) = \{\ v : v \textit{ infinitely differentiable on } \Omega^c \textit{ and with compact support in } \Omega^c\}$, and

$\mathcal{D}'(\Omega^c)$ is the dual space of $\mathcal{D}(\Omega^c)$. Its norm and semi-norm are defined as

$$\|v\|_{W^1(\Omega^c)} = (\|\frac{v}{r}\|^2_{L^2(\Omega^c)} + \|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega^c)} + \|\frac{\partial v}{\partial y}\|^2_{L^2(\Omega^c)} + \|\frac{\partial v}{\partial z}\|^2_{L^2(\Omega^c)})^{\frac{1}{2}} \tag{2.4}$$

and

$$|v|_{W^1(\Omega^c)} = (\|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega^c)} + \|\frac{\partial v}{\partial y}\|^2_{L^2(\Omega^c)} + \|\frac{\partial v}{\partial z}\|^2_{L^2(\Omega^c)})^{\frac{1}{2}}, \tag{2.5}$$

respectively, where $r = \sqrt{x^2+y^2+z^2}$.

We introduce a prolate spheroidal coordinates (μ,θ,φ), such that Γ_0 coincides with the prolate spheroid $\mu=\mu_0$ and $\Omega^c = \{(\mu,\theta,\varphi) : \mu > \mu_0 > 0, \theta \in [0,\pi], \varphi \in [0,2\pi)\}$. Thus, the Cartesian coordinates (x,y,z) are related to the prolate spheroidal coordinates (μ,θ,φ) via

$$\begin{cases} x = f_0 \sinh\mu \sin\theta \cos\varphi, & \mu \ge \mu_0 > 0, \\ y = f_0 \sinh\mu \sin\theta \sin\varphi, & \theta \in [0,\pi], \\ z = f_0 \cosh\mu \cos\theta, & \varphi \in [0,2\pi), \end{cases} \tag{2.6}$$

where $f_0 = \sqrt{a^2-b^2},\ a = f_0\cosh\mu_0,\ b = f_0 \sinh\mu_0$.

We use the method of separation of variables to derive Poisson integral formula and natural integral operator. Through (2.6), we can obtain

$$\Delta\Phi = \frac{1}{f_0^2(\cosh^2\mu - \cos^2\theta)}\left\{\frac{1}{\sinh\mu}\frac{\partial}{\partial\mu}(\sinh\mu\frac{\partial\Phi}{\partial\mu}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Phi}{\partial\theta}) + (\frac{1}{\sin^2\theta}+\frac{1}{\sinh^2\mu})\frac{\partial^2\Phi}{\partial\varphi^2}\right\}.$$

Let $\Phi = F(\mu)G(\theta)H(\varphi)$ and $\Delta\Phi = 0$. Then, we have

$$H''(\varphi) + m^2 H(\varphi) = 0,$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dG(\theta)}{d\theta}) - \frac{m^2G(\theta)}{\sin^2\theta} + n(n+1)G(\theta) = 0,$$
$$\frac{1}{\sinh\mu}\frac{d}{d\mu}(\sinh\mu\frac{dF(\mu)}{d\mu}) - \frac{m^2F(\mu)}{\sinh^2\mu} - n(n+1)F(\mu) = 0,$$

where m,n are both integer. Since we consider the problem in unbounded domain, we can have

$$\Phi_{nm} = \sqrt{\frac{(n-m)!}{(n+m)!}} Q_n^m(\cosh\mu)Y_{nm}(\theta,\varphi), \quad m = 0,\pm1,\pm2,\cdots,\pm n,$$

where

$$Q_n^m(x) = (-1)^m(x^2-1)^{\frac{m}{2}}\frac{d^m}{dx^m}Q_n(x), \quad x>1 \tag{2.7}$$

are the second kind associated Legendre functions,

$$Q_n(x) = \frac{2^n(n!)^2x^{-n-1}}{(2n+1)!}\left(1+\sum_{k=1}^{+\infty} b_k^n x^{-2k}\right) \tag{2.8}$$

are the second kind Legendre functions, $b_k^n = \frac{(\frac{n+1}{2})_k(\frac{n+2}{2})_k}{k!(\frac{2n+3}{2})_k}$, $(s)_k = s(s+1)\cdots(s+k-1)$, and $Y_{nm}(\theta,\varphi)$ are spherical harmonic functions. Therefore,

$$u(\mu,\theta,\varphi) = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\sqrt{\frac{(n-m)!}{(n+m)!}}A_{nm}Q_n^m(\cosh\mu)Y_{nm}(\theta,\varphi) \tag{2.9}$$

and

$$\begin{aligned}\frac{\partial u}{\partial \nu}(\mu,\theta,\varphi) &= -\frac{1}{f_0\sqrt{\cosh^2\mu-\cos^2\theta}}\frac{\partial u}{\partial \mu}\\ &= -\frac{1}{f_0\sqrt{\cosh^2\mu-\cos^2\theta}}\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\sqrt{\frac{(n-m)!}{(n+m)!}}A_{nm}\frac{d}{d\mu}Q_n^m(\cosh\mu)Y_{nm}(\theta,\varphi),\end{aligned} \tag{2.10}$$

where A_{nm} are some constants, $A_{n(-m)}=(-1)^m A_{nm}^*$ and A_{nm}^* are the conjugate complex of A_{nm}.

Suppose that

$$u_0(\mu_0,\theta,\varphi)=\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}U_{nm}Y_{nm}(\theta,\varphi), \tag{2.11}$$

where $U_{nm}=\int_0^{\pi}\int_0^{2\pi}u_0(\theta,\varphi)Y_{nm}^*(\theta,\varphi)\sin\theta d\theta d\varphi$. In (2.9) and (2.10), set $\mu\to\mu_0^+$, together with (2.11), we can obtain

$$A_{nm}=\sqrt{\frac{(n+m)!}{(n-m)!}}\frac{U_{nm}}{Q_n^m(\cosh\mu_0)}.$$

Hence, we have

$$u(\mu,\theta,\varphi)=\mathcal{P}u_0 \doteq \sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\frac{Q_n^m(\cosh\mu)}{Q_n^m(\cosh\mu_0)}U_{nm}Y_{nm}(\theta,\varphi),\ \ \mu\geq\mu_0>0 \tag{2.12}$$

and

$$\frac{\partial u}{\partial \nu}=\mathcal{K}u_0 \doteq -\frac{1}{f_0\sqrt{\cosh^2\mu_0-\cos^2\theta}}\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\frac{\frac{d}{d\mu}Q_n^m(\cosh\mu_0)}{Q_n^m(\cosh\mu_0)}U_{nm}Y_{nm}(\theta,\varphi). \tag{2.13}$$

Here, (2.12) and (2.13) are Poisson integral formula and natural integral operator, respectively. In addition, we set

$$\mathcal{K}_N u_0 \doteq -\frac{1}{f_0\sqrt{\cosh^2\mu_0-\cos^2\theta}}\sum_{n=0}^{N}\sum_{m=-n}^{n}\frac{\frac{d}{d\mu}Q_n^m(\cosh\mu_0)}{Q_n^m(\cosh\mu_0)}U_{nm}Y_{nm}(\theta,\varphi). \tag{2.14}$$

We obtain the solution of problem (2.1) directly from (2.12). In the following parts, we mainly discuss how to solve the problem (2.2).

3. Variational Problem and Its Well-posedness

First, we give the concerned concepts in Sobolev spaces $H^s(\Gamma_0)$ and $H^{-s}(\Gamma_0)(s\geq 0)$. The space $H^s(\Gamma_0)$, its norm and its inner product are defined by

$$\begin{aligned}&H^s(\Gamma_0)=\{v\in\mathcal{D}'(\Gamma_0):\\ &(\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^s|<v,\frac{1}{f_0^2\sinh\mu_0\sqrt{\cosh^2\mu_0-\cos^2\theta}}Y_{nm}^*>_{\Gamma_0}|^2)^{\frac{1}{2}}<+\infty,\mu_0>0\},\end{aligned} \tag{3.1}$$

$$\|v\|_{H^s(\Gamma_0)}=(\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^s|<v,\frac{1}{f_0^2\sinh\mu_0\sqrt{\cosh^2\mu_0-\cos^2\theta}}Y_{nm}^*>_{\Gamma_0}|^2)^{\frac{1}{2}}, \tag{3.2}$$

$$(f,g)_{\Gamma_0}=\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^s F_{nm}G_{nm}^*,\ \forall f,g\in H^s(\Gamma_0), \tag{3.3}$$

respectively, where $G_{nm}^* = \int_0^{2\pi}\int_0^{\pi} g(\theta,\varphi)Y_{nm}(\theta,\varphi)\sin\theta d\theta d\varphi$, $< v,f >_{\Gamma_0}$ denotes L^2 inner product on Γ_0, and $F_{nm}=\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)Y_{nm}^*(\theta,\varphi)\sin\theta d\theta d\varphi$. The space $H^{-s}(\Gamma_0)$ denotes the dual space of $H^s(\Gamma_0)$, whose norm is defined as

$$\|u\|_{H^{-s}(\Gamma_0)}=\sup_{v\in H^s(\Gamma_0)}\frac{|< u,v >_{\Gamma_0}|}{\|v\|_{H^s(\Gamma_0)}},\quad \forall u\in H^{-s}(\Gamma_0). \tag{3.4}$$

From [6] and [11], $H^s(\Gamma_0)$ with regard to inner product (3.3) forms a Hilbert space.

Problem (2.2) is equivalent to problem (2.13) and (2.12), while problem (2.13) is equivalent to the following variational problem:

$$\begin{cases} find\ u_0\in H^{\frac12}(\Gamma_0), such\ that\\ \hat{D}(u_0,v_0)=\int_{\Gamma_0} v_0 g_0 ds, \forall v_0\in H^{\frac12}(\Gamma_0),\end{cases} \tag{3.5}$$

where

$$\hat{D}(u_0,v_0)=\int_{\Gamma_0}\mathcal{K}u_0\cdot v_0 ds. \tag{3.6}$$

Let $V_{nm}=\int_0^{\pi}\int_0^{2\pi} v(\theta,\varphi)Y_{nm}^*(\theta,\varphi)\sin\theta d\theta d\varphi$, $F_{nm}=\int_0^{\pi}\int_0^{2\pi} f(\theta,\varphi)Y_{nm}^*(\theta,\varphi)\sin\theta d\theta d\varphi$. Thus,

$$\begin{aligned}\hat{D}(f,v)\ &=< \mathcal{K}f,v >_{\Gamma_0}=\int_{\Gamma_0}\mathcal{K}f\cdot v ds\\ &=f_0^2\int_0^{2\pi}\int_0^{\pi}\mathcal{K}f\cdot v\sqrt{\cosh^2\mu_0-\cos^2\theta}\sinh\mu_0\sin\theta d\theta d\varphi\\ &=-f_0\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\frac{\frac{d}{d\mu}Q_n^m(\cosh\mu_0)}{Q_n^m(\cosh\mu_0)}\sinh\mu_0 V_{nm}^*F_{nm}.\end{aligned} \tag{3.7}$$

Set

$$\hat{D}_N(f,v)=< \mathcal{K}_N f,v >_{\Gamma_0}=-f_0\sum_{n=0}^{N}\sum_{m=-n}^{n}\frac{\frac{d}{d\mu}Q_n^m(\cosh\mu_0)}{Q_n^m(\cosh\mu_0)}\sinh\mu_0 V_{nm}^*F_{nm}, \tag{3.8}$$

and

$$H_n^m(x_0)=-\frac{(x_0^2-1)\frac{d}{dx}Q_n^m(x_0)}{Q_n^m(x_0)}, \tag{3.9}$$

where $x_0=\cosh\mu_0$. In fact, the problem which we calculate is

$$\begin{cases} find\ u_0^N\in H^{\frac12}(\Gamma_0), such\ that\\ \hat{D}_N(u_0^N,v_0)=\int_{\Gamma_0} v_0 g_0 ds, \forall v_0\in H^{\frac12}(\Gamma_0),\end{cases} \tag{3.10}$$

where$\hat{D}_N(u_0^N,v_0)=\int_{\Gamma_0}\mathcal{K}_N u_0\cdot v_0 ds$.

Now we consider the well-posedness of variational problems (3.5) and (3.10) and give truncation error estimate. First, we give two Lemma.

Lemma 3.1. *There are some following conclusions about $Q_n^m(x)(x>1, m\le n)$:*

$$Q_n^m(x)=(x^2-1)^{\frac{m}{2}}\frac{2^n(n!)^2x^{-n-1-m}}{(2n+1)!}\{\frac{(n+m)!}{n!}+\sum_{k=1}^{+\infty}b_k^n\frac{(n+m+2k)!}{(n+2k)!}x^{-2k}\}, \tag{3.11}$$

$$(x^2-1)\frac{d}{dx}Q_n^m(x)=(n-m+1)Q_{n+1}^m(x)-(n+1)xQ_n^m(x), \tag{3.12}$$

Proof. Formula (3.11) immediately follows from (2.7) and (2.8). we refer [7] and obtain (3.12).

Lemma 3.2. *Let n and m be both non-negative integer.*

(1) If $x=\cosh\mu$, then $H_n^m(x)=-\dfrac{\frac{d}{d\mu}Q_n^m(\cosh\mu)}{Q_n^m(\cosh\mu)}\sinh\mu$.

(2) $H_n^{-m}(x)=H_n^m(x)$.

(3) If $0\le m\le n$ and $1<x$, then

$$\frac{(x^2-1)}{x}(n^2+1)^{\frac12}<H_n^m(x)<\sqrt2(n^2+1)^{\frac12}\ x. \tag{3.13}$$

(4) If $0\le m\le n$ and $1<x_0<x$, then

$$\left(\frac{x_0^2-1}{x^2-1}\right)^{\frac{n+1}{2}}\le\frac{Q_n^m(x)}{Q_n^m(x_0)}\le\left(\frac{x_0}{x}\right)^{n+1}. \tag{3.14}$$

Proof. For (1): Derivation of composite function gives

$$-\frac{\frac{d}{d\mu}Q_n^m(\cosh\mu)}{Q_n^m(\cosh\mu)}\sinh\mu=-\frac{\frac{d}{dx}Q_n^m(x)}{Q_n^m(x)}\sinh^2\mu=-(x^2-1)\frac{\frac{d}{dx}Q_n^m(x)}{Q_n^m(x)}=H_n^m(x).$$

For (2): Since $Q_n^{-m}(x)=\dfrac{(n-m)!}{(m+n)!}Q_n^m(x)$, it is obvious that

$$H_n^m(x)=-\frac{(x^2-1)\frac{d}{dx}Q_n^m(x)}{Q_n^m(x)}=-\frac{(x^2-1)\frac{d}{dx}Q_n^{-m}(x)}{Q_n^{-m}(x)}=H_n^{-m}(x).$$

For (3): First we discuss the right of the inequality. When $0\le m\le n$, (3.11) and (3.12) imply that

$$H_n^m(x)=(n+1)x-(n-m+1)\frac{Q_{n+1}^m(x)}{Q_n^m(x)}<(n+1)x<\sqrt2(n^2+1)^{\frac12}x. \tag{3.15}$$

Now we consider the left. Let

$$f(k)=\frac{(2n+3)(n+1+m+2k)}{(n+1)(2n+3+2k)}.$$

Clearly, when $k\ge0$, $f(k)\le\frac{2n+3}{n+1}$. From (3.11), we have

$$\frac{Q_{n+1}^m(x)}{Q_n^m(x)}=\frac{(n+1)}{x(2n+3)}\frac{\left\{\frac{(n+m)!}{n!}f(0)+\sum_{k=1}^{+\infty}b_k^nf(k)\frac{(n+m+2k)!}{(n+2k)!}x^{-2k}\right\}}{\left\{\frac{(n+m)!}{n!}+\sum_{k=1}^{+\infty}b_k^n\frac{(n+m+2k)!}{(n+2k)!}x^{-2k}\right\}}\le\frac1x.$$

Therefore,

$$H_n^m(x) = (n+1)x - (n-m+1)\frac{Q_{n+1}^m(x)}{Q_n^m(x)} > (n+1)\frac{(x^2-1)}{x} + \frac{m}{x} > (n^2+1)^{\frac{1}{2}}\frac{(x^2-1)}{x}. \quad (3.16)$$

Thus, (3.14) holds.

For (4): Formulas (3.15) and (3.16) imply that

$$-\frac{(n+1)x}{(x^2-1)} \le \frac{\frac{d}{dx}Q_n^m(x)}{Q_n^m(x)} \le -\frac{n+1}{x}.$$

Integration in interval $[x_0, x]$ satisfies

$$\left(\frac{x_0^2-1}{x^2-1}\right)^{\frac{n+1}{2}} \le \frac{Q_n^m(x)}{Q_n^m(x_0)} \le \left(\frac{x_0}{x}\right)^{n+1}.$$

Remark 3.1. When $\frac{a}{b} = \frac{\cosh\mu_0}{\sinh\mu_0} \to 1$, we have $\mu_0 \to +\infty$, i.e. $x_0 = \cosh\mu_0 \to +\infty$. Therefore, from (3.13) and (3.14), we know the limit coincides with those for a sphere boundary [2,6,9].

Theorem 3.1. *$\hat{D}(\cdot,\cdot)$ and $\hat{D}_N(\cdot,\cdot)$ are both the symmetric bilinear form on $H^{\frac{1}{2}}(\Gamma_0)$. $\hat{D}(\cdot,\cdot)$ is continuous and coercive on $H^{\frac{1}{2}}(\Gamma_0)$, i.e.*

$$\alpha\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \le \hat{D}(v,v), \quad \forall v \in H^{\frac{1}{2}}(\Gamma_0) \quad (3.17)$$

and

$$|\hat{D}(f,v)| \le \sqrt{2}f_0x_0\|v\|_{H^{\frac{1}{2}}(\Gamma_0)}\|f\|_{H^{\frac{1}{2}}(\Gamma_0)}, \quad \forall v,f \in H^{\frac{1}{2}}(\Gamma_0). \quad (3.18)$$

$\hat{D}_N(\cdot,\cdot)$ is also continuous on $H^{\frac{1}{2}}(\Gamma_0)$, i.e.

$$|\hat{D}_N(f,v)| \le \sqrt{2}f_0x_0\|v\|_{H^{\frac{1}{2}}(\Gamma_0)}\|f\|_{H^{\frac{1}{2}}(\Gamma_0)}, \quad \forall v,f \in H^{\frac{1}{2}}(\Gamma_0), \quad (3.19)$$

and there exists a positive constant M_0 such that when $N > M_0$,

$$\frac{\alpha}{2}\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \le \hat{D}_N(v,v), \quad \forall v \in H^{\frac{1}{2}}(\Gamma_0), \quad (3.20)$$

where $\alpha = \dfrac{x_0^2-1}{x_0}$ and $x_0 = \cosh\mu_0$.

Proof. Clearly, it follows from (3.7) and (3.8) that $\hat{D}(\cdot,\cdot)$ and $\hat{D}_N(\cdot,\cdot)$ are both the symmetric bilinear form on $H^{\frac{1}{2}}(\Gamma_0)$. For any $f,v \in H^{\frac{1}{2}}(\Gamma_0)$, Schwarz inequality and Lemma 3.2 imply that

$$\begin{aligned}\hat{D}(v,v) &= f_0\sum_{n=0}^{+\infty}\sum_{m=-n}^{n} H_n^m(x_0)|V_{nm}|^2 \\ &\ge \alpha\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(n^2+1)^{\frac{1}{2}}|V_{nm}|^2 = \alpha\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)},\end{aligned}$$

$$|\hat{D}(f,v)| = |f_0\sum_{n=0}^{+\infty}\sum_{m=-n}^{n} H_n^m(x_0)F_{nm}V_{nm}^*| < \sqrt{2}f_0x_0\|v\|_{H^{\frac{1}{2}}(\Gamma_0)}\|f\|_{H^{\frac{1}{2}}(\Gamma_0)},$$

and

$$\begin{aligned}&|\hat{D}_N(f,v)| = |f_0\sum_{n=0}^{N}\sum_{m=-n}^{n} H_n^m(x_0)F_{nm}V_{nm}^*| < \sqrt{2}f_0x_0\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(n^2+1)^{\frac{1}{2}}|F_{nm}||V_{nm}^*| \\ &< \sqrt{2}f_0x_0\|v\|_{H^{\frac{1}{2}}(\Gamma_0)}\|f\|_{H^{\frac{1}{2}}(\Gamma_0)}.\end{aligned}$$

Since $\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)} = \sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(n^2+1)^{\frac{1}{2}}|V_{nm}|^2 < +\infty$, *there exists positive* M_0 *such that when* $N > M_0$,

$$\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(n^2+1)^{\frac{1}{2}}|V_{nm}|^2 < \sum_{n=0}^{N}\sum_{m=-n}^{n}(n^2+1)^{\frac{1}{2}}|V_{nm}|^2, \tag{3.21}$$

and then

$$\begin{aligned}\hat{D}(v,v) &\geq -f_0\sum_{n=0}^{N}\sum_{m=-n}^{n}\frac{\frac{d}{dx}Q_n^m(x_0)}{Q_n^m(x_0)}(x_0^2-1)|V_{nm}|^2 = \hat{D}_N(v,v)\\ &\geq \alpha\sum_{n=0}^{N}\sum_{m=-n}^{n}(n^2+1)^{\frac{1}{2}}|V_{nm}|^2 \geq \frac{\alpha}{2}\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}(n^2+1)^{\frac{1}{2}}|V_{nm}|^2 = \frac{\alpha}{2}\|v\|^2_{H^{\frac{1}{2}}(\Gamma_0)}.\end{aligned}$$

Theorem 3.2. *For any given* $g_0 \in H^{-\frac{1}{2}}(\Gamma_0)$, *there exists a unique solution* u_0 *for the variational problem (3.5) and the solution depends on the given initial value* g_0 *continuously.*

Proof. Since $g_0 \in H^{-\frac{1}{2}}(\Gamma_0)$, we have $| < g_0, v >_{\Gamma_0} | \leq \|v\|_{H^{\frac{1}{2}}(\Gamma_0)}\|g_0\|_{H^{-\frac{1}{2}}(\Gamma_0)}$. Thus $< g_0, v >_{\Gamma_0}$ is bounded linear functional in $H^{\frac{1}{2}}(\Gamma_0)$. Using Theorem 3.1, together with Lax-Milgram theorem, we know that there exists a unique solution for the problem (3.5) and then have

$$\alpha\|u_0\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \leq \hat{D}(u_0,u_0) =< g_0, u_0 >_{\Gamma_0} \leq \|u_0\|_{H^{\frac{1}{2}}(\Gamma_0)}\|g_0\|_{H^{-\frac{1}{2}}(\Gamma_0)}.$$

Hence,

$$\|u_0\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq \frac{1}{\alpha}\|g_0\|_{H^{-\frac{1}{2}}(\Gamma_0)}.$$

Consequently, we have the following conclusion.

Theorem 3.3. *For any given* $g_0 \in H^{-\frac{1}{2}}(\Gamma_0)$,*there is a constant* $M_0 > 0$ *such that when* $N > M_0$, *there exists a unique solution* u_0^N *for the variational problem (3.10) and the solution depends on the given initial value* g_0 *continuously.*

Now we consider truncation error estimate.

Theorem 3.4. *Suppose that* $u_0 \in H^{\frac{1}{2}}(\Gamma_0)$ *and* $u_0^N \in H^{\frac{1}{2}}(\Gamma_0)$ *are solutions of problem (3.5) and problem (3.10) respectively and there exist positive constant* M_0 *and* α *such that when* $N > M_0$, *we have*

$$\|u_0 - u_0^N\|_{H^{\frac{1}{2}}(\Gamma_0)} \leq \frac{2\sqrt{2}f_0x_0}{\alpha}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{\frac{1}{2}}|U_{nm}|^2\right)^{\frac{1}{2}}. \tag{3.22}$$

Proof. It follows from (3.5) and (3.10) that

$$\hat{D}(u_0,v) - \hat{D}_N(u_0^N,v) = 0, \forall v \in H^{\frac{1}{2}}(\Gamma_0).$$

Assume that $Q_{nm} = \int_0^{\pi}\int_0^{2\pi}(u_0 - u_0^N)Y_{nm}^*(\theta,\varphi)\sin\theta d\theta d\varphi$. There is positive M_0 such that when

$N > M_0$, (3.20) and Lemma 3.2 imply that

$$\begin{aligned}
&\frac{\alpha}{2}\|u_0-u_0^N\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \le \hat{D}_N(u_0-u_0^N, u_0-u_0^N) = \hat{D}_N(u_0, u_0-u_0^N) - \hat{D}(u_0, u_0-u_0^N)\\
&= \sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n} H_n^m(x_0)U_{nm}Q_{nm}^* < \sqrt{2}f_0x_0 \sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{\frac{1}{2}}|U_{nm}Q_{nm}^*|\\
&< \sqrt{2}x_0f_0\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{\frac{1}{2}}|U_{nm}|^2\right)^{\frac{1}{2}}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{\frac{1}{2}}|Q_{nm}|^2\right)^{\frac{1}{2}}\\
&< \sqrt{2}f_0x_0\|u_0-u_0^N\|_{H^{\frac{1}{2}}(\Gamma_0)}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{\frac{1}{2}}|U_{nm}|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof.

4. Discrete Variational Problem and Its Error Estimates

We divide the intervals $[0, 2\pi)$, $[0, \pi]$ into N_2, N_1 parts with the same length, respectively, and then make the corresponding finite element partitions in Γ_0. In our computation, continuous piecewise linear elements are used.

Set $\theta_s = \frac{s\pi}{N_1}, \varphi_t = \frac{2t\pi}{N_2}$,

$$L_0(\theta) = \begin{cases} \frac{N_1}{\pi}(\theta_1-\theta), & \theta\in[0,\theta_1],\\ 0, & otherwise, \end{cases} \tag{4.1}$$

$$L_{N_1}(\theta) = \begin{cases} \frac{N_1}{\pi}(\theta-\theta_{N_1-1}), & \theta\in[\theta_{N_1-1},\pi],\\ 0, & otherwise, \end{cases} \tag{4.2}$$

$$L_s(\theta) = \begin{cases} \frac{N_1}{\pi}(\theta-\theta_{s-1}), & \theta\in[\theta_{s-1},\theta_s],\\ \frac{N_1}{\pi}(\theta_{s+1}-\theta), & \theta\in[\theta_s,\theta_{s+1}],\\ 0, & otherwise, \end{cases} \tag{4.3}$$

and

$$M_t(\varphi) = \begin{cases} \frac{N_2}{\pi}(\varphi-\varphi_{t-1}), & \varphi\in[\varphi_{t-1},\varphi_t],\\ \frac{N_2}{\pi}(\varphi_{t+1}-\varphi), & \varphi\in[\varphi_t,\varphi_{t+1}],\\ 0, & otherwise, \end{cases} \tag{4.4}$$

where $s = 1, 2, \cdots, N_1-1;\ t = 1, 2, \cdots, N_2$. Using these formulas, we can construct the bilinear interpolating basis functions on Γ_0, i.e.

$$\begin{cases} B_0(\theta,\varphi) = L_0(\theta), \quad B_1(\theta,\varphi) = L_{N_1}(\theta),\\ B_{st}(\theta,\varphi) = L_s(\theta)M_t(\varphi), \quad s = 1,2,\cdots,N_1-1;\ t = 1,2,\cdots,N_2. \end{cases} \tag{4.5}$$

Sequential coding of $B_{st}(\theta,\varphi)$ are as follows : for the different s, the elements with the bigger s are put back; for the same s, the elements with the bigger t are permutated back.

We set

$$S^h(\Gamma_0) = span\{B_0(\theta,\varphi), B_1(\theta,\varphi), B_{st}(\theta,\varphi) : s = 1,2,\cdots,N_1-1; t = 1,2,\cdots,N_2\}, \tag{4.6}$$

and then obtain

$$S^h(\Gamma_0) \subset H^1(\Gamma_0) \subset H^{\frac{1}{2}}(\Gamma_0).$$

Thus, the approximate variational problem of (3.10) is

$$\begin{cases} find\ u_0^{Nh}\in S^h(\Gamma_0),\ such\ that\\ \hat{D}_N(u_0^{Nh}, v^h) = < g_0, v^h >_{\Gamma_0}, \forall v^h\in S^h(\Gamma_0). \end{cases} \tag{4.7}$$

Using Theorem 3.1 , Theorem 3.3 and $S^h(\Gamma_0) \subset H^{\frac{1}{2}}(\Gamma_0)$, together with Lax-Milgram Theorem, we know that there is $M_0 > 0$ independent of h, such that when $N > M_0$, the variational problem (4.7) exists a unique solution in $S^h(\Gamma_0)$. Again, we set

$$u_0^{Nh}(\theta,\varphi) = U_0 B_0(\theta,\varphi) + U_1 B_1(\theta,\varphi) + \sum_{s=1}^{N_1-1}\sum_{t=1}^{N_2} U_{st} B_{st}(\theta,\varphi). \tag{4.8}$$

When v_h is fetched basis functions $B_0, B_1, B_{st} (s = 1,2,\cdots,N_1-1; t = 1,2,\cdots,N_2)$ in turn, we substitute (4.8) into (4.7) and then derive linear equations:

$$\begin{bmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} & \mathbf{A}_{02} & \cdots & \mathbf{A}_{0,N_1-1} \\ \mathbf{A}_{10} & \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1,N_1-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{N_1-1,0} & \mathbf{A}_{N_1-1,1} & \mathbf{A}_{N_1-1,2} & \cdots & \mathbf{A}_{N_1-1,N_1-1} \end{bmatrix} \begin{bmatrix} \mathbf{U}^{(0)} \\ \mathbf{U}^{(1)} \\ \vdots \\ \mathbf{U}^{(N_1-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{b}^{(0)} \\ \mathbf{b}^{(1)} \\ \vdots \\ \mathbf{b}^{(N_1-1)} \end{bmatrix} \tag{4.9}$$

where

$$\mathbf{A}_{00} = \begin{bmatrix} \mathbf{a}_{00} & \mathbf{a}_{01} \\ \mathbf{a}_{10} & \mathbf{a}_{11} \end{bmatrix}, \mathbf{A}_{0s} = \begin{bmatrix} \mathbf{a}_{01}^{(0s)} & \mathbf{a}_{02}^{(0s)} & \cdots & \mathbf{a}_{0N_2}^{(0s)} \\ \mathbf{a}_{11}^{(0s)} & \mathbf{a}_{12}^{(0s)} & \cdots & \mathbf{a}_{1N_2}^{(0s)} \end{bmatrix}, \mathbf{A}_{s0} = \mathbf{A}_{0s}^T,$$

$$\mathbf{A}_{ss'} = \left(\mathbf{a}_{tt'}^{(ss')}\right)_{N_2\times N_2}, \mathbf{U}^{(0)} = [U_0, U_1]^T, \mathbf{U}^{(s)} = [U_{s1}, U_{s2}, \cdots, U_{sN_2}]^T,$$

$$\mathbf{b}^{(0)} = [\mathbf{b}_0, \mathbf{b}_1]^T, \mathbf{b}_i = f_0^2 \int_0^\pi \int_0^{2\pi} g_0(\theta,\varphi) B_i(\theta,\varphi) \sqrt{cosh^2\mu_0 - \cos^2\theta}\,\sinh\mu_0 \sin\theta d\theta d\varphi,$$

$$\mathbf{b}^{(s)} = [\mathbf{b}_{s1}, \mathbf{b}_{s2}, \cdots, \mathbf{b}_{sN_2}]^T, \mathbf{b}_{st} = f_0^2 \int_0^\pi \int_0^{2\pi} g_0(\theta,\varphi) B_{st}(\theta,\varphi) \sqrt{cosh^2\mu_0 - \cos^2\theta}\,\sinh\mu_0 \sin\theta d\theta d\varphi,$$

$$\mathbf{a}_{ik} = \hat{D}_N(B_k, B_i), \mathbf{a}_{it}^{(0s)} = \hat{D}_N(B_{st}, B_i), \mathbf{a}_{tt'}^{(ss')} = \hat{D}_N(B_{s't'}, B_{st}),$$
$$i,\ k = 0,1; s,\ s' = 1,2,\cdots,N_1-1; t,t' = 1,2,\cdots,N_2.$$

Let

$$G(n,m,s) = \sqrt{\frac{(2n+1)(n-m)!}{(n+m)!}} \int_0^\pi L_s(\theta) P_n^m(\cos\theta) \sin\theta d\theta, s = 0,1,2,\cdots,N_1,$$

where $P_n^m(t)(|t| \le 1, m \le n)$ are the first kind Legendre functions. Using (3.8) yields the computing formulas of elements in stiffness matrix $\mathbf{A}$ as follows:

$$\mathbf{a}_{ik} = f_0\pi \sum_{n=0}^{N} H_n^0(x_0) G(n,0,iN_1) G(n,0,kN_1), i,k = 0,1,$$

$$\mathbf{a}_{it}^{(0s)} = \frac{f_0\pi}{N_2} \sum_{n=0}^{N} H_n^0(x_0) G(n,0,iN_1) G(n,0,s), i = 0,1,$$

$$\begin{aligned} \mathbf{a}_{tt'}^{ss'} = \quad & f_0 \sum_{n=0}^{N} \Bigg[\frac{\pi}{N_2^2} H_n^0(x_0) G(n,0,s) G(n,0,s') \\ & + \sum_{m=1}^{n} \frac{2N_2^2}{\pi^3 m^4} H_n^m(x_0) G(n,m,s) G(n,m,s') \sin^4\frac{m\pi}{N_2} \cos\frac{2m(t-t')\pi}{N_2} \Bigg], \\ & s,s' = 1,2,\cdots,N_1-1;\ t,t' = 1,2,\cdots,N_2. \end{aligned}$$

Now we discuss how to calculate $G(n,m,s)$ and $H_n^m(x_0)$. From [3], we know

$$P_n(\cos\theta)=\frac{1}{4^n}\sum_{k=0}^{n}\frac{(2k)!(2n-2k)!}{(k!)^2(n-k)!(n-k)!}\cos(n-2k)\theta$$

and

$$P_n^1(\cos\theta)=-\frac{1}{4^n}\sum_{k=0}^{n}\frac{(2k)!(2n-2k)!(n-2k)}{(k!)^2(n-k)!(n-k)!}\sin(n-2k)\theta.$$

Thus, we have

$$G(n,0,s)=\frac{\sqrt{2n+1}}{4^n}\sum_{k=0}^{n}\frac{(2k)!(2n-2k)!}{(k!)^2(n-k)!(n-k)!}\int_0^{\pi}L_s(\theta)\cos(n-2k)\theta\sin\theta d\theta,$$

$$G(n,1,s)=-\frac{1}{4^n}\sqrt{\frac{2n+1}{n(n+1)}}\sum_{k=0}^{n}\frac{(2k)!(2n-2k)!(n-2k)}{(k!)^2(n-k)!(n-k)!}\int_0^{\pi}L_s(\theta)\sin(n-2k)\theta\sin\theta d\theta,$$

$$\begin{aligned}G(n,m,s)=\ &\sqrt{\frac{(2n+1)(n-m)(n-m-1)}{(2n-3)(n+m)(n+m-1)}}G(n-2,m,s)\\&-\sqrt{\frac{(n-m+2)(n-m+1)}{(n+m)(n+m-1)}}G(n,m-2,s)\\&+\sqrt{\frac{(2n+1)(n+m-2)(n+m-3)}{(2n-3)(n+m)(n+m-1)}}G(n-2,m-2,s),\quad n\ge m\ge 2,\end{aligned}$$

and when $n<m$, $G(n,m,s)=0$, where $s=0,1,2,\cdots,N_1$.

Computing formulas of $H_n^m(x_0)$ are as follows:

$$H_n^m(x_0)=\begin{cases}2m\dfrac{Q_{m-1}^m(x_0)}{Q_m^m(x_0)}-mx_0, & m=n,\\ \dfrac{(n+m)(n-m)}{nx_0-H_{n-1}^m(x_0)}-nx_0, & n\ge m,\end{cases}$$

where

$$Q_{m-1}^m(x_0)=\begin{cases}\dfrac{1}{\sqrt{x_0^2-1}}, & m=1,\\ 2^{m-1}(m-1)!(x_0^2-1)^{\frac{1-m}{2}}Q_0^1(x_0), & m>1,\end{cases}$$

and

$$Q_m^m(x_0)=\begin{cases}\dfrac{1}{2}ln\dfrac{x_0+1}{x_0-1}, & m=0,\\ x_0Q_{m-1}^m(x_0)-(2m-1)(x_0^2-1)^{\frac{1}{2}}Q_{m-1}^{m-1}(x_0) & m\ge 1.\end{cases}$$

Let u_0^{Nh} denote numerical solution of u_0^N, $\Gamma=\{(\mu,\theta,\varphi):\mu=\mu_1>\mu_0,\theta\in[0,\pi],\varphi\in[0,2\pi)\}$, u is exact solution of problem (2.2) and u^{Nh} is numerical value of u. h is maximum mesh size.

Now we consider error estimates in $H^{\frac{1}{2}}(\Gamma_0)$ and $L^2(\Gamma_0)$.

Theorem 4.1. If $u_0\in H^2(\Gamma_0)$, then there are constant C and M_0 independent of h and N. We have the following conclusions:

(1) for $N>M_0$,

$$\|u_0-u_0^{Nh}\|_{H^{\frac{1}{2}}(\Gamma_0)}\le C\left(h^{\frac{3}{2}}|u_0|_{H^2(\Gamma_0)}+\frac{1}{(N+1)^{\frac{3}{2}}}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^2|U_{nm}|^2\right)^{\frac{1}{2}}\right).\quad(4.10)$$

(2)
$$\|u-u^{Nh}\|_{H^{\frac{1}{2}}(\Gamma)} \le \|u_0-u_0^{Nh}\|_{H^{\frac{1}{2}}(\Gamma_0)}. \tag{4.11}$$

where $U_{nm}=\int_0^{2\pi}\int_0^{\pi} u_0 Y_{nm}^*(\theta,\varphi)\sin\theta d\theta d\varphi$.

Proof. For (1): Note that (3.5) and (4.7) imply

$$\hat{D}(u_0,v^h)-\hat{D}_N(u_0^{Nh},v^h)=0, \forall v^h\in S^h(\Gamma_0).$$

If $N>M_0$, it follows from Theorem 3.1 and Theorem 3.4 that

$$\begin{aligned}
&\frac{\alpha}{2}\|v^h-u_0^{Nh}\|^2_{H^{\frac{1}{2}}(\Gamma_0)} \le \hat{D}_N(v^h-u_0^{Nh},v^h-u_0^{Nh})\\
&=\hat{D}_N(v^h-u_0^{Nh},v^h-u_0)+\hat{D}_N(v^h-u_0^{Nh},u_0-u_0^{Nh})\\
&=\hat{D}_N(v^h-u_0^{Nh},v^h-u_0)+\hat{D}_N(u_0,v^h-u_0^{Nh})-\hat{D}(u_0,v^h-u_0^{Nh})\\
&\le \sqrt{2}f_0x_0\|v^h-u_0^{Nh}\|_{H^{\frac{1}{2}}(\Gamma_0)}\left(\|v^h-u_0\|_{H^{\frac{1}{2}}(\Gamma_0)}+\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{\frac{1}{2}}|U_{nm}|^2\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Since $u_0\in H^2(\Gamma_0)$, we obtain

$$\begin{aligned}
&\|v^h-u_0^{Nh}\|_{H^{\frac{1}{2}}(\Gamma_0)} \le \frac{2\sqrt{2}f_0x_0}{\alpha}\left(\|v^h-u_0\|_{H^{\frac{1}{2}}(\Gamma_0)}+\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^{\frac{1}{2}}|U_{nm}|^2\right)^{\frac{1}{2}}\right)\\
&\le \frac{2\sqrt{2}f_0x_0}{\alpha}\left(\|v^h-u_0\|_{H^{\frac{1}{2}}(\Gamma_0)}+\frac{1}{(N+1)^{\frac{3}{2}}}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^2|U_{nm}|^2\right)^{\frac{1}{2}}\right).
\end{aligned}$$

Using the concerned conclusions in [10] gives

$$\begin{aligned}
&\|u_0-u_0^{Nh}\|_{H^{\frac{1}{2}}(\Gamma_0)} \le \inf_{v^h\in H^{\frac{1}{2}}(\Gamma_0)}(\|v^h-u_0^{Nh}\|_{H^{\frac{1}{2}}(\Gamma_0)}+\|v^h-u_0\|_{H^{\frac{1}{2}}(\Gamma_0)})\\
&\le (\frac{2\sqrt{2}f_0x_0}{\alpha}+1)\left(\inf_{v^h\in H^{\frac{1}{2}}(\Gamma_0)}\|v^h-u_0\|_{H^{\frac{1}{2}}(\Gamma_0)}+\frac{1}{(N+1)^{\frac{3}{2}}}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^2|U_{nm}|^2\right)^{\frac{1}{2}}\right)\\
&\le C\left(h^{\frac{3}{2}}|u_0|_{H^2(\Gamma_0)}+\frac{1}{(N+1)^{\frac{3}{2}}}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^2|U_{nm}|^2\right)^{\frac{1}{2}}\right).
\end{aligned}$$

For (2): Suppose that $P_{nm}=\int_0^{\pi}\int_0^{2\pi}(u_0-u_0^{Nh})Y_{nm}^*(\theta,\varphi)\sin\theta d\theta d\varphi$. Lemma 3.2 and (2.12) imply that

$$\|u-u^{Nh}\|^2_{H^{\frac{1}{2}}(\Gamma)}=\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\left(\frac{Q_n^m(\cosh\mu)}{Q_n^m(\cosh\mu_0)}\right)^2(1+n^2)^{\frac{1}{2}}|P_{nm}|^2 \le \|u_0-u_0^{Nh}\|^2_{H^{\frac{1}{2}}(\Gamma_0)}.$$

Theorem 4.2. *Suppose that $u_0\in H^2(\Gamma_0)$ and $u_0^N-u_0^{Nh}$ satisfies the following regularity assumptions: $u_0^N-u_0^{Nh}\in L^2(\Gamma_0)$; the natural integral equation $\mathcal{K}v_0=u_0^N-u_0^{Nh}$ has solution v_0 and $v_0\in H^1(\Gamma_0)$; there is a positive constant C, such that $\|v_0\|_{H^1(\Gamma_0)}\le C\|u_0^N-u_0^{Nh}\|_{L^2(\Gamma_0)}$. There is a positive constant M_0 such that when $N>M_0$, we have*

$$\|u_0-u_0^{Nh}\|_{L^2(\Gamma_0)}\le C\left(h^2|u_0|_{H^2(\Gamma_0)}+\frac{1}{(N+1)^{\frac{3}{2}}}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^2|U_{nm}|^2\right)^{\frac{1}{2}}\right). \tag{4.12}$$

Proof. Theorem 3.4 and error estimate in [10] imply that

$$\begin{aligned}\|u_0 - u_0^{Nh}\|_{L^2(\Gamma_0)} &\le (\|u_0^N - u_0^{Nh}\|_{L^2(\Gamma_0)} + \|u_0 - u_0^N\|_{L^2(\Gamma_0)}) \\ &\le (\|u_0^N - u_0^{Nh}\|_{L^2(\Gamma_0)} + \|u_0 - u_0^N\|_{H^{\frac{1}{2}}(\Gamma_0)}) \\ &\le C\left(h^2|u_0|_{H^2(\Gamma_0)} + \frac{1}{(N+1)^{\frac{3}{2}}}\left(\sum_{n=N+1}^{+\infty}\sum_{m=-n}^{n}(1+n^2)^2|U_{nm}|^2\right)^{\frac{1}{2}}\right).\end{aligned}$$

Remark 4.1. From Theorem 4.1 and 4.2, our error estimates are based on the mesh size h and also on the terms N. For a given mesh size h, Theorem 4.1 and 4.2 tell us the optimal way for choosing the terms N.

5. Numerical Examples

In this section we present the numerical results which demonstrate the performance of the error estimates(4.10)-(4.12). In our practical computation, continuous piecewise linear elements are used.

Example 1. Let $f_0 = 4$, $\Gamma_0 = \{(\mu_0, \theta, \varphi) : \mu_0 = 1, \theta \in [0, \pi], \varphi \in [0, 2\pi)\}$ and

$$g_0 = -\frac{\sqrt{2}\sin 2\theta\cos\varphi}{4f_0^4\sqrt{\cosh^2\mu_0 - \cos^2\theta}}\frac{(7 - 3\cosh 4\mu_0 + 4\cosh 2\mu_0\cos 2\theta)}{(\cosh 2\mu_0 + \cos 2\theta)^{\frac{7}{2}}}.$$

Solve problem (2.2), whose exact solution is

$$u = \frac{\sqrt{2}\sinh(2\mu)\sin(2\theta)\cos\varphi}{2f_0^3(\cosh 2\mu + \cos 2\theta)^{\frac{5}{2}}}.$$

Then the relationships between the errors of solution and each parameters are as follows:

Table 5.1: The effect of the mesh parameters $(N1, N2)$ for the solution u_0 and u.

mesh(N1,N2)	(10,20)	(20,40)	(40,80)	(80,160)
$\|\|u_0 - u_0^{Nh}\|\|_{L^\infty(\Gamma_0)}$	1.8426e-4	4.7542e-5	1.2072e-5	3.0323e-6
Ratio1	–	3.8758	3.9382	3.9812
$\|\|u - u_0^{Nh}\|\|_{L^2(\Gamma_0)}$	2.3981e-4	6.7109e-5	1.7460e-5	4.4576e-6
Ratio2	–	3.5735	3.8436	3.9169
$\|\|u_0 - u_0^{Nh}\|\|_{H^{\frac{1}{2}}(\Gamma_0)}$	5.5687e-4	1.5728e-4	4.1080e-5	1.4464e-5
Ratio3	–	3.5406	3.8286	2.8402
$\|\|u - u^{Nh}\|\|_{H^{\frac{1}{2}}(\Gamma)}$	5.9056e-5	1.5650e-5	4.0337e-6	1.3467e-6
Ratio4	–	3.7736	3.8798	2.9956

Table 5.2: The effect of the terms N for the solution u_0.

mesh(N1,N2)	(20,40)		(40,80)		(80,160)	
N	15	20	20	40	70	80
$L^\infty(\Gamma_0)$	1.3863	3.0239e-6	3.4257	4.0532e-6	1.1936	3.7576e-6
$L^2(\Gamma_0)$	0.2621	2.8397e-6	0.2912	1.9266e-6	0.0707	2.2484e-6
$H^{\frac{1}{2}}(\Gamma_0)$	1.6084	2.1847e-5	5.4238e-6	9.2469e-6	0.5863	2.9684e-5

First, we test the effect of the mesh parameters $(N1, N2)$ in the error estimate (4.10)-(4.12). In Table 5.1, we give concerned error for large N, where $N = 100$ and $\mu_1 = 1.5$. The results show that the convergent rates of $||u_0 - u_0^{Nh}||_{L^\infty(\Gamma_0)}$ and $||u - u_0^{Nh}||_{L^2(\Gamma_0)}$ with respect to h is 2, while the convergent rates of $||u_0 - u_0^{Nh}||_{H^{\frac{1}{2}}(\Gamma_0)}$ and $||u - u^{Nh}||_{H^{\frac{1}{2}}(\Gamma)}$ with respect to h are more than $\frac{3}{2}$. Secondly, we test the effect of the terms N for the solution u_0. Let u_0^{100h} denote numerical solution of the problem on the boundary Γ_0 with large N, where $N = 100$. In Table 5.2, we calculate the corresponding error of $u_0^{Nh} - u_0^{100h}$. The numerical results indicate the errors are admittable for a given mesh size, only if the terms N are big enough. These results are consistent with theoretical analysis in Theorem 4.1 and 4.2. In numerical experiment, we also find that whether linear equations (4.9) exists unique solution depends on the terms N. Finally, we test the effect of μ for the exterior solution u. Let Ed denote $||u-u^{Nh}||_{H^{\frac{1}{2}}(\Gamma)}$,where $N = 60$. In Fig 5.1, we choose respectively $\mu_1 = 1.2, 1.5, 2.0, 2.5, 3.0$.

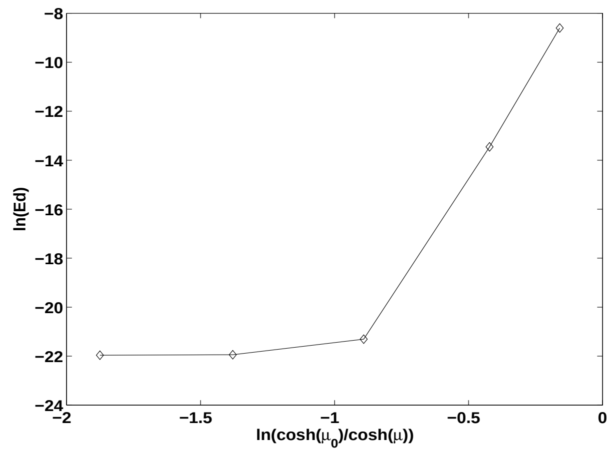

Fig. 5.1. The effect of μ for the exterior solution u.

Example 2. Let $f_0 = 4, \Gamma_0 = \{(\mu_0, \theta, \varphi) : \mu_0 = 1, \theta \in [0, \pi], \varphi \in [0, 2\pi)\}$ and

$$g_0 = -\frac{\sqrt{2}\cos\varphi}{f_0^5\sqrt{\cosh^2\mu_0 - \cos^2\theta}(\cosh 2\mu_0 + \cos 2\theta)^{\frac{9}{2}}}(15\cosh 3\mu_0 \sin 5\theta - 20\sin 3\theta\cosh 5\mu_0 \\ -3\sin 3\theta\cosh\mu_0 - 4\sin\theta\cosh 5\mu_0 + 102\cosh\mu_0\sin\theta - 66\sin\theta\cosh 3\mu_0 + \cosh\mu_0\sin 5\theta).$$

Solve (2.2), whose exact solution is

$$u = \frac{8\sqrt{2}\sinh\mu\sin\theta\cos\varphi(5\cosh 2\mu\cos 2\theta + 3\cosh 2\mu + 3\cos 2\theta + 5)}{f_0^4(\cosh 2\mu + \cos 2\theta)^{\frac{7}{2}}}.$$

The errors of approximate solution are as follows:

Table 5.3: The effect of the mesh parameters $(N1, N2)$ for solution u_0 and u.

mesh(N1,N2)	(10,20)	(20,40)	(40,80)	(80,160)
$\|\|u_0 - u_0^{Nh}\|\|_{L^\infty(\Gamma_0)}$	1.6744e-3	4.64466e-4	1.1700e-4	3.0940e-5
Ratio1	–	3.6050	3.9698	3.7815
$\|\|u - u_0^{Nh}\|\|_{L^2(\Gamma_0)}$	1.6495e-3	4.9582e-4	1.3131e-4	3.4024e-5
Ratio2	–	3.3268	3.7760	3.8594
$\|\|u_0 - u_0^{Nh}\|\|_{H^{\frac{1}{2}}(\Gamma_0)}$	4.2904e-3	1.3018e-3	3.45463e-4	8.9834e-5
Ratio3	–	3.2958	3.7683	3.8456
$\|\|u - u^{Nh}\|\|_{H^{\frac{1}{2}}(\Gamma)}$	2.8951e-4	8.02981e-5	2.0618e-5	5.4415e-6
Ratio4	–	3.6055	3.8947	3.7890

Table 5.4: The effect of the terms N for the solution u_0.

mesh(N1,N2)	(20,40)		(40,80)		(80,160)	
N	15	20	30	40	70	80
$L^\infty(\Gamma_0)$	1.3865	2.4691e-5	1.5963	8.0481e-6	1.1936	7.7843e-6
$L^2(\Gamma_0)$	0.2661	2.2772e-5	0.1587	6.3976e-6	0.0707	2.6323e-6
$H^{\frac{1}{2}}(\Gamma_0)$	1.6087	1.80378e-4	1.3998e-5	1.8740e-5	0.5863	3.1723e-5

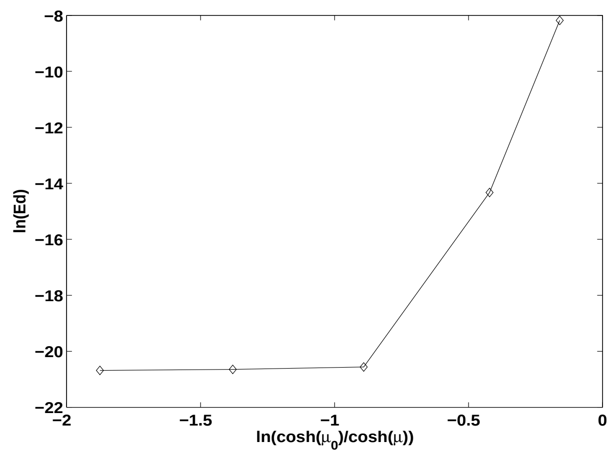

Fig. 5.2. The effect of μ for the exterior solution u.

In Table 5.3, Table 5.4, and Fig.5.2, the corresponding parameters selected are the same as those in Example 1. The numerical results and conclusions are similar to those in Example 1.

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