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CHEBYSHEV WEIGHTED NORM LEAST-SQUARES SPECTRAL METHODS FOR THE ELLIPTIC PROBLEM *1)

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Abstract

We develop and analyze a first-order system least-squares spectral method for the second-order elliptic boundary value problem with variable coefficients. We first analyze the Chebyshev weighted norm least-squares functional defined by the sum of the L^2_w- and H^{-1}_w- norm of the residual equations and then we replace the negative norm by the discrete negative norm and analyze the discrete Chebyshev weighted least-squares method. The spectral convergence is derived for the proposed method. We also present various numerical experiments. The Legendre weighted least-squares method can be easily developed by following this paper.

Mathematics subject classification: 65F10, 65F30.
Key words: Least-squares methods, Spectral method, Negative norm.

1. Introduction

Let Ω be the square $(-1,1)^2$. We consider the second-order elliptic boundary value problem:

$$\begin{cases} -\nabla\cdot A\nabla p + \mathbf{b}\cdot\nabla p + c_0\, p &=& f, & \text{in } \Omega,\\ p &=& 0, & \text{on } \Gamma_D,\\ \mathbf{n}\cdot A\nabla p &=& 0, & \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$ denotes the boundary of Ω, A is a 2×2 symmetric matrix of bounded functions, f is a given continuous function, $\mathbf{b}$ is a bounded vector function and c_0 is a given bounded function, and $\mathbf{n}$ is the outward unit vector normal to the boundary. We assume that the matrix A is uniformly elliptic such as

$$0 < \lambda\,\boldsymbol{\xi}^t\boldsymbol{\xi} \le \boldsymbol{\xi}^t A(x,y)\boldsymbol{\xi} \le \Lambda\,\boldsymbol{\xi}^t\boldsymbol{\xi} < \infty \tag{1.2}$$

for all $\boldsymbol{\xi}\in\Re^2$ and almost all $(x,y)\in\bar{\Omega}$.

In recent years there has been lots of interest in the use of first-order system least-squares method (FOSLS) for numerical approximations of elliptic partial differential equations and systems. Introducing an extra physical meaningful variable, for example the flux $\mathbf{u} = A\nabla p$, the equation (3.1) can be written as an equivalent first-order system of partial differential equations and then one may try to apply mixed methods to find the approximation solution. Among the several mixed methods the least-squares methods have several benefits such that the resulting algebraic system is always positive symmetric and the methods can avoid LBB compatibility condition. For more details we refer to [2] and references therein.

In [4] and [5], Cai, Lazarov, Manteuffel and McCormick developed an L^2-norm least squares for scalar second order elliptic partial differential equations. But the limitation of such L^2-norm FOSLS is the requirement of sufficient smoothness of the underlying problem which guarantees

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the equivalence of norms between $H(\text{div};\Omega) \cap H(\text{curl}A;\Omega)$ and $H^1(\Omega)^d$, where $d = 2$ or 3, so that it can be approximated by standard continuous finite element space. But those two spaces are not equivalent in general when the domain Ω is not smooth or not convex, or the coefficient is not continuous. To overcome such a limitation, in [7] Cai and Shin developed the discrete FOSLS by directly approximating $H(\text{div})\cap H(\text{curl})$-type space based on the Helmholtz decomposition, in which under general assumptions they established error estimates in the L^2- and H^1-norms for the vector and scalar variables, respectively. Also, in [3] Bramble, Lazarov and Pasciak developed the discrete negative norm FOSLS for overcoming the same limitation. An alternative using adjoint FOSLS, so-called FOSLL*, can be found in [6].

In this paper, instead of using finite element method, we adopt Legendre and Chebyshev spectral method to solve the first-order system corresponding to (3.1) in the least-squares approaches. The usage of spectral methods instead of using piecewise polynomials in a finite element approach has been known as very accurate and popular regardless of higher amount of work for approximating solutions of various kinds of differential equations (see [1], [13]). In [9], Kim, Lee and Shin recently developed the fusion method combining the concept of L^2 FOSLS and spectral collocation method, so-called least-squares spectral collocation method, to solve second order elliptic boundary value problems with constant coefficients. They derived the spectral error estimates which allow us to get the spectral convergence for both Legendre and Chebyshev approximations.

In order to apply least-squares spectral method to general elliptic problems having various coefficients, we first develop the Chebyshev weighted negative norm least-squares method and then we develop the discrete Chebyshev weighted negative norm least-squares method. The negative norm least-squares given in [3] is further extended to the Chebyshev weighted negative norm least-squares method. To define the discrete weighted negative norm least-squares functional under the polynomial space, we first define a discrete solution operator $T_N : H_w^{-1}(\Omega) \to \mathcal{Q}_N^0$, where $\mathcal{Q}_N$ is the space consisting of all polynomials of degree less than or equal to N and $\mathcal{Q}_N^0 = \mathcal{Q}_N \cap H^1_{0,w}(\Omega)$, and then we establish the relation between $\|f\|^2_{H_w^{-1}(\Omega)}$ and $(f, T_N f)_w$. Owing to such estimates we show the equivalence between the proposed homogeneous discrete least-squares functional $G_{w,N}(\mathbf{v}, q; 0)$ and the product norm $|||(\mathbf{v}, q)|||_w^2 := \|\mathbf{v}\|^2_{L^2_w(\Omega)} + N^{-1}\|\nabla\cdot\mathbf{v}\|^2_{L^2_w(\Omega)} + \|p\|^2_{H^1_w(\Omega)}$ over $H_w(\text{div};\Omega) \times H^1_{0,w}(\Omega)$. The equivalence is more improved than that of the result given in [3], in which the norm is given by $\|\mathbf{v}\|_{L^2_w(\Omega)} + \|q\|_{H^1_w(\Omega)}$. We also establish the spectral convergence such that for $(\mathbf{u}, p) \in H_w^{s-1}(\Omega)^2 \times H_w^s(\Omega)$ $(s \geq 2)$

$$|||(\mathbf{u} - \mathbf{u}_N, p - p_N)|||_w \leq C\, N^{1-s}(\|\mathbf{u}\|_{H_w^{s-1}(\Omega),w} + \|p\|_{H_w^s(\Omega)}).$$

The spectral Galerkin approximation using least-squares principle developed in this paper is slightly different from the spectral collocation method in the respect of using the $L^2_w(\Omega)$ scalar product instead of the discrete scalar product. The exact computation of spectral Galerkin approach is somewhat complicated. But, if the coefficients are continuous then it can be further approximated by using Gaussian quadrature formula. That is, the $L^2_w(\Omega)$ scalar product of continuous functions can be approximated by the discrete scalar product using Chebyshev-Gauss-Lobatto(CGL) points and the corresponding quadrature weights. Also, based on this paper one may develop the spectral element approximation or high-order element method like *hp*-method for the general elliptic problems and more complicated problems, e.g., Stokes problems and Navier-Stokes problems.

Throughout this paper, we assume that w is the Chebyshev weight function, i.e., we will investigate our theory for only Chebyshev spectral approximation. For Legendre case, one may easily obtain the similar results following the arguments of this paper.

The paper is organized as follows. The definitions, notations and basic facts are presented in the following section 2. The weighted negative norm first-order system least-squares method is introduced and analyzed by showing ellipticity and continuity in section 3. And the spectral Galerkin approximation for the discrete least-squares is discussed with convergence analysis in section 4. The implementation and some numerical experiments for the proposed method with some problems are presented in section 5.

2. Preliminaries

In this paper, we use the standard notations and definitions for the Chebyshev/Legendre weighted Sobolev space $H^s_w(\Omega)^2$, associated to weighted inner products $(\cdot,\,\cdot)_{s,w}$, and respective weighted norms $\|\cdot\|_{s,w}$, $s \geq 0$ where $w(x,y) = \hat{w}(x)\hat{w}(y)$ is either the Legendre weight function with $\hat{w}(t) = 1$ or the Chebyshev weight function with $\hat{w}(t) = \frac{1}{(1-t^2)^{1/2}}$. From now on we assume that w is the Chebyshev weight function. The space $H^0_w(\Omega)$ coincides with $L^2_w(\Omega)$, in which the norm and inner product will be denoted by $\|\cdot\|_w$ and $(\cdot,\,\cdot)_w$, respectively. Let $H^1_{0,w}(\Omega)$ be the subspace of $H^1_w(\Omega)$ consisting of the functions which vanish on the boundary. Denote by $H^{-1}_w(\Omega)$ the dual space of the space $H^1_{0,w}(\Omega)$ equipped with the norm

$$\|g\|_{-1,w} := \sup_{\theta \in H^1_{0,w}} \frac{(g,\theta)_w}{\|\theta\|_{1,w}}.$$

Let

$$H_w(\mathrm{div};\Omega) = \{\mathbf{v} \in L^2_w(\Omega)^2 \,:\, \nabla\cdot\mathbf{v} \in L^2_w(\Omega)\,\}$$

which is Hilbert space under norm

$$\|v\|_{div,w} = \left(\|\mathbf{v}\|^2_w + \|\nabla\cdot\mathbf{v}\|^2_w\right)^{1/2}.$$

Let $\mathcal{P}_N$ be the space of all polynomials of degree less than or equal to N. Let $\{\xi_i\}_{i=0}^N$ be the LGL or CGL points on $[-1,1]$ such that $-1 =: \xi_0 < \xi_1 < \cdots, < \xi_{N-1} < \xi_N := 1$. For Legendre case, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)L'_N(t)$ where L_N is the N^{th} Legendre polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{2}{N(N+1)}, \quad w_j = \frac{2}{N(N+1)}\frac{1}{[L_N(\xi_j)]^2}, \quad 1 \leq j \leq N-1. \tag{2.1}$$

For Chebyshev case, $\{\xi_i\}_{i=0}^N$ are the zeros of $(1-t^2)T'_N(t)$ where T_N is the N^{th} Chebyshev polynomial and the corresponding quadrature weights $\{w_i\}_{i=0}^N$ are given by

$$w_0 = w_N = \frac{\pi}{2N}, \quad w_j = \frac{\pi}{N}, \quad 1 \leq j \leq N-1. \tag{2.2}$$

The Gaussian quadrature rule yields the exactness of numerical integration such that

$$\int_{-1}^{1} p(t)\hat{w}(t)dt = \sum_{i=0}^{N} w_i\, p(\xi_i), \quad \forall\, p \in \mathcal{P}_{2N-1}. \tag{2.3}$$

The two-dimensional LGL or CGL nodes $\{\mathrm{x}_{ij}\}$ and the corresponding weights $\{\mathbf{w}_{ij}\}$ are given by

$$\mathrm{x}_{ij} = (\xi_i,\xi_j), \quad \mathbf{w}_{ij} = w_i w_j, \quad i,j = 0,1,\cdots,N.$$

Let $\mathcal{Q}_N$ be the space of all polynomials of degree less than or equal to N with respect to each single variable x and y. For any continuous functions u and v over $\bar{\Omega}$, the associated discrete scalar product and norm are given by

$$\langle u,v\rangle_{w,N} = \sum_{i,j=0}^{N} \mathbf{w}_{ij}\, u(\mathrm{x}_{ij})\, v(\mathrm{x}_{ij}) \quad \text{and} \quad \|v\|_{w,N} = \langle v,v\rangle^{1/2}_{w,N}. \tag{2.4}$$

Then, we have from (2.3) that

$$\langle u,v\rangle_{w,N} = (u,v)_w \quad \text{for} \quad uv \in \mathcal{Q}_{2N-1} \tag{2.5}$$

and it is well-known that

$$\|v\|_w \leq \|v\|_{w,N} \leq \gamma^*\|v\|_w, \quad \forall\, v \in \mathcal{Q}_N, \tag{2.6}$$

where $\gamma^* = \left(2 + \frac{1}{N}\right)$ for Legendre case and $\gamma^* = 2$ for Chebyshev case. For any continuous function v, we denote by $I_{w,N}v \in \mathcal{Q}_N$ the interpolant of v at the LGL or CGL-points $\{\mathrm{x}_{ij}\}$. Then

$$I_{w,N}v(\mathrm{x}) = \sum_{i,j=0}^{N} v(\mathrm{x}_{ij})\psi_{ij}(\mathrm{x}), \quad \forall\, \mathrm{x} \in \bar{\Omega} \tag{2.7}$$

where ψ_{ij} are the Lagrange interpolation polynomials of degree N for $i, j = 1, 2, \cdots, N$.

The interpolation error estimate is given by

$$\|v - I_{w,N}v\|_{k,w} \le C\, N^{k-s}\|v\|_{s,w}, \quad k = 0, 1, \tag{2.8}$$

provided $v \in H^s_w(\Omega)$ for $s \ge 2$ (see [1, 8, 13]) and using (2.5)–(2.8) yields that for all $u \in H^s_w(\Omega)$, $s \ge 2$, and $v_N \in \mathcal{Q}_N$

$$|(u, v_N)_w - \langle u, v_N\rangle_{w,N}| \le C\, N^{-s}\, \|u\|_{s,w}\, \|v_N\|_w. \tag{2.9}$$

Let $\mathcal{Q}^0_N = \mathcal{Q}_N \cap H^1_{0,w}(\Omega)$. We now recall the orthogonal projection $P^0_{1,N} : H^1_{0,w}(\Omega) \to \mathcal{Q}^0_N$ through

$$(\nabla P^0_{1,N}u, \nabla\theta_N)_w = (\nabla u, \nabla\theta_N)_w, \quad \forall\, \theta_N \in \mathcal{Q}^0_N. \tag{2.10}$$

Then, for all $u \in H^s_w(\Omega) \cap H^1_{0,w}(\Omega)$, with $s \ge 1$, (see [13])

$$\|u - P^0_{1,N}u\|_{k,w} \le C\, N^{k-s}\|u\|_{s,w}, \quad k = 0, 1. \tag{2.11}$$

Let us recall the inverse inequality (see [13])

$$\|v_N\|_{1,w} \le C\, N^2\, \|v_N\|_w, \quad \forall v_N \in \mathcal{Q}_N. \tag{2.12}$$

For any $v_N \in \mathcal{Q}_N$, since

$$\|v_N\|^2_w = (v_N, v_N)_w \le \|v_N\|_{-1,w}\, \|v_N\|_{1,w} \le C\, N^2\, \|v_N\|_{-1,w}\, \|v_N\|_w,$$

we also have the following inverse inequality

$$\|v_N\|_w \le C\, N^2\, \|v_N\|_{-1,w} \quad \forall v_N \in \mathcal{Q}_N. \tag{2.13}$$

3. Minus one Norm Weighted Least-squares Method

In this section, we investigate the minus one norm weighted least-squares method for the first-order system of linear equations equivalent to the problem :

$$\begin{cases} -\nabla\cdot\nabla p + \mathbf{b}\cdot\nabla p + c_0\, p &= f, \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\quad p &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

From now on we assume that the diffusion coefficient A given in (1.1) is the identity matrix and assume that the above problem (3.1) has the unique solution in $H^1_{0,w}(\Omega)$. If A is symmetric and uniformly elliptic with bounded elements, one may easily derive the similar results given in this paper. The minus one norm un-weighted least squares method was explained in [3].

Now we slightly extend such minus one norm and inner product to the weighted Sobolev spaces for a square domain. As did in [3], let $T : H^{-1}_w(\Omega) \longrightarrow H^1_{0,w}(\Omega)$ be defined by $Tg = p$ for $g \in H^{-1}_w(\Omega)$ where $p \in H^1_{0,w}(\Omega)$ is the unique solution satisfying

$$(\nabla p, \nabla\theta)_w + (p, \theta)_w = (g, \theta)_w \quad \text{for all } \theta \in H^1_{0,w}(\Omega). \tag{3.2}$$

Using (3.2), Schwarz inequality and the definition of T, we have the following lemma.

Lemma 3.1. *For all $f \in H^{-1}_w(\Omega)$, we have*

$$\|Tf\|_{1,w} = \left(f, Tf\right)^{1/2}_w = \sup_{\theta\in H^1_{0,w}(\Omega)} \frac{(f,\theta)_w}{\|\theta\|_{1,w}} = \|f\|_{-1,w}. \tag{3.3}$$

We can easily verify from (3.2) that

$$\big(f, Tg\big)_w = \big(g, Tf\big)_w \quad \text{for all } f,\ g \in L^2_w(\Omega). \tag{3.4}$$

This shows that the inner product on $H^{-1}_w(\Omega) \times H^{-1}_w(\Omega)$ can be defined by

$$(f,g)_{-1,w} := (f, Tg)_w \quad \text{for } f, g \in H^{-1}_w(\Omega).$$

We recall the Poincaré-Friedrichs inequality such that (see [8])

$$\|p\|_w \le C\|\nabla p\|_w, \quad \forall\, p \in H^1_{0,w}(\Omega), \tag{3.5}$$

where C is a positive constant.

Lemma 3.2. *We have the followings:*

(a) There are two positive constants c and C such that

$$c\|\phi\|^2_{1,w} \le \int_\Omega \nabla\phi \cdot \nabla(\phi w)\, dxdy \le C\|\phi\|^2_{1,w} \quad \forall\, \phi \in H^1_{0,w}(\Omega).$$

(b) There is a constant C such that, with $t = x$ or y,

$$\Big|\int_\Omega \phi^2(x,y)\frac{t^2}{(1-t^2)^2}\hat{w}(x)\hat{w}(y)\, dxdy\Big| \le C\|\phi\|^2_{1,w} \quad \forall\, \phi \in H^1_{0,w}(\Omega).$$

(c) There is a constant C such that

$$\|\nabla\cdot\mathbf{v}\|_{-1,w} \le C\,\|\mathbf{v}\|_w \quad \forall\, \mathbf{v} \in L^2_w(\Omega)^2.$$

(d) There is a constant C such that

$$\|\nabla p\|_{-1,w} \le C\,\|p\|_w \quad \forall\, p \in L^2_w(\Omega).$$

Proof. The proofs of (a), (b) can be found in [8] and [9], and the proof of (c) is given in [9]. One may easily prove (d) by the similar way given in the proof (c).

Now, we establish the following a priori estimate which overcomes the restriction of assumption (A_0) given in [9].

Lemma 3.3. *For any $p \in H^1_{0,w}(\Omega)$ we have*

$$\|p\|_{1,w} \le C\,\| -\Delta p + \mathbf{b}\cdot\nabla p + c_0\, p\|_{-1,w}, \quad \forall\, p \in H^1_{0,w}(\Omega). \tag{3.6}$$

Proof. It is well-known from Theorem 11.1 in [8] that there exists a constant C such that

$$\|p\|^2_{1,w} \le C\,\big(\nabla p, \nabla(pw)\big) = C\,\big(-\Delta p, p\big)_w, \quad \forall\, p \in H^1_{0,w}(\Omega).$$

Hence, by the definition of $\|\cdot\|_{-1,w}$ we have

$$\|p\|_{1,w} \le C\,\| -\Delta p\|_{-1,w} \quad \forall\, p \in H^1_{0,w}(\Omega).$$

Using the triangle inequality yields

$$\begin{aligned}
\|p\|^2_{1,w} &\le C\,\| -\Delta p\|^2_{-1,w} \\
&\le C\,\big(\| -\Delta p + \mathbf{b}\cdot\nabla p + c_0\, p\|^2_{-1,w} + \|\mathbf{b}\cdot\nabla p + c_0\, p\|^2_{-1,w}\big) \\
&\le C\,\big(\| -\Delta p + \mathbf{b}\cdot\nabla p + c_0\, p\|^2_{-1,w} + \|\mathbf{b}\cdot\nabla p\|^2_{-1,w} + \|c_0\, p\|^2_{-1,w}\big)\,.
\end{aligned}$$

By (d) of Lemma 3.3 we have

$$\|p\|^2_{1,w} \le C\,\big(\| -\Delta p + \mathbf{b}\cdot\nabla p + c_0\, p\|^2_{-1,w} + \|p\|^2_w\big)\,.$$

Applying the standard compactness argument yields the conclusion (see [3]).

Now, we establish the weighted minus one norm least squares method for the elliptic problem (3.1). Setting the flux variable $\mathbf{u} = \nabla p$, we have the first-order system of linear equations equivalent to (3.1):

$$\left\{\begin{array}{rcll}
\mathbf{u} - \nabla p &=& \mathbf{0}, & \text{in } \Omega, \\
-\nabla\cdot\mathbf{u} + \mathbf{b}\cdot\mathbf{u} + c_0\, p &=& f, & \text{in } \Omega, \\
p &=& 0, & \text{on } \partial\Omega.
\end{array}\right. \tag{3.7}$$

Define the least-squares functional corresponding to (3.7):

$$G_w(\mathbf{v}, q; f) = \|f + \nabla\cdot\mathbf{v} - \mathbf{b}\cdot\mathbf{v} - c_0\, q\|_{-1,w}^2 + \|\mathbf{v} - \nabla q\|_w^2 \tag{3.8}$$

for $(\mathbf{v}, q) \in L^2_w(\Omega)^2 \times H^1_{0,w}(\Omega)$. Denote by

$$\boldsymbol{\mathcal{V}} = L^2_w(\Omega)^2 \times H^1_{0,w}(\Omega).$$

Then the first-order system least-squares problem for (3.7) is to minimize the quadratic functional $G_w(\mathbf{v}, q; f)$ over $\boldsymbol{\mathcal{V}}$: find $(\mathbf{u}, p) \in \boldsymbol{\mathcal{V}}$ such that

$$G_w(\mathbf{u}, p; f) = \inf_{(\mathbf{v},q)\in\boldsymbol{\mathcal{V}}} G_w(\mathbf{v}, q; f), \tag{3.9}$$

and the variational problem for (3.9) is to find $(\mathbf{u}, p) \in \boldsymbol{\mathcal{V}}$ such that

$$a_w(\mathbf{u}, p; \mathbf{v}, q) = f_w(\mathbf{v}, q) \quad \forall\, (\mathbf{v}, q) \in \boldsymbol{\mathcal{V}}, \tag{3.10}$$

where the bilinear form $a_w(\cdot;\cdot)$ is given by

$$\begin{aligned} a_w(\mathbf{u}, p; \mathbf{v}, q) =& \Big(T\big(\nabla\cdot\mathbf{u} - \mathbf{b}\cdot\mathbf{u} - c_0\, p\big), \nabla\cdot\mathbf{v} - \mathbf{b}\cdot\mathbf{v} - c_0\, q\Big)_w \\ &+ \big(\mathbf{u} - \nabla p, \mathbf{v} - \nabla q\big)_w \end{aligned} \tag{3.11}$$

and the linear form $f_w(\cdot)$ is given by

$$f_w(\mathbf{v}, q) = -\big(Tf, \nabla\cdot\mathbf{v} - \mathbf{b}\cdot\mathbf{v} - c_0\, q\big)_w. \tag{3.12}$$

Now we establish the coercivity and continuity for the homogeneous weighted least-squares functional $G_w(\cdot; 0)$ over $\boldsymbol{\mathcal{V}}$:

Theorem 3.4. *There exists a positive constant C such that, for any $(\mathbf{v}, q) \in \boldsymbol{\mathcal{V}}$*

$$\frac{1}{C}\left(\|\mathbf{v}\|_w^2 + \|q\|_{1,w}^2\right) \le G_w(\mathbf{v}, q, ; 0) \le C\left(\|\mathbf{v}\|_w^2 + \|q\|_{1,w}^2\right). \tag{3.13}$$

Proof. Triangle inequality yields the upper bound. For the lower bound, let $(\mathbf{v}, q) \in \boldsymbol{\mathcal{V}}$. By (3.6), triangle inequality and (c) of Lemma 3.2, we have

$$\begin{aligned} \|q\|_{1,w}^2 &\le C\, \|\Delta q - \mathbf{b}\cdot\nabla q - c_0\, q\|_{-1,w}^2 \\ &\le C\left(\|\nabla\cdot\mathbf{v} - \mathbf{b}\cdot\mathbf{v} - c_0\, q\|_{-1,w}^2 + \|\nabla\cdot(\mathbf{v} - \nabla q)\|_{-1,w}^2 + \|\mathbf{b}\cdot(\mathbf{v} - \nabla q)\|_{-1,w}^2\right) \\ &\le C\left(\|\nabla\cdot\mathbf{v} - \mathbf{b}\cdot\mathbf{v} - c_0\, q\|_{-1,w}^2 + \|\mathbf{v} - \nabla q\|_w^2\right) \\ &\le C\, G_w(\mathbf{v}, q; 0). \end{aligned}$$

Using triangle inequality together with the last inequality, we have

$$\|\mathbf{v}\|_w^2 \le C\left(\|\mathbf{v} - \nabla q\|_w^2 + \|\nabla q\|_w^2\right) \le C\, G_w(\mathbf{v}, q; 0)$$

which completes the theorem.

4. Spectral Galerkin Approximation for Minus one Norm Least-squares

Let $\mathcal{Q}_N^0 = \mathcal{Q}_N \cap H^1_{0,w}(\Omega)$ and let $T_N : H_w^{-1}(\Omega) \longrightarrow \mathcal{Q}_N^0$ be the discrete solution operator defined by $T_N f = p_N \in \mathcal{Q}_N^0$ for $f \in H_w^{-1}(\Omega)$ where u_N is the solution of the following discrete variational problem

$$(\nabla p_N, \nabla\theta_N)_w + (p_N, \theta_N)_w = (f, \theta_N)_w \quad \forall\theta_N \in \mathcal{Q}_N^0. \tag{4.1}$$

Then we can easily show from the definition of T_N that

$$(f, T_N g)_w = (g, T_N f)_w \quad \text{for all } f,\ g \in H_w^{-1}(\Omega) \tag{4.2}$$

and

$$(f, T_N f)_w = \|T_N f\|_{1,w}^2 \quad \text{for all } f \in H_w^{-1}(\Omega). \tag{4.3}$$

We here enumerate some properties of T_N.

Proposition 4.1. *We have the following properties.*

(1) For all $f \in H_w^{-1}(\Omega)$, we have

$$(f, T_N f)_w^{1/2} = \sup_{0 \neq \theta_N \in \mathcal{Q}_N^0} \frac{(f, \theta_N)_w}{\|\theta_N\|_{1,w}} \leq \|f\|_{-1,w}.$$

(2) There exists a positive constant C such that

$$\|f\|_{-1,w} \leq C \left[N^{-1} \|f\|_w + (f, T_N f)_w^{1/2} \right], \quad \forall f \in L_w^2(\Omega).$$

Proof. For the upper bound of (1), using (4.3) and the fact that $\mathcal{Q}_N^0 \subset H_{0,w}^1(\Omega)$ we have

$$(f, T_N f)_w^{1/2} = \frac{(f, T_N f)_w}{\|T_N f\|_{1,w}} \leq \sup_{0 \neq \theta_N \in \mathcal{Q}_N^0} \frac{(f, \theta_N)_w}{\|\theta_N\|_{1,w}} \leq \sup_{0 \neq \theta \in H_{0,w}^1(\Omega)} \frac{(f, \theta)_w}{\|\theta\|_{1,w}} = \|f\|_{-1,w}.$$

Using the definition of T_N and Cauchy-Schwarz inequality yields that

$$(f, \theta_N)_w = (\nabla T_N f, \nabla \theta_N)_w + (T_N f, \theta_N)_w \leq \|T_N f\|_{1,w} \|\theta_N\|_{1,w}$$

for all $f \in H_w^{-1}(\Omega)$ and all $\theta_N \in Q_N^0$. Hence, we have

$$\sup_{0 \neq \theta_N \in \mathcal{Q}_N^0} \frac{(f, \theta_N)_w}{\|\theta_N\|_{1,w}} \leq \|T_N f\|_{1,w} = (T_N f, f)_w^{1/2}$$

which completes the conclusion (1).

Using the triangle inequality and (2.11) yields that for $f \in L_w^2(\Omega)$ and $\theta \in H_{0,w}^1(\Omega)$

$$\begin{aligned} |(f, \theta)_w| &\leq |(f, \theta - P_{1,N}^0 \theta)| + |(f, P_{1,N}^0 \theta)| \\ &\leq \|\theta - P_{1,N}^0 \theta\|_w \|f\|_w + |(f, P_{1,N}^0 \theta)| \\ &\leq C N^{-1} \|\theta\|_{1,w} \|f\|_w + |(f, P_{1,N}^0 \theta)|. \end{aligned} \tag{4.4}$$

From the fact that $\|P_{1,N}^0 \theta\|_{1,w} \leq C \|\theta\|_{1,w}$ for $\theta \in H_{0,w}^1(\Omega)$ we have

$$\|f\|_{-1,w} = \sup_{0 \neq \theta \in H_{0,w}^1(\Omega)} \frac{(f, \theta)_w}{\|\theta\|_{1,w}} \leq C N^{-1} \|f\|_w + C \sup_{0 \neq \theta \in H_{0,w}^1(\Omega)} \frac{(f, P_{1,N}^0 \theta)_w}{\|P_{1,N}^0 \theta\|_{1,w}}.$$

Now, using (1) of the proposition yields the conclusion (2).

Let

$$\mathbf{W}_N = Q_N^2, \quad V_N = Q_N^0 \quad \text{and} \quad \boldsymbol{\mathcal{V}}_N = \mathbf{W}_N \times V_N.$$

Define $\mathcal{T}_N = \alpha N^{-2} I + \beta T_N$ for fixed positive constants α and β. The parameters α and β could be used to tune the iterative convergence rate. Assume that $\alpha = \beta = 1$ for convenience.

Now we define the discrete counterpart of the least-squares functional $G_w(\cdot;\cdot)$ as follows:

$$\begin{aligned} G_{w,N}(\mathbf{v}, q; f) = &\Big(\mathcal{T}_N \big(f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0\, q \big), f + \nabla \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} - c_0\, q \Big)_w \\ &+ \big(\mathbf{v} - \nabla q, \mathbf{v} - \nabla q \big)_w. \end{aligned} \tag{4.5}$$

The discrete least-squares problem associated to (4.5) is then to minimize the quadratic functional $G_{w,N}(\mathbf{v}_N, q_N; f)$ over $\mathbf{W}_N \times V_N$ and the corresponding variational problem is to find $(\mathbf{u}_N, p_N) \in \mathbf{W}_N \times V_N$ such that

$$a_{w,N}(\mathbf{u}_N, p_N; \mathbf{v}_N, q_N) = f_{w,N}(\mathbf{v}_N, q_N), \quad \forall\, (\mathbf{v}_N, q_N) \in \mathbf{W}_N \times V_N \tag{4.6}$$

where the discrete bilinear form $a_{w,N}(\cdot;\cdot)$ and linear form $f_{w,N}(\cdot)$ are given by

$$\begin{aligned} a_{w,N}(\mathbf{u}_N, p_N; \mathbf{v}_N, q_N) = &\Big(\mathcal{T}_N \big(\nabla \cdot \mathbf{u}_N - \mathbf{b} \cdot \mathbf{u}_N - c_0\, p_N \big), \nabla \cdot \mathbf{v}_N - \mathbf{b} \cdot \mathbf{v}_N - c_0\, q_N \Big)_w \\ &+ \big(\mathbf{u}_N - \nabla p_N, \mathbf{v}_N - \nabla q_N \big)_w \end{aligned} \tag{4.7}$$

and

$$f_{w,N}(\mathbf{v}_N, q_N) = -\big(\mathcal{T}_N f, \nabla \cdot \mathbf{v}_N - \mathbf{b} \cdot \mathbf{v}_N - c_0\, q_N \big)_w. \tag{4.8}$$

Define

$$\|f\|^2_{-1,w,N} := \big(\mathcal{T}_N f, f\big)_w = N^{-2}\|f\|^2_w + \big(T_N f, f\big)_w.$$

Then we have the following inequality:

Lemma 4.1. *There exists a constant C such that*

$$\frac{1}{C}\left(N^{-2}\|f\|^2_w + \|f\|^2_{-1,w}\right) \le \|f\|^2_{-1,w,N} \le C\left(N^{-2}\|f\|^2_w + \|f\|^2_{-1,w}\right), \quad \forall f \in L^2_w(\Omega). \tag{4.9}$$

Proof. Using Proposition 4.1, one may easily prove the lemma.

The continuity and coercivity of the discrete homogeneous functional $G_{w,N}(\cdot;0)$ are shown in the following theorem.

Theorem 4.2. *For all $(\mathbf{v}, q) \in H(\mathrm{div};\Omega) \times H^1_{0,w}(\Omega)$, there exists a constant C such that*

$$\frac{1}{C}\left(\|\mathbf{v}\|^2_w + N^{-2}\|\nabla\cdot\mathbf{v}\|^2_w + \|q\|^2_{1,w}\right) \le G_{w,N}(\mathbf{v}, q; 0) \tag{4.10}$$

and

$$G_{w,N}(\mathbf{v}, q; 0) \le C\left(\|\mathbf{v}\|^2_w + N^{-2}\|\nabla\cdot\mathbf{v}\|^2_w + \|q\|^2_{1,w}\right). \tag{4.11}$$

Proof. Combining the lower bounds of Theorem 3.4 and (4.9) yields

$$\left(\|\mathbf{v}\|^2_w + \|q\|^2_{1,w}\right) \le C\,G_w(\mathbf{v}, q; 0) \le C\,G_{w,N}(\mathbf{v}, q; 0).$$

Then, we can easily show that

$$\begin{aligned} N^{-2}\|\nabla\cdot\mathbf{v}\|^2_w &\le C\,N^{-2}\left(\|\nabla\cdot\mathbf{v} - \mathbf{b}\cdot\mathbf{v} - c_0\,q\|^2_w + \|\mathbf{b}\cdot\mathbf{v} - c_0\,q\|^2_w\right) \\ &\le C\left(G_{w,N}(\mathbf{v}, q; 0) + N^{-2}\left(\|\mathbf{v}\|^2_w + \|q\|^2_{1,w}\right)\right) \\ &\le C\,G_{w,N}(\mathbf{v}, q; 0) \end{aligned}$$

which completes the conclusion (4.10).

Similarly, using the upper bounds of Theorem 3.4 and (4.9), we can derive the conclusion (4.11) :

$$\begin{aligned} G_{w,N}(\mathbf{v}, q; 0) &\le C\left(G_w(\mathbf{v}, q; 0) + N^{-2}\|\nabla\cdot\mathbf{v} - \mathbf{b}\cdot\mathbf{v} - c_0\,q\|^2_w\right) \\ &\le C\,\left(\|\mathbf{v}\|^2_w + \|q\|^2_{1,w} + N^{-2}\|\nabla\cdot\mathbf{v}\|^2_w\right). \end{aligned}$$

Now we have the following spectral convergence for the weighted spectral least-squares method.

Theorem 4.3. *Let $(\mathbf{u}, p) \in H^{s-1}_w(\Omega)^2 \times H^s_w(\Omega)$ $(s \ge 2)$ be the solution to the problem (3.7) and let $(\mathbf{u}_N, p_N) \in \mathbf{W}_N \times V_N$ be the discrete solution to the problem (4.6). Then there exists a constant C such that*

$$\|\mathbf{u} - \mathbf{u}_N\|_w + N^{-1}\|\nabla\cdot(\mathbf{u} - \mathbf{u}_N)\|_w + \|p - p_N\|_{1,w} \le C\,N^{1-s}\left(\|\mathbf{u}\|_{s-1,w} + \|p\|_{s,w}\right). \tag{4.12}$$

Proof. It is easy to show that the error $(\mathbf{u} - \mathbf{u}_N, p - p_N)$ is orthogonal to $\mathbf{W}_N \times V_N$ with respect to the scalar product $a_{w,N}(\cdot;\cdot)$ corresponding to the functional $G_{w,N}(\cdot;0)$. For any given $(\mathbf{w}_N, r_N) \in \mathbf{W}_N \times V_N$, set $(\mathbf{v}_N, q_N) = (\mathbf{u}_N, p_N) - (\mathbf{w}_N, r_N)$. From (4.10), the orthogonality and Cauchy-Schwarz inequality we have

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_N\|^2_w &+ N^{-2}\|\nabla\cdot(\mathbf{u} - \mathbf{u}_N)\|^2_w + \|p - p_N\|^2_{1,w} \\ &\le C\,a_{w,N}(\mathbf{u} - \mathbf{u}_N, p - p_N; \mathbf{u} - \mathbf{u}_N, p - p_N) \\ &\le C\,a_{w,N}(\mathbf{u} - \mathbf{u}_N, p - p_N; \mathbf{u} - \mathbf{u}_N + \mathbf{v}_N, p - p_N + q_N) \\ &\le C\,a_{w,N}(\mathbf{u} - \mathbf{u}_N, p - p_N; \mathbf{u} - \mathbf{w}_N, p - r_N) \\ &\le C\,G_{w,N}(\mathbf{u} - \mathbf{u}_N, p - p_N; 0)^{1/2} G_{w,N}(\mathbf{u} - \mathbf{w}_N, p - r_N; 0)^{1/2}. \end{aligned}$$

Using (4.11) and the approximation property (2.11) yields

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_N\|_w + N^{-1}\|\nabla\cdot(\mathbf{u}-\mathbf{u}_N)\|_w + \|p-p_N\|_{1,w} \\
&\quad\le C \inf_{(\mathbf{w}_N,r_N)\in\mathbf{W}_N\times V_N} G_{w,N}(\mathbf{u}-\mathbf{w}_N, p-r_N;0)^{1/2} \\
&\quad\le C \inf_{(\mathbf{w}_N,r_N)\in\mathbf{W}_N\times V_N} \Big(\|\mathbf{u}-\mathbf{w}_N\|_w + N^{-1}\|\nabla\cdot(\mathbf{u}-\mathbf{w}_N)\|_w + \|p-r_N\|_{1,w}\Big) \\
&\quad\le C\, N^{1-s}\Big(\|\mathbf{u}\|_{s-1,w} + \|p\|_{s,w}\Big).
\end{aligned}$$

5. Implementation and Numerical Results

The spectral Galerkin approximation developed in this paper is slightly different from the spectral collocation method in the respect of using the $L^2_w(\Omega)$ scalar product $(\cdot,\cdot)_w$ instead of the discrete scalar product $\langle\cdot,\cdot\rangle_w$. However, if the coefficients are continuous then it can be further approximated by using Gaussian quadrature formula. That is, the $L^2_w(\Omega)$ scalar product of continuous functions can be approximated by the discrete scalar product using Gauss-Lobatto points and the corresponding quadrature weights.

In this section, to find an approximate solution to the problem (4.5), we replace the $L^2_w(\Omega)$ scalar product $(\cdot,\cdot)_w$ by the discrete scalar product $\langle\cdot,\cdot\rangle_w$ in the definition of $a_{w,N}(\cdot,\cdot)$ and $f_{w,N}(\cdot)$. Also, the evaluation of $T_N f$ is based on the discrete scalar product $\langle\cdot,\cdot\rangle_w$.

Let us consider the resulting algebraic system written by the matrix problem

$$\mathcal{A}_N\mathbf{U}_N = \mathbf{F}_N.$$

Here, that the matrix $\mathcal{A}_N$ itself is never assembled because the operator T_N appeared in the bilinear form $a_{w,N}(\cdot,\cdot)$. In our experiments, we will use the preconditioning conjugate gradient method(PCGM) to solve the resulting system with the block diagonal preconditioner

$$\mathcal{P}_N = \begin{pmatrix} W_N & 0 & 0 \\ 0 & W_N & 0 \\ 0 & 0 & S_N \end{pmatrix}$$

where W_N is the diagonal weight matrix and S_N is the stiffness matrix for the Poisson problem based on the continuous bilinear functions with Gauss-Lobatto points as the nodal points.

For the computation of $\mathbf{F}_N$ we need only one calculation of $T_N f$. But, for each k-th iteration of PCGM we have to evaluate $T_N g^k$ appeared in $a_{w,N}(\cdot,\cdot)$ to compute $\mathcal{A}_N\mathbf{U}^k_N$, in which g^k is updated for the k-th iteration of PCGM. That is, for each iteration we have to solve the Poisson problem $T_N g^k$ for each g^k. In this case, the use of direct method like LU-decomposition is very efficient to solve the problem. Now we are ready to find an approximate solution to the problem (4.5). See [9] for more implementations of spectral collocation method.

Example 1. We now present the discretization errors along with various coefficients $\mathbf{b}$ and c_0. The tested exact solutions p and $\mathbf{u}=\nabla p$ are given by

$$p = \sin\left(\frac{5\pi}{2}(x+1)\right)\,\sin\left(\frac{5\pi}{2}(y+1)\right),$$

$$\mathbf{u} = \begin{pmatrix} \dfrac{5\pi}{2}\cos\left(\dfrac{5\pi}{2}(x+1)\right)\sin\left(\dfrac{5\pi}{2}(y+1)\right) \\ \dfrac{5\pi}{2}\sin\left(\dfrac{5\pi}{2}(x+1)\right)\cos\left(\dfrac{5\pi}{2}(y+1)\right) \end{pmatrix}.$$

The presented solutions are sufficiently smooth and satisfy the given boundary conditions, and by substituting the solutions into (3.1) we have the right hand side f along with several coefficients $\mathbf{b}$ and several c_0. We now present the discretization errors along with various coefficients $\mathbf{b}$ and c_0. The numerical tests in Table 1 reveal the spectral convergence in all variables for Chebyshev approximation. For Legendre approximation, we had the similar result.

$\mathbf{b}$	c_0	N	$\|e_p\|_N$	$\|\nabla e_p\|_N$	$\|e_{\mathbf{u}}\|_N$	$\|\nabla e_{\mathbf{u}}\|_N$
$\mathbf{0}$	0	4	3.5220e+01	1.1157e+02	1.2122e+02	4.1163e+02
		8	3.6143e-01	7.8727e+00	1.0615e+01	2.2053e+02
		12	2.7088e-03	1.0315e-01	2.8673e-01	1.3209e+01
		16	1.4157e-05	5.8509e-04	1.7285e-03	1.2032e-01
		20	2.8630e-08	1.2031e-06	3.4496e-06	3.3039e-04
		24	5.8809e-10	4.6548e-09	3.3221e-09	4.7251e-07
$\mathbf{0}$	-10	4	3.7720e+01	1.2392e+02	1.1547e+02	2.6911e+02
		8	5.4535e-01	1.0395e+01	1.0293e+01	2.1004e+02
		12	3.5045e-03	1.1181e-01	2.8211e-01	1.3064e+01
		16	1.8343e-05	6.3756e-04	1.7045e-03	1.1912e-01
		20	3.4943e-08	1.3075e-06	3.4099e-06	3.2751e-04
		24	3.5833e-10	6.2521e-09	4.0418e-09	2.5759e-07
$(6,9)^t$	0	4	4.6486e+00	2.7567e+01	3.1039e+01	1.6131e+02
		8	5.6884e-01	7.4639e+00	8.8815e+00	1.9018e+02
		12	6.1864e-03	1.2676e-01	2.2626e-01	1.1290e+01
		16	2.8261e-05	7.1076e-04	1.4749e-03	1.1005e-01
		20	4.7522e-08	1.4226e-06	3.1816e-06	3.2083e-04
		24	4.9357e-10	4.8166e-09	4.9100e-09	3.9322e-07

Table 1. Discretization errors for Chebyshev approximation

Example 2. Now let us consider the following problem:

$$\begin{cases} -\nabla\cdot(a\nabla p)+\mathbf{b}\cdot\nabla p+c_0\,p &= f, \quad \text{in } \Omega,\\ p &= 0, \quad \text{on } \partial\Omega. \end{cases}$$

The discontinuous diffusion coefficient $a(x,y)$ is defined by

$$a(x,y)=\begin{cases} 1, & x\le 0,\\ \sigma, & x>0. \end{cases}$$

We construct an exact solution such that

$$p(x,y)=\begin{cases} \Big((\sigma-2)(x+1)^2+(4-\sigma)(x+1)\Big)\sin\left(\frac{\pi}{2}(y+1)\right), & x\le 0,\\ \Big(-3(x+1)^2+7(x+1)-2\Big)\sin\left(\frac{\pi}{2}(y+1)\right), & x>0. \end{cases}$$

Then $\mathbf{u}=a(x,y)\nabla p$ is not a $H^1_w(\Omega)^2$ function if $\sigma\ne 1$ so that $p\notin H^2_w(\Omega)$. If $\sigma=1$, then $\mathbf{u}$ belongs to $H^1_w(\Omega)^2$ but not $H^2_w(\Omega)^2$ so that $p\in H^2_w(\Omega)$ but $p\notin H^3_w(\Omega)$. The numerical tests were performed with $\sigma=1,10$ and 100. When $\sigma=1$, as given in Theorem 4.3, Table 2 shows that the convergence rate is $O(N^{1-s})=O(N^{-1})$ with $s=2$. The convergence rates deteriorate as the σ becomes larger and larger. For the large σ having big jump at $x=0$ it is hard to see the spectral convergence. The spectral convergence will occur when the given problem has enough regularity. But, if we divide full domain Ω into two elements $(-1,0)\times(-1,1)$ and $(0,1)\times(-1,1)$ and then apply the developed spectral method, i.e., spectral element least-squares method, then we may get full spectral convergence. Such approaches for elliptic problem and Stokes problem having discontinuous variable coefficients are the subjects of forthcoming papers.

$\mathbf{b}$	c_0	N	$\|e_p\|_N$	$\|\nabla e_p\|_N$	$\|e_{\mathbf{u}}\|_N$	$\|\nabla e_{\mathbf{u}}\|_N$
$\mathbf{0}$	0	3	9.7361e-02	3.3358e-01	5.5379e-01	2.7500e+00
		7	9.7096e-03	7.7014e-02	1.2171e-01	1.3985e+00
		11	4.2600e-03	3.3178e-02	6.0676e-02	8.7918e-01
		15	2.3930e-03	1.8841e-02	3.8669e-02	6.3789e-01
		19	1.5331e-03	1.2370e-02	2.7541e-02	5.0799e-01
		23	1.0663e-03	8.8622e-03	2.0930e-02	4.2919e-01
		27	7.8471e-04	6.7279e-03	1.6610e-02	3.7684e-01
		31	6.0170e-04	5.3221e-03	1.3601e-02	3.3956e-01
$\mathbf{0}$	-10	3	1.9291e-01	8.5955e-01	6.1092e-01	2.2908e+00
		7	3.4282e-02	1.7354e-01	1.8573e-01	1.2565e+00
		11	1.9961e-02	9.0728e-02	1.0964e-01	9.1298e-01
		15	1.0961e-02	4.9855e-02	6.3808e-02	6.6097e-01
		19	6.9110e-03	3.1590e-02	4.2299e-02	5.2240e-01
		23	4.7489e-03	2.1850e-02	3.0429e-02	4.3842e-01
		27	3.4612e-03	1.6041e-02	2.3138e-02	3.8298e-01
		31	2.6335e-03	1.2297e-02	1.8311e-02	3.4381e-01
$(6,9)^t$	0	3	1.1094e-01	4.3818e-01	5.5240e-01	2.9492e+00
		7	2.1039e-02	1.3116e-01	1.1845e-01	1.2393e+00
		11	9.8310e-03	6.5187e-02	6.9844e-02	1.1932e+00
		15	5.0937e-03	3.5018e-02	4.3470e-02	1.1683e+00
		19	3.1178e-03	2.1861e-02	3.0490e-02	1.1229e+00
		23	2.1084e-03	1.4982e-02	2.2847e-02	1.0525e+00
		27	1.5227e-03	1.0942e-02	1.7892e-02	9.7154e-01
		31	1.1521e-03	8.3685e-03	1.4477e-02	8.8995e-01

Table 4. Discretization errors for Chebyshev approximation

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