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# RECENT ADVANCES OF UPSCALING METHODS FOR THE SIMULATION OF FLOW TRANSPORT THROUGH HETEROGENEOUS POROUS MEDIA *[1)]

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**Dedicated to the 70th birthday of Professor Lin Qun**

**Abstract**

We review some of our recent efforts in developing upscaling methods for simulating the flow transport through heterogeneous porous media. In particular, the steady flow transport through highly heterogeneous porous media driven by extraction wells and the flow transport through unsaturated porous media will be considered.

*Mathematics subject classification:* 65F10, 65F30.
*Key words*: Upscaling, Flow transport, Heterogeneous porous media.

## 1. Introduction

The central difficulty in the modeling of subsurface flow and transport is the accounting for the spatial variability in the parameters used to characterize the relevant physical properties of the natural porous media. In realistic situations, the precise spatial distribution of the parameters required to characterize the problem is never available due to the lack of enough data. Thus sophisticated geological and geostatistical modeling tools are used in practice to generate highly detailed medium parameters based on some site-specific measurements and experience from other sites. There exists a vast literature on the upscaling or homogenization techniques that lump the small-scale details of the medium into a few representative macroscopic parameters on a coarse scale which preserve the large-scale behavior of the medium and are more appropriate for reservoir simulations. We refer to the book of Christakos [7] for more information on the random field modeling of the natural porous medium parameters and the recent review paper [24] on the existent upscaling techniques in the engineering literature.

The recently introduced multiscale finite element method [15, 16] for solving elliptic equations with oscillating coefficients provides an effective way to capture the large scale structures of the solutions on a coarse mesh without resolving all the fine scale structures. The central idea of the method is to incorporate the local small scale information of the leading order differential operator into the finite element bases. It is through these multiscale bases and the finite element formulation that the effect of small scales on the large scales is correctly captured. We also refer to the related analysis of the heterogeneous multiscale method (HMM) [19] for solving the elliptic problem with oscillating coefficients. In section 2 we will describe one engineering upscaling technique and discuss its relation with the multiscale finite element method.

The study of steady flow through highly heterogeneous porous medium driven by extraction wells is of great importance in hydrology, petroleum reservoir engineering. It is observed in the

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engineering literature (cf. e.g. [8] and [21] and the references therein) that in the near-well region, many of the existing upscaling methods do not provide satisfactory results. The reason may be explained as the standard upscaling methods generally assume the pressure field is slowly varying, that is clearly not true in the vicinity of the flowing wells [8]. This fact may also be explained mathematically from the homogenization theory for the second order elliptic equations with periodic coefficients. In the homogenization theory, multiscale convergence is ensured under the assumption that the source should be at least in $H^{-1}$ so that the solution is bounded in Sobolev space $H^1$. As we will see below, however, well singularities can be modeled as Dirac sources and thus in the near-well region, the solution behaves like Green function which is not uniformly bounded in $H^1$. In section 3 we will describe an upscaling technique for dealing with well singularities.

The nonlinear Richards equation which models the flow transport in unsaturated porous media is of significant importance in engineering applications. We consider the following nonlinear partial differential equations

$$\frac{\partial \theta}{\partial t} - \frac{\partial K}{\partial x_3} - \nabla \cdot (K \nabla u) = f,$$

where $\theta$ is volumetric water content, $K$ is the absolute permeability tensor, $u$ is the fluid pressure, $x_3$ denotes the vertical coordinate in the medium, and $f$ stands for possible sources/sinks. The sources of nonlinearity of Richards equation come from the moisture retention function $\theta(u)$ and relative hydraulic conductivity function $K(\theta)$, respectively. Based on experimental results, many different functional relations have been proposed in the literature through various combinations of the dependent variables $\theta$, $u$ and $K$, and a certain number of fitting parameters (e.g., [13, 14]). There are several widely known formulations of the constitutive relations such as the van Genuchten-Mualem model [14], or the Garder model [13]. For example, in the Garder model, also called exponential model,

$$\theta(u) = \theta_r + (\theta_s - \theta_r) e^{-\beta |u|}, \quad K(u) = K_s e^{-\alpha |u|},$$

where $\theta_r$ and $\theta_s$ represent the residual water content and saturated water content respectively, $K_s$ is the saturated hydraulic conductivity, and $\alpha, \beta$ are parameters of the porous media. In section 4 we develop an upscaling method for a class of nonlinear parabolic equations which includes the Richards equation in the parabolic range as a special case.

## 2. Upscaling of the Permeability

The purpose of this section is to show that one of the well-known engineering upscaling techniques (see e.g. [20]) is equivalent to the multiscale finite element method proposed in [10, 15]. We remark that multiscale finite element method is shown to be convergent under the condition that the permeability is locally periodic $K_\varepsilon(x) = K(x, x/\varepsilon)$, where $K(x, \cdot)$ is periodic with respect to the second variable. As a consequence, the convergence of the engineering approach described in this section is guaranteed.

Let $\mathcal{M}_H$ be a finite element mesh of $\Omega$ with the mesh size $H$ much larger than the $\varepsilon$, the characteristic length representing the small scale variability of the media. Usually, $\varepsilon$ is equal to the correlation length in the statistical random field modeling of the media. Let $W_H$ be the standard conforming linear finite element space over $\mathcal{M}_H$ and $W_H^0 = W_H \cap H_0^1(\Omega)$. In the engineering literature, the problem

$$-\mathrm{div}\,(K_\varepsilon(x) \nabla u_\varepsilon) = f \quad \text{in } \Omega, \quad u_\varepsilon = 0 \quad \text{on } \Gamma \tag{2.1}$$

is approximated by the homogenized or upscaled problem: Find $u_H^* \in W_H^0$ such that

$$\int_\Omega K^*(x) \nabla u_H^* \nabla v_H dx = \int_\Omega f v_H dx \quad \forall v_H \in W_H^0 \tag{2.2}$$

with $K^*$ being piecewise constant on the coarse mesh $\mathcal{M}_H$. The so-called effective permeability matrix $K^*$ on each $T \in \mathcal{M}_H$ is defined as follows. For any $G \in \mathbf{R}^2$, let $\theta_\varepsilon$ be the solution of the problem

$$-\mathrm{div}\,(K_\varepsilon \nabla \theta_\varepsilon) = 0 \text{ in } T, \quad \theta_\varepsilon|_{\partial T} = G \cdot x.$$

Simple integration by parts implies that

$$\overline{\nabla\theta_\varepsilon} = \frac{1}{|T|}\int_T \nabla\theta_\varepsilon dx = \frac{1}{|T|}\int_{\partial T} Gx\cdot\nu ds = \frac{1}{|T|}\int_T \nabla(Gx)dx = G. \tag{2.3}$$

On the other hand, since $Q = \overline{K_\varepsilon\nabla\theta_\varepsilon} = \frac{1}{|T|}\int_T K_\varepsilon\nabla\theta_\varepsilon dx$ is linear in $G$, there exists a matrix $K^*$ such that

$$Q = K^*\cdot G \iff -\overline{K_\varepsilon\nabla\theta_\varepsilon} = -K^*\overline{\nabla\theta_\varepsilon}. \tag{2.4}$$

The multiscale finite element method introduced in [10, 15] is based on multiscale finite element base functions. For any $T\in\mathcal{M}_H$, let $\psi_i$, $i=1,2,3$, be the linear nodal bases. Define $\phi_i^T$, $i=1,2,3$, as the solution of the local problem

$$-\mathrm{div}\,(K_\varepsilon\nabla\phi_i^T) = 0 \ \text{ in } T, \quad \phi_i^T|_{\partial T} = \psi_i.$$

Denote by $V(T) = \mathrm{span}\,\{\phi_i^T, i=1,2,3\}$ and $V_H$ the finite element space

$$V_H = \{v_H\in H^1(\Omega) : v_H|_T\in V(T)\}.$$

Then the multiscale finite element approximation to (2.1) is: Find $u_H\in V_H^0 = V_H\cap H_0^1(\Omega)$ such that

$$\int_\Omega K_\varepsilon\nabla u_H\nabla v_H dx = \int_\Omega f v_H dx \ \ \forall v_H\in V_H^0. \tag{2.5}$$

It is easy to show that the stiffness matrices corresponding to the discrete problems (2.2) and (2.5) are identical.

## 3. Upscaling of the Well Singularity

Let $\Omega\subset\mathbf{R}^2$ be a bounded domain with Lipschitz boundary $\Gamma$. $B(x_0,r)$ will be denoted as the disk centered at $x_0$ with radius $r>0$. Let $B_j = B(\bar{x}_j,\delta), 1\le j\le N$, be mutually disjoint subdomains inside $\Omega$ that are occupied by the wells. Denote by $\Omega_\delta = \Omega\backslash(\cup_{j=1}^N\bar{B}_j)$. We consider the following single phase pressure equation which is formed by combining Darcy's law with the conservation of mass

$$-\mathrm{div}\,(K(x)\nabla u_\delta) = 0 \quad \text{in } \Omega_\delta, \tag{3.1}$$

where $u_\delta$ is pressure, $K$ is the permeability which is typically highly variable in space. We will impose homogeneous Dirichlet boundary condition $u_\delta|_\Gamma = 0$ on the exterior boundary. The other types of boundary conditions can be treated similarly without any essential difficulties. On the well boundary $\Gamma_j = \partial B_j$, two quantities are of particular importance in practical applications: the well bore pressure $u_\delta|_{\Gamma_j}$ and the well flow rate $\displaystyle\int_{\Gamma_j} K\frac{\partial u_\delta}{\partial\nu}ds$, where $\nu$ is the unit outer normal to $\partial\Omega_\delta$. The boundary condition to be imposed on $\Gamma_j$ is either the Dirichlet boundary condition which fixes the well bore pressure $\alpha_j$ (assume to be constant)

$$u_\delta|_{\Gamma_j} = \alpha_j, \tag{3.2}$$

or the following mixed boundary condition which fixes the well flow rate $\mathrm{q}_j$

$$u_\delta|_{\Gamma_j} = c_j = \mathrm{const}\,, \qquad \int_{\Gamma_j} K\frac{\partial u_\delta}{\partial\nu}ds = \mathrm{q}_j. \tag{3.3}$$

The constant $c_j$ in (3.3) is unknown.

The purpose of this section is to develop a complete coarse grid algorithm for solving steady flow problem involving well singularities in heterogeneous porous medium based on the upscaling method in the last section. The additional well singularities of the problem are resolved locally by adding finite element base functions. The final coarse grid model is based on a variational formulation which is different from the heuristic techniques in [8, 21]. We emphasize that as pointed out in [9], the spatial periodicity assumption does not a priori restrict the applicability of the results only to media which do exhibit such strict repetitive spatial ordering in the properties of interest. The numerical experiments carried out for random log-normal relative permeabilities in [6] using the over-sampling multiscale finite element method demonstrate clearly the applicability of our method beyond the periodic structures.

Since the size of the wells $\delta$ is negligible in practical situations, the first approximation to be made is to replace (3.1)-(3.3) by the following problem

$$-\text{div}\,(K(x)\nabla u) = \sum_{j=1}^{N} \mathrm{q}_j \delta_{\bar{x}_j} \quad \text{in } \Omega \tag{3.4}$$

with the boundary condition $u|_\Gamma = 0$. Here $\delta_{\bar{x}_j}$ is the Dirac measure at $\bar{x}_j$, $1 \le j \le N$. Denote by $K_j = K(\bar{x}_j)$, $\phi_j = -\dfrac{1}{2\pi K_j} \ln|x - \bar{x}_j|$ and $U = u - \sum_{j=1}^{N} \mathrm{q}_j \phi_j$. Note that the flow rates $\mathrm{q}_i$ for $i \in I_D$ in (3.4) are unknown. They are determined through the following additional conditions which are obtained by requiring $u|_{\Gamma_i} \approx \alpha_i$ as the approximation of the boundary conditions (3.2):

$$-\frac{\mathrm{q}_i}{2\pi K_i} \ln\delta + \sum_{j \ne i} \mathrm{q}_j \phi_j(\bar{x}_i) + U(\bar{x}_i) = \alpha_i, \quad i \in I_D. \tag{3.5}$$

The following error estimate is proved in [6] between the solution $u_\delta$ of the original problem (3.1)-(3.3) and the solution $u$ of (3.4)-(3.5)

$$\max_{x \in \bar{\Omega}_\delta} |u - u_\delta| \le C\delta |\ln\delta| \sum_{j=1}^{N} |\mathrm{q}_j|.$$

To introduce our multiscale algorithm, we first introduce an equivalent variational formulation for (3.4) which is suitable for our multiscale approximation. Let $\Omega_j, 1 \le j \le N$, be mutually disjoint subdomains inside $\Omega$ such that $\bar{B}_j \subset\subset \Omega_j$. Let $G_j$ be the Green function associated with the domain $\Omega_j$

$$-\text{div}\,(K(x)\nabla G_j) = \delta_{\bar{x}_j} \quad \text{in } \Omega_j, \quad G_j|_{\Sigma_j} = 0, \tag{3.6}$$

where $\Sigma_j = \partial\Omega_j$. Now for any $v \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} v(\bar{x}_j) &= \int_{\Omega_j} -\text{div}\,(K(x)\nabla G_j) v dx \\ &= \int_{\Omega_j} K(x)\nabla G_j \nabla v dx - \int_{\Sigma_j} K \frac{\partial G_j}{\partial \nu} v ds. \end{aligned}$$

On the other hand, from (3.4) we know that

$$\sum_{j=1}^{N} \mathrm{q}_j v(\bar{x}_j) = \int_\Omega K(x)\nabla u \nabla v dx.$$

Let $G_j = 0$ for $x \in \Omega \backslash \bar{\Omega}_j$ and $\zeta = u - \sum_{j=1}^{N} \mathrm{q}_j G_j$, then we know that $\zeta \in H_0^1(\Omega)$ satisfies the following variational form

$$\int_\Omega K(x)\nabla\zeta\nabla v dx = -\sum_{j=1}^{N} \mathrm{q}_j \int_{\Sigma_j} K \frac{\partial G_j}{\partial \nu} v dx \quad \forall v \in C_0^\infty(\Omega). \tag{3.7}$$

Denote by $w_j = G_j + \dfrac{1}{2\pi K_j} \ln|x - \bar{x}_j|$ in $\Omega_j$, then the complementary condition (3.5) to determine $\mathrm{q}_i$ for $i \in I_D$ becomes

$$-\frac{\mathrm{q}_i}{2\pi K_i} \ln\delta + \mathrm{q}_i w_i(\bar{x}_i) + \zeta(\bar{x}_i) = \alpha_i, \quad i \in I_D. \tag{3.8}$$

We note that for $\zeta$ the singularities of the original solution $u$ are removed and we can use the upscaling method in Section 2 to discretize it on a coarse grid.

Let $\mathcal{M}_H$ be a finite element mesh of $\Omega$ with the mesh size $H$ much larger than the $\varepsilon$, the characteristic length representing the small scale variability of the media. Let $W_H$ be the standard conforming linear finite element space over $\mathcal{M}_H$ and $W_H^0 = W_H \cap H_0^1(\Omega)$. Then we

introduce the following discrete problem: Find $\zeta_H \in W_H^0$ and $\{\mathrm{q}_i^H\}_{i\in I_D}$ such that

$$\int_\Omega K^*(x)\nabla\zeta_H\nabla\chi_H dx = -\sum_{j=1}^N \mathrm{q}_j^H \int_{\Sigma_j} K_\varepsilon \frac{\partial G_j^\varepsilon}{\partial\nu}\chi_H ds \quad \forall\chi_H \in W_H^0, \tag{3.9}$$

$$-\frac{\mathrm{q}_i^H}{2\pi K_i}\ln\delta + \mathrm{q}_i^H w_i(\bar{x}_i) + \zeta_H(\bar{x}_i) = \alpha_i \quad \forall i \in I_D, \tag{3.10}$$

where $K^*$ is defined elementwise as in (2.4), and we set $\mathrm{q}_j^H = \mathrm{q}_j$ for $j \in I_M$ to simplify the notation.

In [6], the problem (3.7)-(3.8) is solved by the over-sampling multiscale finite element method and the convergence of the physically interested quantities like the well bore pressure for the wells that prescribe the flow rate and the flow rate for the wells that prescribe the well bore pressure has been established. A similar convergence analysis can also be proved for the upscaling method in (3.9)-(3.10).

In the practical computation, the following algorithm can be used to the problem (3.1)-(3.3) which is a good approximation of the original problem (3.4)-(3.5) when the size of the wells is negligible. The algorithm adapts the corresponding algorithm proposed and studied in [6] in which the reduced problem (3.7)-(3.8) is solved by the over-sampling multiscale finite element method.

**Algorithm.** Given the well bore pressure $\alpha_i$ for $i \in I_D$ and the well flow rate $\mathrm{q}_j$ for $j \in I_M$. The following procedure finds the approximate well bore pressure $\alpha_j^H$ for $j \in I_M$, the approximate well flow rate $\mathrm{q}_i^H$ for $i \in I_D$, and the approximate pressure $u_H = \zeta_H + \sum_{j=1}^N \mathrm{q}_j^H G_j^h$, where $\mathrm{q}_j^H = \mathrm{q}_j$ for $j \in I_M$.

- For $j = 1, \cdots, N$, compute the discrete Green function $G_j^h$ on each subdomain $\Omega_j$, i.e. $G_j^h \in V_h^0(\Omega_j)$ such that

$$\int_{\Omega_j} K(x)\nabla G_j^h \nabla v_h dx = v_h(\bar{x}_j) \quad \forall v_h \in V_h^0(\Omega_j). \tag{3.11}$$

  Here $V_h^0(\Omega_j) = V_h(\Omega_j) \cap H_0^1(\Omega_j)$, and $V_h(\Omega_j)$ is the standard conforming linear finite element space over the mesh $\mathcal{M}_h(\Omega_j)$ of $\Omega_j$ with the mesh size $h$ resolving the scale of the permeability field $K(x)$. Compute the associate effective radius $\bar{r}_j$ from $G_j^h$ according to the following relation

$$\frac{1}{2\pi K_j}\ln\bar{r}_j = \int_{\Omega_j}(K - K_j)\nabla\phi\nabla G_j^h dx - \int_{\Omega_j} K\nabla G_j^h \nabla\psi_h, \tag{3.12}$$

  where $\phi = -\frac{1}{2\pi K_j}\ln|x - x_j|$ and $\psi_h \in V_h(\Omega_j)$ whose nodal values are defined by

$$\psi_h(x_k) = \begin{cases} \phi(x_k) & \text{if } x_k \in \partial\Omega_j, \\ 0 & \text{otherwise,} \end{cases}$$

  Thus the approximate value of the Green function $G_j$ on $\Gamma_j$ is

$$\alpha_j^h = G_j^h(\bar{x}_j) - \frac{1}{2\pi K_j}\ln\frac{\delta}{\bar{r}_j}, \quad K_j = K(\bar{x}_j).$$

- Find $\zeta_H \in W_H^0$ and $\{\mathrm{q}_i^H\}_{i\in I_D}$ such that

$$\int_\Omega K^*(x)\nabla\zeta_H\nabla\chi_H dx = -\sum_{j=1}^N \mathrm{q}_j^H \int_{\Sigma_j} K\frac{\partial G_j^h}{\partial\nu}\chi_H ds \quad \forall\chi_H \in W_H^0,$$

$$\mathrm{q}_i^H\alpha_i^h + \zeta_H(\bar{x}_i) = \alpha_i \quad \text{for } i \in I_D,$$

  where $K^*$ is computed elementwise according to (2.4).

- Compute the approximate well bore pressure $\alpha_j^H$ for $j \in I_M$ through the relation

$$\alpha_j^H = \zeta_H(\bar{x}_j) + \mathrm{q}_j\alpha_j^h.$$

The formula (3.12) defining the effective radius extends the well-known Peaceman method [22, 23] in the engineering literature for five point finite difference discretization and constant permeability to the general case and is convergent for any finite element meshes and any heterogeneous porous media, as proved in [6].

## 4. Upscaling of Nonlinear Parabolic Equations

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$ be a bounded polyhedral domain with boundary $\partial\Omega$. We set $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$ for $0 < T < \infty$. Consider the following parabolic equation

$$\begin{aligned} \partial_t b(u_\varepsilon) - \nabla \cdot (\mathbf{g}^\varepsilon(x, u_\varepsilon) + \mathbf{a}^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) &= f(x,t) \quad \text{in } Q_T, \\ u_\varepsilon(x,t) &= 0 \qquad \text{on } S_T, \\ u_\varepsilon(x,0) &= u_0(x) \quad \text{in } \Omega, \end{aligned} \tag{4.1}$$

where $\mathbf{a}^\varepsilon(x, u_\varepsilon) = (a^\varepsilon_{ij}(x, u_\varepsilon))$ is a symmetric, positive definite, bounded tensor:

$$\lambda|\xi|^2 \le a^\varepsilon_{ij}(x, s)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d,\ x \in \bar{\Omega},\ s \in \mathbb{R} \tag{4.2}$$

for some positive constants $\lambda$ and $\Lambda$, and $\mathbf{g}^\varepsilon(x, u_\varepsilon) = (g^\varepsilon_i(x, u_\varepsilon))$ is a bounded vector. $\varepsilon$ is the characteristic length representing the small scale variability of the media. We also assume that $\frac{\partial}{\partial s}a^\varepsilon_{ij}(x, s)$, $\frac{\partial}{\partial s}g^\varepsilon_i(x, s)$ are uniformly bounded and $b(s)$ satisfies

$$0 < b_1 \le b'(s) \le b_2 < \infty, \quad b''(s) < C \quad \forall s \in \mathbb{R}. \tag{4.3}$$

Define the space

$$\mathcal{W} = \{u : u \in L^2(0, T; H^1_0(\Omega)), u \in H^1(0, T; H^{-1}(\Omega))\}.$$

The variational problem of (4.1) is to seek $u_\varepsilon(x, t) \in \mathcal{W}$, for almost every $t \in (0, T)$, $u_\varepsilon(\cdot, t) \in H^1_0(\Omega)$ such that $u_\varepsilon(x, 0) = u_0(x)$ in $\Omega$, and

$$(\partial_t b(u_\varepsilon), w) + (\mathbf{g}^\varepsilon(x, u_\varepsilon) + \mathbf{a}^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon, \nabla w) = (f, w) \quad \forall w \in H^1_0(\Omega). \tag{4.4}$$

Here and henceforth, $(\cdot, \cdot)$ stands for the inner product of $L^2(\Omega)$ or the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

Instead of solving (4.4) on a fine mesh with a mesh size resolving the small scale variability $\varepsilon$, the basic idea of the upscaling methods is to solve the homogenized or upscaled equation

$$(\partial_t b(u), w) + (\mathbf{g}^*(x, u) + \mathbf{a}^*(x, u)\nabla u, \nabla w) = (f, w) \quad \forall w \in H^1_0(\Omega).$$

The homogenized coefficients $\mathbf{a}^*(x, s)$, $\mathbf{g}^*(x, s)$, for $s \in \mathbb{R}$, can be computed analytically from $\mathbf{a}^\varepsilon$, $\mathbf{g}^\varepsilon$ if they are periodic with respect to the second variable, see e.g. [17, 2, 4]. Unfortunately, for practical natural porous media, such analytical formulae do not exist. In the following we shall develop a way to compute the nonlinear relations $\mathbf{a}^*(x, \cdot)$, $\mathbf{g}^*(x, \cdot)$ numerically.

Let $\mathcal{M}_H$ be a regular triangulation of $\Omega$ with mesh size $H$ and $\tau = T/N$ be the time step length, $t^n = n\tau$, $n = 0, 1, \cdots, N$. Further, let $W_H$ be the standard conforming linear finite element space over $\mathcal{M}_H$ and $W^0_H = W_H \cap H^1_0(\Omega)$. For any $K \in \mathcal{M}_H$, denote

$$\langle\cdot\rangle_K = \frac{1}{|K|}\int_K (\cdot)dx$$

as the volume average over $K$.

Set $v = b(u)$. For $n = 1, \cdots, N$, our discrete problem is to seek $v^n_H \in W^0_H$, the approximate solution of $v$ at time $t = t^n$, such that

$$\left(\frac{v^n_H - v^{n-1}_H}{\tau}, w_H\right) + (\tilde{\mathbf{g}}(x, \tilde{u}^n) + \tilde{\mathbf{a}}(x, \tilde{u}^n)\nabla\tilde{u}^n, \nabla w_H) = (\bar{f}^n, w_H) \quad \forall w_H \in W^0_H, \tag{4.5}$$

where $\tilde{u}^n = b^{-1}(v^n_H)$, $v^0_H = b(u_0)$ and $\bar{f}^n = \tau^{-1}\int_{t^{n-1}}^{t^n} f(x, t)dt$. For any $s \in \mathbb{R}$, the nonlinear functions $\tilde{\mathbf{a}}(x, s)$ and $\tilde{\mathbf{g}}(x, s)$ are piecewise constant over $\mathcal{M}_H$ defined as follows.

For any $K \in \mathcal{M}_H$, $s \in \mathbb{R}$, let $p^\varepsilon_i, i = 1, 2, \cdots, d$, be the solution of the problem

$$\begin{aligned} -\nabla \cdot (\mathbf{a}^\varepsilon(x, s)\nabla p^\varepsilon_i) &= 0 \quad \text{in } K, \\ p^\varepsilon_i &= x_i \quad \text{on } \partial K. \end{aligned} \tag{4.6}$$

Then, on $K$, $\tilde{\mathbf{a}}$ is a constant tensor determined by the following system

$$\tilde{\mathbf{a}}\langle\nabla p_i^\varepsilon\rangle_K = \langle\mathbf{a}^\varepsilon(x,s)\nabla p_i^\varepsilon\rangle_K, \quad i = 1, 2, \cdots, d. \tag{4.7}$$

It is well-defined since by using Green formula

$$\langle\nabla p_i^\varepsilon\rangle_K = \frac{1}{|K|}\int_{\partial K} x_i \mathbf{n} d\sigma = \mathbf{e}_i, \tag{4.8}$$

where $\mathbf{e}_i$ is the unit vector in the $i$th direction. Similar to the argument in [3, 25] for the linear case, we know that $\tilde{\mathbf{a}}$ is symmetric, bounded, and positive definite. Moreover,

$$\mathbf{e}_i \cdot (\tilde{\mathbf{a}}\mathbf{e}_j) = \langle\nabla p_i^\varepsilon \cdot (\mathbf{a}^\varepsilon(x,s)\nabla p_j^\varepsilon)\rangle_K. \tag{4.9}$$

Further, $\tilde{\mathbf{g}}(x,s)$ is a constant vector in $K$ determined by

$$\tilde{g}_i(x,s) = \langle\mathbf{g}^\varepsilon(x,s)\cdot\nabla p_i^\varepsilon\rangle_K, \quad i = 1, 2, \cdots, d. \tag{4.10}$$

Recently a number of multiscale numerical methods, such as multiscale finite element method [11], heterogeneous multiscale method [19], and numerical homogenization method [12] have been proposed to solve the nonlinear problems. The key idea of our method is that we upscale the nonlinear constitutive relations such as the relationship between hydraulic conductivity versus pressure before we solve the nonlinear problems. We stress that the real significance of the method lies in its ability to solve the problems in coarse meshes. This is particularly advantageous when multiple simulations or realizations are necessary due to changes of boundary conditions or source functions for certain given fine micro-structures of the highly heterogeneous permeability of the porous media. Based on the homogenization theory, a sharp error estimate of the method can be established under the periodicity assumption in [4]. This assumption allows us to use the homogenization theory to obtain the asymptotic structure of the solutions. We emphasize that as pointed out in [9], the spatial periodicity assumption does not a priori restrict the applicability of the results only to media which do exhibit such strict repetitive spatial ordering in the properties of interest. The numerical experiments in [4] indicate that our method works fine for the well-accepted random log-normal permeability models in the engineering literature.

Another new feature of our method is the way by which we deal with the nonlinear convection term. Our numerical procedure, as we show in the paper, shares a common element with the other multiscale methods, that is, the local information is coupled in the global formulation. The difference is the coupling way we use. Our local problem does not involve the convection term which is different from the multiscale finite element method and the numerical homogenization approach introduced in [11, 12]. This idea has been introduced in a previous paper for the solute transport model in [3].

## References


[1] A. Bensoussan, J.L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structure*, volume 5 of Studies in Mathematics and Its Application, North-Holland, 1978.

[2] A. Bourgeat,S. M. Kozlov, A. Mikelic, Effective equations of two-phase flow in random media, *Calc. Var. Partial Differential Equations*, **3** (1995), 385–406.

[3] Z. Chen, W.B. Deng and H. Ye, A new upscaling method for the solute transport equations, *Continuous and Discrete Dynamical Systems,* **13** (2005), 941-962.

[4] Z. Chen, W.B. Deng and H. Ye, Upscaling of a class of nonlinear parabolic equations for the flow transport in heterogeneous porous media, *Communications in Mathematical Sciences,* **3** (2005), 493-515.

[5] Z. Chen and T.Y. Hou, A mixed multiscale finite element method for elliptic problem with oscillating coefficients, *Math. Comp.,* **72** (2002), 541-576.

[6] Z. Chen and X. Yue, Numerical homogenization of well singularities in the flow transport through heterogeneous porous media, *Multiscale Modeling and Simulation,* **1** (2002), 260-303.

[7] G. Christakos, Random Field Models in Earth Sciences, Academic Press, San Diego, 1992.

[8] L.J. Durlofsky, W.J. Milliken and A. Bernath, Scale up in the Near-Well Region, SPE 51940, Proceedings of the 15th SPE Symposium on Reservoir Simulation, Houston, February 14-17, 1999.

[9] B.B. Dykaar and P.K. Kitanidis, Transmissivity of a heterogeneous formation, *Water Resource Research,* **29** (1993), 985-1001.

[10] Y.R. Efendiev, T.Y. Hou, and X.H. Wu, The convergence of non-conforming multiscale finite element methods, *SIAM J. Numer. Anal.,* **37** (2000), 888–910.

[11] Y. Efendiev, T. Y. Hou, and V. Ginting, Multiscale finite element methods for nonlinear problems and their applications, *Comm. Math. Sci.,* **2** (2004), to appear.

[12] Y. Efendiev and A. Pankov, Numerical homogenization of nonlinear random parabolic operators, *SIAM Multiscale Modeling and Simulation,* **2** (2004), 237–268.

[13] W.R. Gardner, Some steady state solutions of unsaturared mositure flow equations with apllication to evaporation from a water table, *Soil Sci.,* **85** (1958), 228–232.

[14] M. Th. van Genuchten, A closed-form equation for predicting the hydraulic conductivity of unsaturated soils, *Soil. Sci. Soc. Am. J.,* **44** (1980), 892–898.

[15] T.Y. Hou and X.H. Wu, A multiscale finite element method for elliptic problems in composite materials and porous media, *J. Comput. Phys.,* **134** (1997), 169–189.

[16] T.Y. Hou, X.H. Wu, and Z. Cai, Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients, *Math. Comp.,* **68** (1999), 913–943.

[17] H. Jian, On the homogenization of degenerate parabolic equations, *Acta Math. Appl. Sinica,* **16** (2000), 100–110.

[18] V.V. Jikov,S.M. Kozlov and O.A. Oleinik, Homogenization of Differential and Integral Functionals, Springer, Berlin, 1994.

[19] W. E, P. Ming and P. Zhang, Analysis of the heterogeneous multiscale method for elliptic homogenization problems, *J. AMS,* **18** (2005), 121-156.

[20] K. Markov and L. Preziosi, eds., Heterogeneous Media: Micromechanics Modeling, Methods and Simulations, Birkhäuser, Boston, 2000.

[21] O. Mascarenhas and L.J. Durlofsky, Scale up in the vicinity of horizontal wells, Proceedings of the 20th Annual International Energy Agency Workshop and Symposium, Paris, September 22-24, 1999.

[22] D.W. Peaceman, Interpretation of well-block pressures in numerical reservoir simulations, SPEJ, 183-194, June 1978.

[23] D.W. Peaceman, Interpretation of well-block pressures in numerical reservoir simulations with non-square grid blocks and anisotropic permeability, SPEJ, 531-543, June 1983.

[24] Ph. Renard and G. de Marsily, Calculating equivalent permeability: A review, *Advances in Water Resources,* **20** (1997), 253-278.

[25] X.H. Wu, Y. Edendiev and T.Y. Hou, Analysis of upscaling absolute permeability, *Discrete and Continuous Dynamical Systems-series B*, **2** (2002), 185–204.