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TWO-SCALE FINITE ELEMENT DISCRETIZATIONS FOR PARTIAL DIFFERENTIAL EQUATIONS *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

Some two-scale finite element discretizations are introduced for a class of linear partial differential equations. Both boundary value and eigenvalue problems are studied. Based on the two-scale error resolution techniques, several two-scale finite element algorithms are proposed and analyzed. It is shown that this type of two-scale algorithms not only significantly reduces the number of degrees of freedom but also produces very accurate approximations.

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1. Introduction

It is a challenging task to solve 3−dimensional ($3d$) partial differential equations by conventional discretization methods, due to storage requirements and computational complexity. Usually, both storage requirements and running time grow tremendously when the number of degrees of freedom for approximate solutions increases. Thus, for $3d$ applications such as problems from computational materials science, computational chemistry and computational biology, the most elaborate solver routines like multigrid or multilevel methods should be applied in order to obtain numerical solutions with satisfactory accuracy. Additionally, the code should be implemented on a high-performance computer.

To reduce the computational cost, including the computational time and the storage requirement, some new two-scale finite element discretizations for solving partial differential equations in $3d$ are introduced in this paper. The main idea of our new discretizations is to use a coarse grid to approximate the low frequencies and to combine some univariate fine and coarse grids to handle the high frequencies by some parallel procedures. These discretizations are based on our understanding of the frequency resolution of a finite element solution to some elliptic problem. For a solution to an elliptic problem, it is shown that low frequency components can be approximated well on a relatively coarse grid and high frequency components can be computed on a fine grid (see, e.g., [4, 17, 25, 31]). It is also observed that for elliptic problems on tensor product domains, a part of high frequencies results from the tensor product of the univariate low frequencies, which can then be damped out by the tensor product of some fine and coarse grids.

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We now give a somewhat more detailed but informal (and hopefully informative) description of the main ideas and results in this paper. Consider an elliptic boundary value problem in domain $\Omega = (0,1)^3$. Let $P_{h_{x_1},h_{x_2},h_{x_3}}u$ be the standard trilinear finite element solution, that is, the Ritz-Galerkin approximation, of a partial differential equation on a uniform grid $T^{h_{x_1},h_{x_2},h_{x_3}}$ with mesh size h_{x_1} in x_1-direction, h_{x_2} in x_2-direction and h_{x_3} in x_3-direction, respectively. Then, a two-scale finite element approximation, which is nothing but a simple combination of different standard finite element solutions of the original problem over different scale meshes, is constructed as follows (see Section 3):

$$P^h_{H,H,H}u \equiv P_{h,H,H}u + P_{H,h,H}u + P_{H,H,h}u - 2P_{H,H,H}u,$$

where $H \gg h$.

In this two-scale approximate scheme, only partially refined meshes are involved, and the following result for a class of partial differential equations can be established (see Theorem 3.1)

$$\|u - P^h_{H,H,H}u\|_{1,\Omega} = O(h + H^2), \tag{1.1}$$

where u is the exact solution of the partial differential equation.

This is a very satisfactory result in many ways. Consequently, for example, we obtain an asymptotically optimal approximation $P^h_{H,H,H}u$ in parallel by taking $H = O(\sqrt{h})$ and the number of degrees of freedom for obtaining $P^h_{H,H,H}u$ is only of $O(h^{-2})$, while that for the standard finite element solution $P_{h,h,h}u$ with the same approximate accuracy is of $O(h^{-3})$.

We may also design efficient two-scale approximate schemes for other problems. For instance, consider the following eigenvalue problem posed on Ω:

$$\begin{cases} -\nabla(a\nabla u) &= \lambda u, \text{ in } \quad \Omega, \\ \qquad\quad u &= 0, \text{ on } \quad \partial\Omega, \end{cases} \tag{1.2}$$

where a is a positive smooth function on $\bar{\Omega}$. We may employ the following algorithm to approximate (1.2) (see Section 4):

1. Solve (1.2) on a coarse grid: find $(u_{H,H,H}, \lambda_{H,H,H}) \in S_0^{H,H,H}(\Omega) \times R^1$ such that $\int_\Omega a|\nabla u_{H,H,H}|^2 = 1$ and
$$\int_\Omega a\nabla u_{H,H,H}\nabla v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{H,H,H}(\Omega). \tag{1.3}$$
2. Compute the linear boundary value problems on partially fine grids in parallel:
find $u^{h,H,H} \in S_0^{h,H,H}(\Omega)$ such that
$$\int_\Omega a\nabla u^{h,H,H}\nabla v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{h,H,H}(\Omega);$$
find $u^{H,h,H} \in S_0^{H,h,H}(\Omega)$ such that
$$\int_\Omega a\nabla u^{H,h,H}\nabla v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{H,h,H}(\Omega);$$
find $u^{H,H,h} \in S_0^{H,H,h}(\Omega)$ such that
$$\int_\Omega a\nabla u^{H,H,h}v = \lambda_{H,H,H}\int_\Omega u_{H,H,H}v, \quad \forall v \in S_0^{H,H,h}(\Omega).$$
3. Set
$$u^h_{H,H,H} = u^{h,H,H} + u^{H,h,H} + u^{H,H,h} - 2u_{H,H,H}$$
and
$$\lambda^h_{H,H,H} = \frac{\int_\Omega a|\nabla u^h_{H,H,H}|^2}{\int_\Omega |u^h_{H,H,H}|^2},$$

where $S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega)$ is the standard trilinear finite element space associated with $T^{h_{x_1},h_{x_2},h_{x_3}}$.

If, for example, $\lambda_{H,H,H}$ is the first eigenvalue of (1.3) at the first step, then we can establish the following results (see Theorem 4.1)

$$\left(\int_\Omega a|\nabla(u-u^h_{H,H,H})|^2\right)^{1/2}=O(h+H^2) \text{ and } |\lambda-\lambda^h_{H,H,H}|=O(h^2+H^4).$$

These estimates mean that we can obtain asymptotically optimal approximations by taking $H=O(\sqrt{h})$. Note that what need to be solved at the second step are linear boundary value problems on partially fine grids only!

Our two-scale finite element discretization method is related to the sparse grid method developed by Zenger [32], where the multi-level basis of Yserentant [30] was used. Zenger's sparse grid method is proposed for solving partial differential equations and has been known for many years in interpolation, approximation, recovery theory and numerical quadrature under the different names "hyperbolic crosses" [1], "Boolean methods" [10], and "discrete blending" [5, 14, 18]. A general framework for approximating tensor product problems has been presented in Smolyak [24]. Zenger's sparse grid method has turned out to be a powerful approach for satisfactory numerical solutions, see, e.g., [6, 7, 13, 15, 21, 22, 23]. Similar to the sparse grid method, multi-level bases are also used in wavelets (see, e.g.,[6, 9, 11] and references cited therein). Instead of the multi-level basis approach, in this paper, we adopt a two-level basis approach. It is shown that the two-level basis approach is more flexible than the multi-level basis approach (c.f. [21]), which is a key for us to introduce the multiscale techniques to eigenvalue problems and nonlinear equations. Moreover, since the two-scale finite element approximations are computed on regular meshes, existing solvers can be used without any need for an explicit discretization on a sparse grid. Our multiscale finite element approach is also different from other multiscale/upscaling methods in the literature. The multiscale/upscaling methods are proposed for the homogenization of multiscale problems (see, e.g., [12, 19] and references cited therein) while ours is set for the discretization of partial differential equations.

Our approach turns out to be advantageous in two respects. First, the possibility of using existing codes allows the straightforward application of two-scale combination discretization to large scale problems. Second, since the different subproblems can be solved fully in parallel, there is a very elegant and efficient inherent coarse-grain parallelism that makes the two-scale combination discretization perfectly suitable for modern high-performance computers. Our technical tools for analyzing two-scale finite element approximations are some superconvergence techniques developed in [21, 23].

The remainder of this paper is organized as follows. In the coming section, basic notation and assumptions are described, and the two-scale interpolations are introduced. In section 3, some two-scale finite element discretizations are proposed and analyzed for $3d$ linear elliptic boundary value partial differential equations. These two-scale approaches are then generalized to a class of elliptic eigenvalue problems in section 4. In section 5, several numerical experiments, which support our theory, are reported. Finally in section 6, some further remarks are presented.

2. Preliminaries

Let $\Omega=(0,1)^3$. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms, see, e.g., [8]. For $p=2$, we denote $H^s(\Omega)=W^{s,2}(\Omega)$ and $H^1_0(\Omega)=\{v\in H^1(\Omega): v\,|_{\partial\Omega}=0\}$, where $v\,|_{\partial\Omega}=0$ is in the sense of trace, $\|\cdot\|_{s,\Omega}=\|\cdot\|_{s,2,\Omega}$ and $\|\cdot\|_\Omega=\|\cdot\|_{0,2,\Omega}$. The space $H^{-1}(\Omega)$, the dual of $H^1_0(\Omega)$, will also be used.

Throughout this paper, we shall use the letter C (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences. For convenience, the symbols $\lesssim, \gtrsim$ and $\eqsim$ will be used in this paper. That $x_1\lesssim y_1, x_2\gtrsim y_2$ and $x_3\eqsim y_3$, mean that $x_1\le C_1y_1, \quad x_2\ge c_2y_2$ and $c_3x_3\le y_3\le C_3x_3$ for some constants C_1,c_2,c_3

and C_3 that are independent of mesh parameters.

To get error estimations, some so-called mixed Sobolev spaces are introduced as follows (c.f. [23]):

$$W_{\infty}^{G,2}(\Omega) = \{w \in W^{1,\infty}(\Omega) : \ \partial_{x_i}\partial_{x_j}(w) \in L^{\infty}(\Omega), \ x_i \neq x_j, \ i,j = 1,2,3\},$$

$$W_2^{G,3}(\Omega) = \{w \in H^2(\Omega) : \ \partial_{x_i}\partial_{x_j}\partial_{x_k}(w) \in L^2(\Omega), \ x_i \neq x_j \text{ or } x_i \neq x_k, \ i,j,k = 1,2,3\}$$

and

$$W_2^{K,4}(\Omega) = \{w \in H^3(\Omega) : \ \partial_{x_i}^2\partial_{x_j}\partial_{x_k}(w) \in L^2(\Omega), \ x_i \neq x_j \text{ or } x_i \neq x_k, \ i,j,k = 1,2,3\}$$

with their natural norms $\|\cdot\|_{W_{\infty}^{G,2}}$, $\|\cdot\|_{W_2^{G,3}}$, and $\|\cdot\|_{W_2^{K,4}}$, respectively.

Assume that T^h is a uniform mesh with mesh size h on $[0,1]$ and $S^h[0,1] \subset H^1[0,1]$ is the associated piecewise linear finite element space. Set $S_0^h[0,1] = S^h[0,1] \cap H_0^1[0,1]$, $T^{h_{x_1},h_{x_2},h_{x_3}} = T^{h_{x_1}} \times T^{h_{x_2}} \times T^{h_{x_3}}$, $S^{h_{x_1},h_{x_2},h_{x_3}}(\Omega) = S^{h_{x_1}}[0,1] \times S^{h_{x_2}}[0,1] \times S^{h_{x_3}}[0,1]$, and $S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega) = S_0^{h_{x_1}}[0,1] \times S_0^{h_{x_2}}[0,1] \times S_0^{h_{x_3}}[0,1]$.

Let $I_h : C[0,1] \to S^h[0,1]$ be the standard Lagrangian interpolation operator defined on T^h and $I_{h_{x_1},h_{x_2},h_{x_3}}$ be the usual trilinear interpolation operator on partition $T^{h_{x_1},h_{x_2},h_{x_3}}$. One sees that $I_{h_{x_1},0,0}$ is the interpolation operator which interpolates only in x_1-direction on lines of mesh size h_{x_1}, etc. Obviously,

$$I_{h_{x_1},h_{x_2},h_{x_3}} = I_{h_{x_1},0,0} \cdot I_{0,h_{x_2},0} \cdot I_{0,0,h_{x_3}}.$$

We shall introduce two-scale interpolations on $3d$. Given $H \in (0,1)$ with $H \gg h$, and define a two-scale interpolation by

$$I_{H,H,H}^h u = I_{h,H,H}u + I_{H,h,H}u + I_{H,H,h}u - 2I_{H,H,H}u.$$

It is shown in the following theorem that a one-scale interpolation on a fine grid can be obtained by some combination of multi-scale interpolations asymptotically. It should be pointed out that a part of the following Theorem 2.1 has been provided in [18].

Theorem 2.1. *There hold*

$$H\|I_{H,H,H}^h u - I_{h,h,h}u\|_{1,\Omega} + \|I_{H,H,H}^h u - I_{h,h,h}u\|_{0,\Omega} \lesssim H^3\|u\|_{W_2^{G,3}}, \ \text{if } u \in W^{G,3}(\Omega), \quad (2.1)$$

$$H\|I_{H,H,H}^h u - I_{h,h,h}u\|_{1,\Omega} + \|I_{H,H,H}^h u - I_{h,h,h}u\|_{0,\Omega} \lesssim H^4\|u\|_{W_2^{K,4}}, \ \text{if } u \in W^{K,4}(\Omega). \quad (2.2)$$

Proof. It can be verified by a direct calculation that

$$\begin{aligned} I_{h,h,h} \ = \ & I_{H,H,H} + I_{H,H,0}(I_{0,0,h} - I_{0,0,H}) + I_{H,0,H}(I_{0,h,0} - I_{0,H,0}) \\ & + I_{0,H,H}(I_{h,0,0} - I_{H,0,0}) + I_{H,0,0}(I_{0,h,0} - I_{0,H,0})(I_{0,0,h} - I_{0,0,H}) \\ & + I_{0,H,0}(I_{h,0,0} - I_{H,0,0})(I_{0,0,h} - I_{0,0,H}) + I_{0,0,H}(I_{h,0,0} - I_{H,0,0})(I_{0,h,0} - I_{0,H,0}) \\ & + (I_{h,0,0} - I_{H,0,0})(I_{0,h,0} - I_{0,H,0})(I_{0,0,h} - I_{0,0,H}) \end{aligned}$$

and

$$\begin{aligned} I_{H,H,H}^h \ = \ & I_{H,H,H} + I_{H,H,0}(I_{0,0,h} - I_{0,0,H}) + I_{H,0,H}(I_{0,h,0} - I_{0,H,0}) \\ & + I_{0,H,H}(I_{h,0,0} - I_{H,0,0}). \end{aligned}$$

Hence for $\|\cdot\| = \|\cdot\|_{1,\Omega}$ or $\|\cdot\| = \|\cdot\|_{0,\Omega}$, we have

$$\begin{aligned} & \|I_{H,H,H}^h u - I_{h,h,h}u\| \\ \lesssim \ & \|I_{H,0,0}(I_{0,h,0} - I_{0,H,0})(I_{0,0,h} - I_{0,0,H})u\| + \|I_{0,H,0}(I_{h,0,0} - I_{H,0,0})(I_{0,0,h} - I_{0,0,H})u\| \\ & + \|I_{0,0,H}(I_{h,0,0} - I_{H,0,0})(I_{0,h,0} - I_{0,H,0})u\| \\ & + \|(I_{h,0,0} - I_{H,0,0})(I_{0,h,0} - I_{0,H,0})(I_{0,0,h} - I_{0,0,H})u\|, \end{aligned}$$

from which we immediately obtain (2.1). If $u \in W_2^{K,4}(\Omega)$, then

$$\|I^h_{H,H,H}u - I_{h,h,h}u\|_{0,\Omega} \lesssim H^4\|u\|_{W_2^{K,4}},$$

$$\begin{aligned}
&\|\partial_{x_1}(I^h_{H,H,H}u - I_{h,h,h}u)\|_{0,\Omega}\\
\lesssim\ & H^3(\|\partial_{x_1}\partial_{x_2}\partial^2_{x_3}u\|_{0,\Omega} + \|\partial^2_{x_1}\partial^2_{x_3}u\|_{0,\Omega} + \|\partial^2_{x_2}\partial^2_{x_1}u\|_{0,\Omega} + \|\partial^2_{x_1}\partial_{x_2}\partial_{x_3}u\|_{0,\Omega})\\
\lesssim\ & H^3\|u\|_{W_2^{K,4}}
\end{aligned}$$

and the similar estimations for $\|\partial_{x_2}(I^h_{H,H,H}u - I_{h,h,h}u)\|_{0,\Omega}$ and $\|\partial_{x_3}(I^h_{H,H,H}u - I_{h,h,h}u)\|_{0,\Omega}$ can be obtained. This completes the proof.

It is observed from the estimation of $I^h_{H,H,H}u - I_{h,h,h}u$ that the two-scale interpolation $I^h_{H,H,H}u$, which is a simple combination of four standard interpolations over different scale meshes, is a more economic approximation to u than $I_{h,h,h}u$ in terms of computational cost. Indeed, the approximation accuracy of the two-scale interpolation $I^h_{H,H,H}u$ is the same as that of the standard interpolation $I_{h,h,h}u$ when $H = O(h^{1/2})$ is chosen while the number degrees of freedom of $I^h_{H,H,H}u$ is only of $O(h^{-2})$ and that of $I_{h,h,h}u$ is of $O(h^{-3})$.

3. Boundary Value Problems

In this section, we shall design and analyze some two-scale finite element discretizations for a class of linear elliptic boundary value problems. Consider a homogeneous boundary value problem

$$\left\{\begin{array}{rll} Lu &= f, & \text{in} \quad \Omega,\\ u &= 0, & \text{on} \quad \partial\Omega. \end{array}\right. \tag{3.1}$$

Here L is a general linear second order elliptic operator:

$$Lu = -\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}(a_{ij}\frac{\partial u}{\partial x_i}) + \sum_{i=1}^{3} b_i\frac{\partial u}{\partial x_i} + cu$$

satisfying $a_{ij} \in W^{1,\infty}(\Omega), b_i, c \in L^\infty(\Omega)$, and (a_{ij}) is uniformly positive definite on $\bar{\Omega}$.

The weak form of (3.1) is as follows: find $u \equiv L^{-1}f \in H^1_0(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H^1_0(\Omega), \tag{3.2}$$

where

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{3} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{i=1}^{3} b_i\frac{\partial u}{\partial x_i}v + cuv,$$

$$(f,v) = \int_\Omega fv.$$

Note that for $a_0(\cdot,\cdot)$ defined by

$$a_0(u,v) = \sum_{i,j=1}^{3}\int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}, \tag{3.3}$$

we have

$$\|w\|^2_{1,\Omega} \lesssim a_0(w,w), \quad \forall w \in H^1_0(\Omega),$$

and

$$a_0(u,v) \lesssim \|u\|_{1,\Omega}\|v\|_{1,\Omega}, \quad |a(u,v) - a_0(u,v)| \lesssim \|u\|_{0,\Omega}\|v\|_{1,\Omega}, \quad \forall u,v \in H^1_0(\Omega).$$

Our basic assumption is that (3.2) is well-posed, namely (3.2) is uniquely solvable for any $f \in H^{-1}(\Omega)$. (A simple sufficient condition for this assumption to be satisfied is that $c \geq 0$.) An application of the open-mapping theorem yields

$$\|w\|_{1,\Omega} \lesssim \|Lw\|_{-1,\Omega}, \quad \forall w \in H^1_0(\Omega). \tag{3.4}$$

A sufficient and necessary condition for the well-posedness of (3.2) is

$$\|w\|_{1,\Omega} \lesssim \sup_{\phi\in H_0^1(\Omega)} \frac{a(w,\phi)}{\|\phi\|_{1,\Omega}}, \quad \forall w\in H_0^1(\Omega) \tag{3.5}$$

and

$$\|w\|_{1,\Omega} \lesssim \sup_{\phi\in H_0^1(\Omega)} \frac{a(\phi,w)}{\|\phi\|_{1,\Omega}}, \quad \forall w\in H_0^1(\Omega). \tag{3.6}$$

We have (c.f. [16]) the following estimate for the regularity of the solution of (3.1) or (3.2)

$$\|u\|_{2,\Omega} \lesssim \|f\|_{0,\Omega},\ \forall f\in L^2(\Omega). \tag{3.7}$$

The standard Galerkin projection $P_{h_{x_1},h_{x_2},h_{x_3}}: H_0^1(\Omega)\mapsto S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega)$ is defined by

$$a(u-P_{h_{x_1},h_{x_2},h_{x_3}}u,v)=0,\quad \forall v\in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega), \tag{3.8}$$

which is well-posed when $\max\{h_{x_1},h_{x_2},h_{x_3}\}\ll 1$ (c.f. [25]). Here and hereafter, we assume that any mesh size involved is small enough so that the associated discrete problem is well-posed.

We provide different assumptions to the exact solution u and the coefficients a_{ij}, b_i and c so that different convergence results can be obtained.

Assumption **A**: $u\in H_0^1(\Omega)\cap W_2^{G,3}(\Omega)$, $a_{ij}\in W^{1,\infty}(\Omega)$, $b_i\in L^\infty(\Omega)$, and $c\in L^\infty(\Omega)(i,j=1,2,3)$.

Assumption **B**: $u\in H_0^1(\Omega)\cap W_2^{G,3}(\Omega)$, $a_{ii}\in W^{1,\infty}(\Omega)$, $a_{ij}\in W_\infty^{G,2}(\Omega)(i\neq j)$, $b_i\in L^\infty(\Omega)$, $\partial_{x_i}b_i\in L^\infty(\Omega)$, and $c\in L^\infty(\Omega)(i,j=1,2,3)$.

Assumption **C**: $u\in H_0^1(\Omega)\cap W_2^{K,4}(\Omega)$, $a_{ii}\in W^{2,\infty}(\Omega)$, $a_{ij}\in W_\infty^{G,2}(\Omega)(i\neq j)$, $b_i\in L^\infty(\Omega)$, $\partial_{x_i}b_i\in L^\infty(\Omega)$, and $c\in L^\infty(\Omega)(i,j=1,2,3)$.

Using the arguments in [22] and [23], we can generalize the results concerning the semi-discrete solutions of $2d$ problems in [22] to $3d$ cases, which are stated as follows:

Proposition 3.1. *(1) Let $g\in L^2(\Omega)$. If $q\in S_0^{h_{x_1},h_{x_2},0}(\Omega)$ satisfies*

$$a(v,q)=\int_\Omega vg,\quad \forall v\in S_0^{h_{x_1},h_{x_2},0}(\Omega), \tag{3.9}$$

then

$$\|\partial_{x_3}q\|_{1,\Omega}\lesssim \|g\|_{0,\Omega}. \tag{3.10}$$

A similar result holds when $S_0^{h_{x_1},h_{x_2},0}(\Omega)$ is replaced by $S_0^{h_{x_1},0,h_{x_3}}(\Omega)$ or $S_0^{0,h_{x_2},h_{x_3}}(\Omega)$.
(2)If $w\in H_0^1(\Omega)$ and $\partial_{x_3}w\in H^1(\Omega)$, then

$$\|\partial_{x_3}P_{h_{x_1},h_{x_2},0}w\|_{1,\Omega}\lesssim \|\partial_{x_3}w\|_{1,\Omega}. \tag{3.11}$$

Similar estimates hold for $\partial_{x_1}P_{0,h_{x_2},h_{x_3}}w$ and $\partial_{x_2}P_{h_{x_1},0,h_{x_3}}w$.

To analyze the convergence of the multiscale finite element approximations, we need a number of propositions. For $P_{h,h_{x_2},h_{x_3}}w\in S_0^{h,h_{x_2},h_{x_3}}(\Omega)$, $P_{h_{x_1},h,h_{x_3}}w\in S_0^{h_{x_1},h,h_{x_3}}(\Omega)$, $P_{h_{x_1},h_{x_2},h}w\in S_0^{h_{x_1},h_{x_2},h}(\Omega)$, set

$$\begin{aligned}
\delta_H^{x_1}P_{h,h_{x_2},h_{x_3}}w &= P_{h,h_{x_2},h_{x_3}}w-P_{H,h_{x_2},h_{x_3}}w,\\
\delta_H^{x_2}P_{h_{x_1},h,h_{x_3}}w &= P_{h_{x_1},h,h_{x_3}}w-P_{h_{x_1},H,h_{x_3}}w,\\
\delta_H^{x_3}P_{h_{x_1},h_{x_2},h}w &= P_{h_{x_1},h_{x_2},h}w-P_{h_{x_1},h_{x_2},H}w,
\end{aligned}$$

where $H,h,h_{x_1},h_{x_2},h_{x_3}\in(0,1)$.

Proposition 3.2. *(1)If $w\in H_0^1(\Omega)$, then*

$$\|\delta_H^{x_3}P_{h_{x_1},h_{x_2},h}w\|_{0,\Omega}\lesssim H\|w\|_{1,\Omega} \tag{3.12}$$

and similar results hold for $\delta_H^{x_1}P_{h,h_{x_2},h_{x_3}}w$ and $\delta_H^{x_2}P_{h_{x_1},h,h_{x_3}}w$.
(2)If $w\in H_0^1(\Omega)$ and $\partial_{x_3}w\in H^1(\Omega)$, then

$$H\|\delta_H^{x_3}P_{h_{x_1},h_{x_2},h}w\|_{1,\Omega}+\|\delta_H^{x_3}P_{h_{x_1},h_{x_2},h}w\|_{0,\Omega}\lesssim H^2\|\partial_{x_3}w\|_{1,\Omega} \tag{3.13}$$

and similar results hold for $\delta_H^{x_1}P_{h,h_{x_2},h_{x_3}}w$ and $\delta_H^{x_2}P_{h_{x_1},h,h_{x_3}}w$.

Proof. It suffices to prove the case of $\delta_H^{x_3} P_{h_{x_1},h_{x_2},h} w$. Note that for $\|\cdot\| = \|\cdot\|_{0,\Omega}$ or $\|\cdot\|_{1,\Omega}$,

$$\|\delta_H^{x_3} P_{h_{x_1},h_{x_2},h} w\| \lesssim \|P_{h_{x_1},h_{x_2},h} w - P_{h_{x_1},h_{x_2},0} w\| + \|P_{h_{x_1},h_{x_2},0} w - P_{h_{x_1},h_{x_2},H} w\|. \quad (3.14)$$

We shall estimate $\|P_{h_{x_1},h_{x_2},h} w - P_{h_{x_1},h_{x_2},0} w\|$ and $\|P_{h_{x_1},h_{x_2},0} w - P_{h_{x_1},h_{x_2},H} w\|$. Let $g_k \in S_0^{h_{x_1},h_{x_2},0}(\Omega)$ be the solution of

$$a(v, g_k) = \int_\Omega v(P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w), \qquad \forall v \in S_0^{h_{x_1},h_{x_2},0}(\Omega)$$

for $k = h$ or H. Set $v = P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w$ in the above equation, and we get

$$\begin{aligned} \|P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w\|_{0,\Omega}^2 &= a(P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w, g_k) \\ &= a(P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w, g_k - I_{0,0,k} g_k) \\ &\lesssim \|P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w\|_{1,\Omega} k \|\partial_{x_3} g_k\|_{1,\Omega}. \end{aligned}$$

Note the regularity of the semi-discrete solution stated in Proposition 3.1 implies

$$\|\partial_{x_3} g_k\|_{1,\Omega} \lesssim \|P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w\|_{0,\Omega}.$$

Hence, we have

$$\|P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w\|_{0,\Omega} \lesssim k \|P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w\|_{1,\Omega} \lesssim k\|w\|_{1,\Omega}. \quad (3.15)$$

Combining (3.14) and (3.15), we get (3.12).

If $w \in H_0^1(\Omega)$ and $\partial_{x_3} w \in H^1(\Omega)$, then Proposition 3.1 and Cea's Lemma give

$$\begin{aligned} \|P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w\|_{1,\Omega} &= \|P_{h_{x_1},h_{x_2},k} P_{h_{x_1},h_{x_2},0} w - P_{h_{x_1},h_{x_2},0} w\|_{1,\Omega} \\ \lesssim k\|\partial_{x_3} P_{h_{x_1},h_{x_2},0} w\|_{1,\Omega} &\lesssim k\|\partial_{x_3} w\|_{1,\Omega}, \end{aligned} \quad (3.16)$$

which together with (3.15) produces

$$\|P_{h_{x_1},h_{x_2},k} w - P_{h_{x_1},h_{x_2},0} w\|_{0,\Omega} \lesssim k^2 \|\partial_{x_3} w\|_{1,\Omega}. \quad (3.17)$$

Finally, we obtain (3.13) from (3.16) and (3.17). This completes the proof.

Proposition 3.3. *(1) If Assumption* **A** *holds, then there exists* $\alpha_{h_{x_1}} \in H_0^1(\Omega) \bigcap H^2(\Omega)$ *such that*

$$P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})u) = P_{h_{x_1},h_{x_2},h_{x_3}}(\alpha_{h_{x_1}}),$$
$$\|\alpha_{h_{x_1}}\|_{2,\Omega} \lesssim h_{x_1} \|u\|_{W_2^{G,3}}.$$

(2) If Assumption **B** *holds, then there exists* $\alpha_{h_{x_1}} \in H_0^1(\Omega)$ *such that*

$$P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})u) = P_{h_{x_1},h_{x_2},h_{x_3}}(\alpha_{h_{x_1}}),$$
$$\|\alpha_{h_{x_1}}\|_{1,\Omega} \lesssim h_{x_1}^2 \|u\|_{W_2^{G,3}}.$$

(3) If Assumption **C** *holds, then there exists* $\alpha_{h_{x_1}} \in H_0^1(\Omega) \bigcap H^2(\Omega)$ *such that*

$$P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})u) = P_{h_{x_1},h_{x_2},h_{x_3}}(\alpha_{h_{x_1}}),$$
$$\|\alpha_{h_{x_1}}\|_{2,\Omega} \lesssim h_{x_1}^2 \|u\|_{W_2^{K,4}}.$$

Proof. Integration by parts gives that $\forall v \in S_0^{h_{x_1},0,0}(\Omega)$, there holds

$$\begin{aligned} & a((I - I_{h_{x_1},0,0})u, v) \\ = & \int_\Omega \left(-\partial_{x_1} a_{11} (I - I_{h_{x_1},0,0}) u \partial_{x_1} v + \sum_{i=2}^3 a_{i1} (I - I_{h_{x_1},0,0}) \partial_{x_i} u \partial_{x_1} v \right) \\ & - \int_\Omega \sum_{j=2}^3 \left(\partial_{x_j} a_{1j} \partial_{x_1} (I - I_{h_{x_1},0,0}) uv + a_{1j} \partial_{x_1} (I - I_{h_{x_1},0,0}) \partial_{x_j} uv \right) \\ & - \int_\Omega \sum_{i=2}^3 \sum_{j=2}^3 \left(\partial_{x_j} a_{ij} (I - I_{h_{x_1},0,0}) \partial_{x_i} uv + a_{ij} (I - I_{h_{x_1},0,0}) \partial_{x_i} \partial_{x_j} uv \right) \\ & + \int_\Omega \left(b_1 \partial_{x_1} (I - I_{h_{x_1},0,0}) uv + \sum_{i=2}^3 b_i (I - I_{h_{x_1},0,0}) \partial_{x_i} uv + c(I - I_{h_{x_1},0,0}) uv \right). \end{aligned}$$

Thus, there exists $f_{h_{x_1}} \in L^2(\Omega)$ such that(c.f. Lemma 3 in [23])

$$a((I - I_{h_{x_1},0,0})u, v) = (f_{h_{x_1}}, v), \qquad \forall v \in S_0^{h_{x_1},0,0}(\Omega), \tag{3.18}$$

$$\|f_{h_{x_1}}\|_{0,\Omega} \lesssim h_{x_1}\|u\|_{W_2^{G,3}}. \tag{3.19}$$

Note that there exists $\alpha_{h_{x_1}} \in H_0^1(\Omega)\bigcap H^2(\Omega)$ satisfying

$$a(\alpha_{h_{x_1}}, v) = (f_{h_{x_1}}, v), \qquad \forall v \in H_0^1(\Omega), \tag{3.20}$$

$$\|\alpha_{h_{x_1}}\|_{2,\Omega} \lesssim \|f_{h_{x_1}}\|_{0,\Omega}. \tag{3.21}$$

Combining (3.18), (3.19), (3.20) and (3.21), we obtain (1). Analogously, we obtain (3).

Integration by parts again leads to

$$\int_\Omega \left(\sum_{j=2}^{3} \left(\partial_{x_j} a_{1j} \partial_{x_1}(I - I_{h_{x_1},0,0})uv + a_{1j}\partial_{x_1}(I - I_{h_{x_1},0,0})\partial_{x_j} uv\right) - b_1\partial_{x_1}(I - I_{h_{x_1},0,0})uv \right)$$

$$= \int_\Omega \left(-\sum_{j=2}^{3} \left((I - I_{h_{x_1},0,0})u\partial_{x_1}(\partial_{x_j} a_{1j} v) + (I - I_{h_{x_1},0,0})\partial_{x_j} u\partial_{x_1}(a_{1j}v)\right) + (I - I_{h_{x_1},0,0})u\partial_{x_1}(b_1 v) \right).$$

Namely,

$$a((I - I_{h_{x_1},0,0})u, v) \lesssim h_{x_1}^2\|u\|_{W_2^{G,3}}\|v\|_{1,\Omega}, \qquad \forall v \in S_0^{h_{x_1},0,0}(\Omega).$$

Hence, there exists $\alpha_{h_{x_1}} \in H_0^1(\Omega)$ satisfying (2). This completes the proof.

Proposition 3.4. *(1) Assumption* **A** *implies*

$$\|\delta_H^{x_3} P_{h_{x_1},h_{x_2},h}(I - I_{h_{x_1},0,0})u\|_{1,\Omega} \lesssim H^2\|u\|_{W_2^{G,3}}.$$

(2) Assumption **B** *implies*

$$\|\delta_H^{x_3} P_{h_{x_1},h_{x_2},h}(I - I_{h_{x_1},0,0})u\|_{0,\Omega} \lesssim H^3\|u\|_{W_2^{G,3}}.$$

(3) Assumption **C** *implies*

$$H\|\delta_H^{x_3} P_{h_{x_1},h_{x_2},h}(I - I_{h_{x_1},0,0})u\|_{1,\Omega} + \|\delta_H^{x_3} P_{h_{x_1},h_{x_2},h}(I - I_{h_{x_1},0,0})u\|_{0,\Omega} \lesssim H^4\|u\|_{W_2^{K,4}}.$$

Similar results hold for $\delta_H^{x_1} P_{h,h_{x_2},h_{x_3}} w$ *and* $\delta_H^{x_2} P_{h_{x_1},h,h_{x_3}} w$.

Proof. By Propositions 3.2 and 3.3,

$$\begin{aligned} \|\delta_H^{x_3} P_{h_{x_1},h_{x_2},h}(I - I_{h_{x_1},0,0})u\|_{1,\Omega} &= \|\delta_H^{x_3} P_{h_{x_1},h_{x_2},h}(\alpha_{h_{x_1}})\|_{1,\Omega} \\ &\lesssim H\|\partial_{x_3}(\alpha_{h_{x_1}})\|_{1,\Omega} \lesssim H^2\|u\|_{W_2^{G,3}}. \end{aligned}$$

Thus (1) follows. Analogously, we obtain (2) and (3).

Proposition 3.5. *Let* $r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u, v) = a((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})u, v)$.

(1)Assumption **A** *gives*

$$|r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u, v)| \lesssim h_{x_1}h_{x_2}\|u\|_{W_2^{G,3}}\|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega).$$

(2) Assumption **B** *and* $a_{ij} = 0(i \neq j, i, j = 1, 2, 3)$ *give*

$$|r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u, v)| \lesssim h_{x_1}h_{x_2}\min(h_{x_1}, h_{x_2})\|u\|_{W_2^{G,3}}||v||_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega).$$

(3)Assumption **C** *and* $a_{ij} = 0(i \neq j, i, j = 1, 2, 3)$ *give*

$$|r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u, v)| \lesssim h_{x_1}^2h_{x_2}^2\|u\|_{W_2^{K,4}}\|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega).$$

Proof. Obviously, (1) is true. It is only necessary to prove (2) and (3).

Set $w = (I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})u$. Integration by parts leads to

$$r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u,v) = -\int_\Omega \left(\sum_{i=1}^{3} w\partial_{x_i} a_{ii}\partial_{x_i} v + \sum_{i,j=1,i\neq j}^{3} w\partial_{x_i}(a_{ij}\partial_{x_j} v) + \sum_{i=1}^{3} w\partial_{x_i}(b_i v) - cwv \right).$$

Hence, when $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$, we have

$$|r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u,v)| \lesssim h_{x_1} h_{x_2}^2 \|u\|_{W_2^{G,3}} |v|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega),$$
$$|r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u,v)| \lesssim h_{x_1}^2 h_{x_2} \|u\|_{W_2^{G,3}} |v|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega)$$

and

$$|r^{1,2}_{h_{x_1},h_{x_2},h_{x_3}}(u,v)| \lesssim h_{x_1}^2 h_{x_2}^2 \|u\|_{W_2^{K,4}} |v|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega),$$

which imply (2) and (3). This completes the proof.

Proposition 3.6. *(1) If Assumption* **A** *holds, then*

$$\|P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})u)\|_{1,\Omega} \lesssim h_{x_1} h_{x_2} \|u\|_{W_2^{G,3}}.$$

(2) If Assumption **B** *holds and* $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$*, then*

$$\|P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})u)\|_{1,\Omega} \lesssim h_{x_1} h_{x_2} \min(h_{x_1}, h_{x_2}) \|u\|_{W_2^{G,3}}.$$

(3) If Assumption **C** *holds and* $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$*, then*

$$\|P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})u)\|_{1,\Omega} \lesssim h_{x_1}^2 h_{x_2}^2 \|u\|_{W_2^{K,4}}.$$

Proof. The estimates follow directly from Proposition 3.5.

Using a similar argument, we can prove the following two results.

Proposition 3.7. *Let* $r_{h_{x_1},h_{x_2},h_{x_3}}(u,v) = a((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})(I - I_{0,0,h_{x_3}})u, v)$*.*

(1)If Assumption **A** *holds, then*

$$|r_{h_{x_1},h_{x_2},h_{x_3}}(u,v)| \lesssim \min(h_{x_1}h_{x_2}, h_{x_1}h_{x_3}, h_{x_2}h_{x_3}) \|u\|_{W_2^{G,3}} \|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega).$$

(2)If Assumption **B** *holds and* $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$*, then*

$$\begin{aligned} &|r_{h_{x_1},h_{x_2},h_{x_3}}(u,v)| \\ \lesssim\ & \min(h_{x_1}h_{x_2}, h_{x_1}h_{x_3}, h_{x_2}h_{x_3}) \min(h_{x_1}, h_{x_2}, h_{x_3}) \|u\|_{W_2^{G,3}} \|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega). \end{aligned}$$

(3)If Assumption **C** *holds and* $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$*, then*

$$|r_{h_{x_1},h_{x_2},h_{x_3}}(u,v)| \lesssim \min(h_{x_1}^2 h_{x_2}^2, h_{x_1}^2 h_{x_3}^2, h_{x_2}^2 h_{x_3}^2) \|u\|_{W_2^{K,4}} \|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega).$$

Proposition 3.8. *(1) If Assumption* **A** *holds, then*

$$\begin{aligned} &\|P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})(I - I_{0,0,h_{x_3}})u)\|_{1,\Omega} \\ \lesssim\ & \min(h_{x_1}h_{x_2}, h_{x_1}h_{x_3}, h_{x_2}h_{x_3}) \|u\|_{W_2^{G,3}}. \end{aligned}$$

(2) If Assumption **B** *holds and* $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$*, then*

$$\begin{aligned} &\|P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})(I - I_{0,0,h_{x_3}})u)\|_{1,\Omega} \\ \lesssim\ & \min(h_{x_1}h_{x_2}, h_{x_1}h_{x_3}, h_{x_2}h_{x_3}) \min(h_{x_1}, h_{x_2}, h_{x_3}) \|u\|_{W_2^{G,3}}. \end{aligned}$$

(3) If Assumption **C** *holds and* $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$*, then*

$$\begin{aligned} &\|P_{h_{x_1},h_{x_2},h_{x_3}}((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})(I - I_{0,0,h_{x_3}})u)\|_{1,\Omega} \\ \lesssim\ & \min(h_{x_1}^2 h_{x_2}^2, h_{x_1}^2 h_{x_3}^2, h_{x_2}^2 h_{x_3}^2) \|u\|_{W_2^{K,4}}. \end{aligned}$$

Now we are ready to present and analyze the error estimations of the two-scale finite element approximation.

Theorem 3.1. *Let $P^h_{H,H,H}u$ be the two-scale finite element approximation defined by*

$$P^h_{H,H,H}u = P_{h,H,H}u + P_{H,h,H}u + P_{H,H,h}u - 2P_{H,H,H}u.$$

(1) If Assumption **A** *holds, then*

$$\|P^h_{H,H,H}u - P_{h,h,h}u\|_{1,\Omega} \lesssim H^2\|u\|_{W_2^{G,3}}.$$

(2) If Assumption **B** *holds and $a_{ij} = 0(i \neq j, i, j = 1, 2, 3)$, then*

$$\|P^h_{H,H,H}u - P_{h,h,h}u\|_{0,\Omega} \lesssim H^3\|u\|_{W_2^{G,3}}.$$

(3) If Assumption **C** *holds and $a_{ij} = 0(i \neq j, i, j = 1, 2, 3)$, then*

$$H\|P^h_{H,H,H}u - P_{h,h,h}u\|_{1,\Omega} + \|P^h_{H,H,H}u - P_{h,h,h}u\|_{0,\Omega} \lesssim H^4\|u\|_{W_2^{K,4}}.$$

Proof. For $H \gg h$, define

$$\delta_H P_{h,h,h}w = P_{h,H,H}w + P_{H,h,H}w + P_{H,H,h}w - 2P_{H,H,H}w - P_{h,h,h}w$$

and

$$\delta_H I_{h,h,h}w = I_{h,H,H}w + I_{H,h,H}w + I_{H,H,h}w - 2I_{H,H,H}w - I_{h,h,h}w.$$

Let $\|\cdot\|$ be $\|\cdot\|_{1,\Omega}$ or $\|\cdot\|_{0,\Omega}$. From the identity

$$\begin{aligned} I \;=\;& I_{h,h,h} + (I - I_{h,0,0}) + (I - I_{0,h,0}) + (I - I_{0,0,h}) - (I - I_{h,0,0})(I - I_{0,h,0}) \\ & -(I - I_{h,0,0})(I - I_{0,0,h}) - (I - I_{0,h,0})(I - I_{0,0,h}) + (I - I_{h,0,0})(I - I_{0,h,0})(I - I_{0,0,h}) \end{aligned}$$

and the estimations

$$\begin{aligned} \|\delta_H P_{h,h,h}w\| &\leq \|P_{h,H,H}w - P_{h,H,h}w\| + \|P_{h,H,h}w - P_{h,h,h}w\| \\ &\quad + \|P_{H,h,H}w - P_{H,H,H}w\| + \|P_{H,H,h}w - P_{H,H,H}w\| \\ &= \|\delta_H^{x_3} P_{h,H,h}w\| + \|\delta_H^{x_2} P_{h,h,h}w\| + \|\delta_H^{x_2} P_{H,h,H}w\| + \|\delta_H^{x_3} P_{H,H,h}w\|, \\ \|\delta_H P_{h,h,h}w\| &\leq \|\delta_H^{x_1} P_{h,h,H}w\| + \|\delta_H^{x_3} P_{h,h,h}w\| + \|\delta_H^{x_1} P_{h,H,H}w\| + \|\delta_H^{x_3} P_{H,H,h}w\| \end{aligned}$$

and

$$\|\delta_H P_{h,h,h}w\| \leq \|\delta_H^{x_2} P_{H,h,h}w\| + \|\delta_H^{x_1} P_{h,h,h}w\| + \|\delta_H^{x_1} P_{h,H,H}w\| + \|\delta_H^{x_2} P_{H,h,H}w\|,$$

we obtain

$$\begin{aligned} \|\delta_H P_{h,h,h}u\| &\lesssim \|\delta_H P_{h,h,h} I_{h,h,h}u\| + \|\delta_H P_{h,h,h}(I - I_{h,0,0})u\| + \|\delta_H P_{h,h,h}(I - I_{0,h,0})u\| \\ &\quad + \|\delta_H P_{h,h,h}(I - I_{0,0,h})u\| + \|\delta_H P_{h,h,h}(I - I_{h,0,0})(I - I_{0,h,0})u\| \\ &\quad + \|\delta_H P_{h,h,h}(I - I_{h,0,0})(I - I_{0,0,h})u\| + \|\delta_H P_{h,h,h}(I - I_{0,h,0})(I - I_{0,0,h})u\| \\ &\quad + \|\delta_H P_{h,h,h}(I - I_{h,0,0})(I - I_{0,h,0})(I - I_{0,0,h})u\|. \end{aligned}$$

Consequently,

$$\begin{aligned} \|\delta_H P_{h,h,h}u\| &\lesssim \|\delta_H I_{h,h,h}u\| + \max_{\tilde h \in \{h,H\}} (\|\delta_H^{x_3} P_{\tilde h,H,h}(I - I_{\tilde h,0,0})u\| + \|\delta_H^{x_2} P_{\tilde h,h,\tilde h}(I - I_{\tilde h,0,0})u\|) \\ &\quad + \max_{\tilde h \in \{h,H\}} (\|\delta_H^{x_1} P_{h,\tilde h,H}(I - I_{0,\tilde h,0})u\| + \|\delta_H^{x_3} P_{\tilde h,\tilde h,h}(I - I_{0,\tilde h,0})u\|) \\ &\quad + \max_{\tilde h \in \{h,H\}} (\|\delta_H^{x_2} P_{H,h,\tilde h}(I - I_{0,0,\tilde h})u\| + \|\delta_H^{x_1} P_{h,\tilde h,\tilde h}(I - I_{0,0,\tilde h})u\|) \\ &\quad + \max_{\tilde h,\bar h,\hat h \in \{h,H\}} \|P_{\tilde h,\bar h,\hat h}(I - I_{\tilde h,0,0})(I - I_{0,\bar h,0})u\| \\ &\quad + \max_{\tilde h,\bar h,\hat h \in \{h,H\}} \|P_{\tilde h,\bar h,\hat h}(I - I_{\tilde h,0,0})(I - I_{0,0,\hat h})u\| \\ &\quad + \max_{\tilde h,\bar h,\hat h \in \{h,H\}} \|P_{\tilde h,\bar h,\hat h}(I - I_{0,\hat h,0})(I - I_{0,0,\bar h})u\| \\ &\quad + \max_{\tilde h,\bar h,\hat h \in \{h,H\}} \|P_{\tilde h,\bar h,\hat h}(I - I_{\tilde h,0,0})(I - I_{0,\hat h,0})(I - I_{0,0,\bar h})u\|. \end{aligned}$$

Applying Theorem 2.1, Proposition 3.4, Proposition 3.6 and Proposition 3.8, we finish the proof.

Similar to the two-scale interpolation on $\bar{\Omega}$, it may be concluded that the two-scale finite element approximation $P^h_{H,H,H}u$ is a much more economic approximate solution in terms of computational cost than $P_{h,h,h}u$. In fact, with the same approximate accuracy, the number of degrees of freedom for getting $P^h_{H,H,H}u$ is only of $O(h^{-2})$ when $H = O(h^{1/2})$ is chosen while that for the standard finite element solution $P_{h,h,h}u$ is of $O(h^{-3})$. In addition, it may be very important that the two-scale finite element approximation $P^h_{H,H,H}u$ can be carried out in parallel. As a result, both the computational time and the storage can be reduced.

For the nonsymmetric problem, the following parallel two-scale symmetric scheme is more efficient. In this scheme, we solve a nonsymmetric problem on a coarse grid firstly and then solve three symmetric problems on partially fine grids in parallel.

Algorithm 3.1.

1. *Solve (3.2) on a coarse grid: find* $u_{H,H,H} \in S_0^{H,H,H}(\Omega)$*:*
$$a(u_{H,H,H}, v) = (f, v), \quad \forall v \in S_0^{H,H,H}(\Omega).$$
2. *Compute symmetric problems on partially fine grids in parallel:*
find $e_{h,H,H} \in S_0^{h,H,H}(\Omega)$ *such that*
$$a_0(e_{h,H,H}, v) = (f, v) - a(u_{H,H,H}, v), \quad \forall v \in S_0^{h,H,H}(\Omega);$$
find $e_{H,h,H} \in S_0^{H,h,H}(\Omega)$ *such that*
$$a_0(e_{H,h,H}, v) = (f, v) - a(u_{H,H,H}, v), \quad \forall v \in S_0^{H,h,H}(\Omega);$$
find $e_{H,H,h} \in S_0^{H,H,h}(\Omega)$ *such that*
$$a_0(e_{H,H,h}, v) = (f, v) - a(u_{H,H,H}, v), \quad \forall v \in S_0^{H,H,h}(\Omega).$$
3. *Set* $u^h_{H,H,H} = u_{H,H,H} + e_{h,H,H} + e_{H,h,H} + e_{H,H,h}$ *in* Ω.

It is shown that Algorithm 3.1 is efficient. For instance, we have

Theorem 3.2. *If Assumption* **A** *holds and* $u^h_{H,H,H} \in S_0^{h,h,h}(\Omega)$ *is obtained by Algorithm 3.1, then*
$$\|u_{h,h,h} - u^h_{H,H,H}\|_{1,\Omega} \lesssim H^2 \|u\|_{W_2^{G,3}}.$$
Consequently,
$$\|u - u^h_{H,H,H}\|_{1,\Omega} \lesssim H^2 \|u\|_{W_2^{G,3}} + h\|u\|_{2,\Omega}.$$

Proof. Set $u^{h,H,H} = u_{H,H,H} + e_{h,H,H}$, $u^{H,h,H} = u_{H,H,H} + e_{H,h,H}$ and $u^{H,H,h} = u_{H,H,H} + e_{H,H,h}$. Then from the definition of $u^h_{H,H,H}$, we have
$$u^h_{H,H,H} = u^{h,H,H} + u^{H,h,H} + u^{H,H,h} - 2u_{H,H,H}.$$
Hence,
$$\begin{aligned} & \|u_{h,h,h} - u^h_{H,H,H}\|_{1,\Omega} \\ \lesssim\ & \|u^{h,H,H} - u_{h,H,H}\|_{1,\Omega} + \|u^{H,h,H} - u_{H,h,H}\|_{1,\Omega} + \|u^{H,H,h} - u_{H,H,h}\|_{1,\Omega} \\ & + \|u_{h,H,H} + u_{H,h,H} + u_{H,H,h} - 2u_{H,H,H} - u_{h,h,h}\|_{1,\Omega}. \end{aligned}$$
Note that Theorem 4.2 of [25] implies
$$\|u^{h,H,H} - u_{h,H,H}\|_{1,\Omega} + \|u^{H,h,H} - u_{H,h,H}\|_{1,\Omega} + \|u^{H,H,h} - u_{H,H,h}\|_{1,\Omega} \lesssim H^2 \|u\|_{2,\Omega}. \quad (3.22)$$
Thus, combining (3.22) and Theorem 3.1, we complete the proof.

4. Eigenvalue Problems

Let $a(\cdot,\cdot) = a_0(\cdot,\cdot)$ be defined by (3.3) with regularity (3.7) for the solution of (3.2). A number λ is called an eigenvalue of the form $a(\cdot,\cdot)$ relative to the form $(\cdot,\cdot)$ if there is a nonzero vector $u \in H_0^1(\Omega)$, called an associated eigenvector, satisfying

$$a(u,v) = \lambda(u,v), \qquad \forall v \in H_0^1(\Omega). \tag{4.1}$$

Define Galerkin-projections $P_h : H_0^1(\Omega) \mapsto S_0^h(\Omega)$ by

$$a(u - P_h u, v) = 0, \qquad \forall v \in S_0^h(\Omega).$$

Here and hereafter, we assume that (a_{ij}) is symmetric. It is known that (4.1) has a countable sequence of real eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$$

and the corresponding eigenvectors

$$u_1, u_2, u_3, \cdots,$$

which can be assumed to satisfy

$$a(u_i, u_j) = \lambda_j (u_i, u_j) = \delta_{ij}, i, j = 1, 2, \cdots$$

In the sequence $\{\lambda_j\}$, the λ_j's are repeated according to their geometric multiplicity.

A standard finite element scheme for (4.1) is: find a pair of (λ_h, u_h), where λ_h is a number and $0 \ne u_h \in S_0^h(\Omega)$, satisfying

$$a(u_h, v) = \lambda_h (u_h, v), \qquad \forall v \in S_0^h(\Omega), \tag{4.2}$$

and use λ_h and u_h as approximations to λ and u (as $h \to 0$), respectively. One sees that (4.2) has a finite sequence of eigenvalues

$$0 < \lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{n_h,h}, \quad n_h = \dim\, S_0^h(\Omega),$$

and the corresponding eigenvectors

$$u_{1,h}, u_{2,h}, \cdots, u_{n_h,h},$$

which can be assumed to satisfy

$$a(u_{i,h}, u_{j,h}) = \lambda_{j,h}(u_{i,h}, u_{j,h}) = \delta_{ij}.$$

It follows directly from the minimum-maximum principle (see, e.g., [3]) that

$$\lambda_i \le \lambda_{i,h}, \quad i = 1, 2, \cdots, n_h.$$

Moreover, we have the following proposition (see [2, 3]).

Proposition 4.1. *(i) For any $u_{i,h}$ of (4.2) $(i = 1, 2, \cdots, n_h)$, there is an eigenfunction u^i of (4.1) corresponding to λ_i satisfying $a(u^i, u^i) = 1$ and*

$$\|u^i - u_{i,h}\|_{1,\Omega} \le C_i h. \tag{4.3}$$

Moreover,

$$\|u^i - u_{i,h}\|_{0,\Omega} \le C_i h \|u^i - u_{i,h}\|_{1,\Omega}. \tag{4.4}$$

(ii) For an eigenvalue,

$$\lambda_i \le \lambda_{i,h} \le \lambda_i + C_i h^2. \tag{4.5}$$

Here and hereafter C_i is some constant depending on i but not depending on the mesh parameter h.

There is some superclose relationship between the Ritz-Galerkin projection of the eigenvector and the finite element approximation to the eigenvector, which was presented in [28]:

Proposition 4.2.

$$\|P_h u^i - u_{i,h}\|_{1,\Omega} \lesssim \lambda_{i,h} - \lambda_i + \lambda_i \|u^i - u_{i,h}\|_{0,\Omega}. \tag{4.6}$$

The two-scale analysis for the eigenvalues is based on the following crucial (but straightforward) property of eigenvalue and eigenvector approximation (see [3, 28]).

Proposition 4.3. *Let (λ, u) be an eigenpair of (4.1). For any $w \in H_0^1(\Omega) \setminus \{0\}$,*

$$\frac{a(w,w)}{(w,w)} - \lambda = \frac{a(w-u,w-u)}{(w,w)} - \lambda\frac{(w-u,w-u)}{(w,w)}. \tag{4.7}$$

It is noted that some two-scale discretizations, in which globally refined grids are involved, were proposed in [28, 29] to solve (4.1). In the coming discussions, we shall combine the techniques in [28, 29] with our multiscale discretization approaches discussed above to establish new two-scale discretizations for (4.1) on tensor product grids.

For clarity, we consider the approximation of any eigenvalue λ of (4.1) with its corresponding eigenvector u satisfying $a(u,u) = 1$ and Assumption **A**. We assume that $(\lambda_{h_{x_1},h_{x_2},h_{x_3}}, u_{h_{x_1},h_{x_2},h_{x_3}})$ is an associated finite element approximation to (λ, u) of (4.1) on $S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega)$, namely, there holds

$$a(u_{h_{x_1},h_{x_2},h_{x_3}}, v) = \lambda_{h_{x_1},h_{x_2},h_{x_3}}(u_{h_{x_1},h_{x_2},h_{x_3}}, v), \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega),$$
$$\lambda_{h_{x_1},h_{x_2},h_{x_3}} - \lambda + h\|u_{h_{x_1},h_{x_2},h_{x_3}} - u\|_{1,\Omega} \lesssim h^2,$$

where $a(u_{h_{x_1},h_{x_2},h_{x_3}}, u_{h_{x_1},h_{x_2},h_{x_3}}) = 1$ and $h = \max(h_{x_1}, h_{x_2}, h_{x_3}) \ll 1$.

Algorithm 4.1.

1. *Solve (4.1) on a coarse grid: find $(u_{H,H,H}, \lambda_{H,H,H}) \in S_0^{H,H,H}(\Omega) \times R^1$ such that $a(u_{H,H,H}, u_{H,H,H}) = 1$ and*
$$a(u_{H,H,H}, v) = \lambda_{H,H,H}(u_{H,H,H}, v), \quad \forall v \in S_0^{H,H,H}(\Omega).$$
2. *Compute linear boundary value problems on partially fine grids in parallel:*
find $u^{h,H,H} \in S_0^{h,H,H}(\Omega)$ such that
$$a(u^{h,H,H}, v) = \lambda_{H,H,H}(u_{H,H,H}, v), \quad \forall v \in S_0^{h,H,H}(\Omega);$$
find $u^{H,h,H} \in S_0^{H,h,H}(\Omega)$ such that
$$a(u^{H,h,H}, v) = \lambda_{H,H,H}(u_{H,H,H}, v), \quad \forall v \in S_0^{H,h,H}(\Omega);$$
find $u^{H,H,h} \in S_0^{H,H,h}(\Omega)$ such that
$$a(u^{H,H,h}, v) = \lambda_{H,H,H}(u_{H,H,H}, v), \quad \forall v \in S_0^{H,H,h}(\Omega).$$
3. *Set*
$$u_{H,H,H}^h = u^{h,H,H} + u^{H,h,H} + u^{H,H,h} - 2u_{H,H,H}$$
and
$$\lambda_{H,H,H}^h = \frac{a(u_{H,H,H}^h, u_{H,H,H}^h)}{(u_{H,H,H}^h, u_{H,H,H}^h)}.$$

Theorem 4.1. *If Assumption* **A** *holds and $(\lambda_{H,H,H}^h, u_{H,H,H}^h)$ is obtained by Algorithm 4.1, then*

$$\|u_{h,h,h} - u_{H,H,H}^h\|_{1,\Omega} \lesssim H^2 \tag{4.8}$$

and

$$|\lambda_{h,h,h} - \lambda_{H,H,H}^h| \lesssim H^4. \tag{4.9}$$

Consequently,

$$\|u - u_{H,H,H}^h\|_{1,\Omega} \lesssim H^2 + h \tag{4.10}$$

and

$$|\lambda - \lambda_{H,H,H}^h| \lesssim H^4 + h^2. \tag{4.11}$$

Proof. Because of $\|u-u_{h,h,h}\|_{1,\Omega}\lesssim h$ and the triangle inequality

$$\|u-u^h_{H,H,H}\|_{1,\Omega}\lesssim\|u^h_{H,H,H}-u_{h,h,h}\|_{1,\Omega}+\|u-u_{h,h,h}\|_{1,\Omega},$$

we shall only estimate $\|u^h_{H,H,H}-u_{h,h,h}\|_{1,\Omega}$. From the following identity

$$\begin{aligned}u^h_{H,H,H}-u_{h,h,h} &= u^{h,H,H}-P_{h,H,H}u+u^{H,h,H}-P_{H,h,H}u+u^{H,H,h}-P_{H,H,h}u\\ &\quad -2(u_{H,H,H}-P_{H,H,H}u)-(u_{h,h,h}-P_{h,h,h}u)\\ &\quad +P_{h,H,H}u+P_{H,h,H}u+P_{H,H,h}u-2P_{H,H,H}u-P_{h,h,h}u,\end{aligned}$$

we have

$$\begin{aligned}&\|u^h_{H,H,H}-u_{h,h,h}\|_{1,\Omega}\\ \lesssim\ &\|u^{h,H,H}-P_{h,H,H}u\|_{1,\Omega}+\|u^{H,h,H}-P_{H,h,H}u\|_{1,\Omega}+\|u^{H,H,h}-P_{H,H,h}u\|_{1,\Omega}\\ &+2\|u_{H,H,H}-P_{H,H,H}u\|_{1,\Omega}+\|u_{h,h,h}-P_{h,h,h}u\|_{1,\Omega}\\ &+\|P_{h,H,H}u+P_{H,h,H}u+P_{H,H,h}u-2P_{H,H,H}u-P_{h,h,h}u\|_{1,\Omega}.\end{aligned}\tag{4.12}$$

Note that there hold

$$a(P_{h,H,H}u-u^{h,H,H},v)=(\lambda-\lambda_{H,H,H})(u_{H,H,H},v)+\lambda(u-u_{H,H,H},v),\ \forall v\in S_0^{h,H,H}(\Omega),$$
$$a(P_{H,h,H}u-u^{H,h,H},v)=(\lambda-\lambda_{H,H,H})(u_{H,H,H},v)+\lambda(u-u_{H,H,H},v),\ \forall v\in S_0^{H,h,H}(\Omega),$$
$$a(P_{H,H,h}u-u^{H,H,h},v)=(\lambda-\lambda_{H,H,H})(u_{H,H,H},v)+\lambda(u-u_{H,H,H},v),\ \forall v\in S_0^{H,H,h}(\Omega),$$

so we obtain

$$\begin{aligned}&\|P_{h,H,H}u-u^{h,H,H}\|_{1,\Omega}+\|P_{H,h,H}u-u^{H,h,H}u\|_{1,\Omega}+\|P_{H,H,h}u-u^{H,H,h}u\|_{1,\Omega}\\ \lesssim\ &\lambda_{H,H,H}-\lambda+\lambda\|u-u_{H,H,H}\|_{0,\Omega}\lesssim H^2.\end{aligned}\tag{4.13}$$

Obviously, Propositions 4.1 and 4.2 imply

$$\|u_{k,k,k}-P_{k,k,k}u\|_{1,\Omega}\lesssim H^2\tag{4.14}$$

for $k=h$ or H and Theorem 3.1 leads to

$$\|P_{h,H,H}u+P_{H,h,H}u+P_{H,H,h}u-2P_{H,H,H}u-P_{h,h,h}u\|_{1,\Omega}\lesssim H^2.\tag{4.15}$$

Thus, we get (4.10) by combining (4.12), (4.13), (4.14) and (4.15). And (4.11) follows from (4.10) and Proposition 4.3. This completes the proof.

Remark. By using the arguments in [20] and the results obtained above, we have that, for the two-scale finite element approximations

$$\tilde u^h_{H,H,H}=u_{h,H,H}+u_{H,h,H}+u_{H,H,h}-2u_{H,H,H},$$
$$\tilde\lambda^h_{H,H,H}=\lambda_{h,H,H}+\lambda_{H,h,H}+\lambda_{H,H,h}-2\lambda_{H,H,H},$$

there hold

$$\|u_{h,h,h}-\tilde u^h_{H,H,H}\|_{1,\Omega}\lesssim H^2\tag{4.16}$$

and

$$|\lambda_{h,h,h}-\tilde\lambda^h_{H,H,H}|\lesssim H^4\tag{4.17}$$

if some assumption is provided.

5. Numerical Experiments

We have presented and analyzed several two-scale finite element discretizations in Section 3 and Section 4. In this section, we shall report some numerical experiments that illustrate the features of our approaches. The numerical experiments were carried out by SGI Origin 3800 in the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences.

We use five piecewise trilinear finite elements with the mesh sizes $h \times H \times H$, $H \times h \times H$, $H \times H \times h$, $H \times H \times H$ and $h \times h \times h$, respectively. In the first two examples, the two-scale finite element approximation is defined by

$$P^h_{H,H,H}u = P_{h,H,H}u + P_{H,h,H}u + P_{H,H,h}u - 2P_{H,H,H}u.$$

In all of our numerical experiments, we choose $h = H^2$.

Note that the finite element meshes involved in these two-scale discretizations are anisotropic (but well-structured). For reasons of efficiency, in general, an appropriate multigrid solver should be used although the implementation of the discretizations can be based on any "lack box solver". In our numerical experiments, however, it is enough to apply the conjugate gradient method to produce pretty good results.

Example 1. Consider a linear problem of three-dimensional case:

$$\begin{cases} -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}(x_i^i \frac{\partial u}{\partial x_i}) &= f, \text{ in } \quad \Omega = (1,3) \times (1,2) \times (1,2), \\ u &= 0, \text{ on } \partial\Omega \end{cases} \tag{5.1}$$

with the exact solution $u = (1-x)^2(3-x)\sin y(1-y)(2-y)e^z(1-z)(2-z)$.

The numerical results, presented in Tables 1 and 2, support our theory (Theorem 3.1). Because of the memory limit, we break our rule of doubling the number of unknowns in the last two rows of the tables. The number of degrees of freedom for obtaining $P_{h,h,h}u$ in Example 1 is equal to, for instance, $2 \times 256 \times 256 \times 256 = 33,554,432 \approx 3.4 \times 10^7$ when $h = 1/256$. It is so large that it is difficult to carry out the computation. However, it is still relatively easy to compute $P^h_{H,H,H}u$ when $h = /256$ since the corresponding number of degrees of freedom is only $2 \times 256 \times 16 \times 16 = 131,072 \approx 1.3 \times 10^5$.

$2/h \times 1/H \times 1/H$	$\|P^h_{H,H,H}u - P_{h,h,h}u\|_1$	$\|u - P_{h,h,h}u\|_1$
8×2×2	0.079664	0.231665
32×4×4	0.011148	0.057932
128×8×8	0.001428	0.014483
162×9×9	0.001005	0.011443
200×10×10	0.000733	0.009269
convergence rate	$O(H^3)$	$O(h)$

Table 1: Example 1: H^1-estimates

$2/h \times 1/H \times 1/H$	$\|P^h_{H,H,H}u - P_{h,h,h}u\|_0$	$\|u - P_{h,h,h}u\|_0$
8×2×2	0.005878	0.010879
32×4×4	0.000375	0.000679
128×8×8	0.000023	0.000042
162×9×9	0.000014	0.000026
200×10×10	0.000009	0.000017
convergence rate	$O(H^4)$	$O(h^2)$

Table 2: Example 1: L^2-estimates

It is seen from Tables 1 and 2 that not only the two-scale finite element approximation $P^h_{H,H,H}u$ has high accuracy but also the number of degrees of freedom for obtaining the two-scale finite element approximation $P^h_{H,H,H}u$ is only of $O(1/h \times 1/H \times 1/H) = O(h^{-2})$ while that for the standard finite element solution $P_{h,h,h}u$ is of $O(h^{-3})$ when $h = H^2$. For instance, the approximate accuracy of the two-scale finite element approximation $P^h_{H,H,H}u$ with 2×10^4 degrees of freedom is asymptotically the same as that of the standard finite element solution $P_{h,h,h}u$ with 2×10^6 degrees of freedom. Hence $P^h_{H,H,H}u$ is a much better approximate solution in terms of computational cost. Moreover, the major computation can be carried out in parallel and the computational time can be reduced further.

$1/h\times1/H\times1/H$	$\|P^h_{H,H,H}u-P_{h,h,h}u\|_1$	$\|u-P_{h,h,h}u\|_1$
4×2×2	0.064850	0.116891
16×4×4	0.009881	0.029317
64×8×8	0.001246	0.007330
81×9×9	0.000874	0.005792
100×10×10	0.000636	0.004691
convergence rate	$O(H^3)$	$O(h)$

Table 3: Example 2: H^1-estimates

$1/h\times1/H\times1/H$	$\|P^h_{H,H,H}u-P_{h,h,h}u\|_0$	$\|u-P_{h,h,h}u\|_0$
4×2×2	0.005251	0.006504
16×4×4	0.000384	0.000404
64×8×8	0.000023	0.000025
81×9×9	0.000014	0.000016
100×10×10	0.000009	0.000010
convergence rate	$O(H^4)$	$O(h^2)$

Table 4: Example 2: L^2-estimates

Example 2. Consider a problem with singular coefficient in three dimensions:

$$\begin{cases} -\Delta u-\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}u+x_1x_2x_3u &= f, \quad \text{in} \quad \Omega=(0,1)\times(0,1)\times(0,1), \\ u &= 0, \quad \text{on} \quad \partial\Omega \end{cases} \tag{5.2}$$

with the exact solution $u=2xyz(1-x)(1-y)(1-z)e^{x+y+z}$. It is shown by Tables 3 and 4 that our two-scale finite element approximation $P^h_{H,H,H}u$ of the three-dimensional singular problem is an economic approximation, too.

Example 3. Consider an eigenvalue problem in three-dimensions:

$$\begin{cases} -\sum_{i=1}^{3}\frac{\partial}{\partial x_i}(x_i^2\frac{\partial u}{\partial x_i}) &= \lambda u, \quad \text{in} \quad \Omega=(1,3)\times(1,2)\times(1,2), \\ u &= 0, \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

The first eigenvalue is $\lambda=\frac{3}{4}+(\frac{2}{\ln^2 2}+\frac{1}{\ln^2 3})\pi^2\simeq 50.01212422$ and the associated eigenfunction is $u=\prod_{i=1}^3(x_i{}^{-\frac{1}{2}}sin(\frac{\pi\ln x_i}{\ln\beta_i}))$, where $\beta_1=3,\ \beta_2=\beta_3=2$.

Here the two-scale finite element approximations are constructed as

$$u^h_{H,H,H}=u^{h,H,H}+u^{H,h,H}+u^{H,H,h}-2u_{H,H,H}$$

and

$$\lambda^h_{H,H,H}=\frac{a(u^h_{H,H,H},u^h_{H,H,H})}{(u^h_{H,H,H},u^h_{H,H,H})}.$$

The numerical results obtained by Algorithm 4.1 are shown in Tables 5 and 6, which support our theory (See Theorem 4.1), too.

$2/h \times 1/H \times 1/H$	$\|u_{h,h,h} - u^h_{H,H,H}\|_1$	$\|u - u_{h,h,h}\|_1$
8×2×2	1.498618	1.520100
32×4×4	0.373862	0.363271
128×8×8	0.091613	0.090570
162×9×9	0.072297	0.071557
200×10×10	0.058508	0.057959
convergence rate	$O(H^2)$	$O(h)$

Table 5: Example 3: estimates for the first eigenvector

$2/h \times 1/H \times 1/H$	$\|\lambda_{h,h,h} - \lambda^h_{H,H,H}\|$	$\|\lambda - \lambda_{h,h,h}\|$
8×2×2	1.894330	3.363422
32×4×4	0.081407	0.203307
128×8×8	0.003940	0.012432
162×9×9	0.002414	0.007662
200×10×10	0.001568	0.004935
convergence rate	$O(H^4)$	$O(h^2)$

Table 6: Example 3: estimates for the first eigenvalue

6. Some Further Remarks

In this paper, we have proposed and analyzed several two-scale finite element discretizations for a class of elliptic boundary value and eigenvalue problems. The main philosophy behind this paper is that to approximate multi-dimensional partial differential equations, we should use a group of finite element discretizations of different mesh scales rather than one-scale finite element discretization only. It is shown by both theory and numerics that the number of degrees of freedom of the two-scale finite element approximations, which are some simple combinations of the standard finite element solutions on different scale meshes, is much less than that of the standard finite element solution. For instance, the number of degrees of freedom of the two-scale finite element approximation in $\Omega = (0,1)^3$ is only of $O(h^{-2})$ while that of the standard finite element solution is of $O(h^{-3})$ when the large scale $H = O(h^{-1/2})$ is chosen, where h is the small scale. However, it is proved in this paper that the corresponding two-scale finite element approximation still processes the same approximate accuracy as that of the standard finite element solution. Hence the two-scale finite element approximation is a much more economic solution in terms of the computational cost.

It is noted that in some part of discussions above, an additional assumption that $a_{ij} = 0(i \neq j, i,j = 1,2,3)$ is required. Indeed this assumption is unnecessary when the exact solution u has higher regularity. As an illustration, let us introduce a new mixed Sobolev space:

$$W_2^{M,5}(\Omega) = \{w \in H^4(\Omega) :\ \partial^3_{x_i}\partial^2_{x_j}(w) \in L^2(\Omega),\ x_i \neq x_j,\ i,j = 1,2,3\}$$

and two assumptions to the exact solution u and the coefficients of (3.1):

Assumption $\tilde{\mathbf{B}}$: $u \in H^1_0(\Omega) \cap W_2^{K,4}(\Omega)$, $a_{ii} \in W^{1,\infty}(\Omega)$, $a_{ij} \in W^{G,2}_\infty(\Omega)(i \neq j)$, $b_i \in L^\infty(\Omega)$, $\partial_{x_i} b_i \in L^\infty(\Omega)$, and $c \in L^\infty(\Omega)(i,j = 1,2,3)$.

Assumption $\tilde{\mathbf{C}}$: $u \in H^1_0(\Omega) \cap W_2^{M,5}(\Omega)$, $a_{ii} \in W^{2,\infty}(\Omega)$, $a_{ij} \in W^{G,2}_\infty(\Omega)(i \neq j)$, $b_i \in L^\infty(\Omega)$, $\partial_{x_i} b_i \in L^\infty(\Omega)$, and $c \in L^\infty(\Omega)(i,j = 1,2,3)$.

It is derived from Lemma 3 in [23] that if $\phi \in L^\infty(\Omega)$, then

$$\begin{aligned} & \left|\int_\Omega \phi(I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})w\partial_{x_1}\partial_{x_2}v\right| \\ \lesssim\ & h_{x_1}h_{x_2}\min(h_{x_1},h_{x_2})||w||_{W_2^{K,4}}||v||_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},0}(\Omega) \end{aligned}$$

and if $\phi, \partial_{x_1}\phi$ (or $\partial_{x_2}\phi$) $\in L^\infty(\Omega)$, then

$$|\int_\Omega \phi(I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})w\partial_{x_1}\partial_{x_2}v| \lesssim h_{x_1}^2 h_{x_2}^2 ||w||_{W_2^{M,5}}||v||_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},0}(\Omega).$$

Thus Assumption $\tilde{\mathbf{B}}$ implies

$$\begin{aligned} &|a((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})u, v)| \\ \lesssim\; & h_{x_1}h_{x_2}\min(h_{x_1}, h_{x_2})\|u\|_{W_2^{K,4}}\|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega), \end{aligned}$$

$$\begin{aligned} &|a((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})(I - I_{0,0,h_{x_3}})u, v)| \\ \lesssim\; & \min(h_{x_1}h_{x_2}, h_{x_2}h_{x_3}, h_{x_3}h_{x_1})\min(h_{x_1}, h_{x_2}, h_{x_3})\|u\|_{W_2^{K,4}}\|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega), \end{aligned}$$

and Assumption $\tilde{\mathbf{C}}$ leads to

$$\begin{aligned} &|a((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})u, v)| \\ \lesssim\; & h_{x_1}^2 h_{x_2}^2\|u\|_{W_2^{M,5}}\|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega) \end{aligned}$$

and

$$\begin{aligned} &|a((I - I_{h_{x_1},0,0})(I - I_{0,h_{x_2},0})(I - I_{0,0,h_{x_3}})u, v)| \\ \lesssim\; & \min(h_{x_1}^2 h_{x_2}^2, h_{x_2}^2 h_{x_3}^2, h_{x_3}^2 h_{x_1}^2)\|u\|_{W_2^{M,5}}\|v\|_{1,\Omega}, \quad \forall v \in S_0^{h_{x_1},h_{x_2},h_{x_3}}(\Omega). \end{aligned}$$

We then conclude that the condition $a_{ij} = 0 (i \neq j, i, j = 1, 2, 3)$ in Theorem 3.1 can be removed when Assumption **B**, Assumption **C** is replaced by Assumption $\tilde{\mathbf{B}}$ and Assumption $\tilde{\mathbf{C}}$, respectively.

In the case of $\Omega = (0, 1)^2$, for instance, we may construct a two-scale finite element approximation

$$P_{H,H}^h u = P_{h,H}u + P_{H,h}u - P_{H,H}u$$

to the boundary value problem, where $P_{h_{x_1},h_{x_2}}u$ is the standard finite element solution in the bilinear finite element space $S_0^{h_{x_1},h_{x_2}}(\Omega)$. For this new finite element approximation, we obtain the following conclusions from Proposition 3, 4 and 5 in [23]:

(1) If Assumption **A** holds, then

$$\|P_{H,H}^h u - P_{h,h}u\|_{1,\Omega} \lesssim H^2\|u\|_{W_2^{G,3}}.$$

(2) If Assumption **B** holds, then

$$\|P_{H,H}^h u - P_{h,h}u\|_{0,\Omega} \lesssim H^3\|u\|_{W_2^{G,3}}.$$

(3) If Assumption **C** holds, then

$$H\|P_{H,H}^h u - P_{h,h}u\|_{1,\Omega} + \|P_{H,H}^h u - P_{h,h}u\|_{0,\Omega} \lesssim H^4\|u\|_{W_2^{K,4}}.$$

(We refer to [23] for the definitions of Assumption **A**, **B** and **C** and spaces $W_2^{G,3}(\Omega)$ and $W_2^{K,4}(\Omega)$ when $\Omega = (0, 1)^2$ for details.) We may also obtain similar efficient two-scale finite element approximations for eigenvalue problems on $(0, 1)^2$.

We should finally mention that these two-scale approaches can be generalized to a class of nonlinear elliptic boundary value problems. Based on the local two-scale methodologies as presented in [26, 27, 28, 29, 33], a type of local and parallel two-scale algorithms may also be devised. These topics will be addressed elsewhere. We should also point out that our two-scale approach requires the regularities of both the exact solution and the finite element mesh. Anyway, we believe that this is a powerful two-scale discretizing technique which can be used for a variety of partial differential equations and integral equations.

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