Journal of Computational Mathematics, Vol.26, No.1, 2008, 98–111.

# COMPACT FOURTH-ORDER FINITE DIFFERENCE SCHEMES FOR HELMHOLTZ EQUATION WITH HIGH WAVE NUMBERS*

Yiping Fu
*School of Mathematical Sciences, South China University of Technology, Guangzhou 510640, China*
*Email: fuyiping@scut.edu.cn*

**Abstract**

In this paper, two fourth-order accurate compact difference schemes are presented for solving the Helmholtz equation in two space dimensions when the corresponding wave numbers are large. The main idea is to derive and to study a fourth-order accurate compact difference scheme whose leading truncation term, namely, the $\mathcal{O}(h^4)$ term, is independent of the wave number and the solution of the Helmholtz equation. The convergence property of the compact schemes are analyzed and the implementation of solving the resulting linear algebraic system based on a FFT approach is considered. Numerical results are presented, which support our theoretical predictions.

*Mathematics subject classification:* 65M06, 65N12.
*Key words:* Helmholtz equation, Compact difference scheme, FFT algorithm, Convergence.

## 1. Introduction

In this paper, we consider two-dimensional Helmholtz equation

$$\nabla^2 u + k^2 u = f(x, y), \tag{1.1}$$

where $k$ is wave number, together with some appropriate boundary conditions. Boundary value problems governed by the Helmholtz equation describe many physical phenomena and have important applications in acoustic and electromagnetic waves.

When the wave number $k$ is very large, Eq. (1.1) has a great difficulty in computation because in this case the solutions of Eq. (1.1) are highly oscillatory. There exist many different numerical methods for solving the Helmholtz equation, such as Galerkin finite element method [1], spectral method [8,12] and finite difference method [2,4,14]. Many of the proposed schemes can provide very accurate approximations to the highly oscillatory solutions under the condition that $kh$ is very small, where $h$ is a characteristic spatial grid size. This condition shows that in order to attain accurate approximate solutions, it is required to significantly decrease $h$ with large wave number $k$.

On the other hand, in recent years, high-order accurate compact finite difference methods have been used widely for solving convection-diffusion problems, the Navier-Stokes equations and the Helmholtz equation [3-7,10,11,13,14]. This class of methods is attractive since they offer a means to obtain high accuracy solutions with less computational costs. In this paper, we will use the compact finite difference methods to deal with Eq. (1.1). We first derive two fourth-order compact finite difference schemes for the problem (1.1), and then provide some convergence analysis for the two methods. The main difference of the two proposed schemes is

* Received January 5, 2007 / Revised version received June 25, 2007 / Accepted July 4, 2007 /

about the coefficient of the leading truncation errors: the coefficient of one of the schemes is independent of the wave number $k$ and the solution of (1.1) (the solution is in general depends on $k$ also). Consequently, it is expected that this scheme will be useful for solving Eq. (1.1) with large wave number $k$. Moreover, in this work we also apply the fast Fourier transform (FFT) algorithm to solve the algebraic system resulting from the compact finite difference discretizations. This significantly speeds up the computational efficiency.

The rest of the paper is organized as follows. In Section 2, two fourth-order compact finite difference schemes are presented. In Section 3, the convergence analysis of the proposed schemes for one- and two-dimensional Helmholtz equation is provided. Numerical implementation based on a FFT approach is given in Section 4. In Section 5, numerical experiments are carried out to verify the theoretical predictions obtained in this work.

## 2. Fourth-order Compact Schemes

We consider Eq. (1.1) with Dirichlet and Neumann boundary conditions. For ease of notations, we only consider a simple square domain $\Omega = (0,1)\times(0,1)$ with $\Delta x = \Delta y$, but the main ideas in this work can be extended to rectangular domains with $\Delta x \neq \Delta y$. Divide uniformly $\Omega$ with lines $\{(x_i, y_j) : x_i = ih, y_j = jh, i, j = 0, 1, \cdots, J\}$, where $h$ is the spacial mesh-size. Use the notation $\delta_x^2$, $\delta_y^2$ to denote the second-order central difference with respect to $x$, $y$, respectively:

$$\delta_x^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \quad \delta_y^2 u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}.$$

By the Taylor series expansion, we get for every sufficiently smooth $u$

$$\delta_x^2 u = u_{xx} + \frac{h^2}{12} u_{x^4} + \frac{h^4}{360} u_{x^6} + \mathcal{O}(h^6),$$

here and below for simplicity, we omit subscripts $i, j$ whenever confusions will not occur. Adding a similar expression for $\delta_y^2$ and rearranging the resulting terms give

$$u_{xx} + u_{yy} = (\delta_x^2 + \delta_y^2)u - \frac{h^2}{12}(u_{x^4} + u_{y^4}) - \frac{h^4}{360}(u_{x^6} + u_{y^6}) + \mathcal{O}(h^6). \tag{2.1}$$

Similarly, there is the expression

$$u_{x^2y^2} = \delta_x^2 \delta_y^2 u - \frac{h^2}{12}(u_{x^4y^2} + u_{x^2y^4}) + \mathcal{O}(h^4). \tag{2.2}$$

Inserting (2.1) into the Helmholtz equation (1.1) gives

$$(\delta_x^2 + \delta_y^2)u - \frac{h^2}{12}(u_{x^4} + u_{y^4}) - \frac{h^4}{360}(u_{x^6} + u_{y^6}) + k^2 u = f + \mathcal{O}(h^6). \tag{2.3}$$

In order to obtain the fourth-order accuracy, we need to approximate the term $u_{x^4} + u_{y^4}$ to $\mathcal{O}(h^2)$. By using the original equation (1.1) and the expressions (2.1), (2.2), we have

$$\begin{aligned} u_{x^4} + u_{y^4} &= \Delta^2 u - 2u_{x^2y^2} = \Delta(f - k^2 u) - 2u_{x^2y^2} = \Delta f - k^2(\delta_x^2 + \delta_y^2)u \\ &\quad -2\delta_x^2\delta_y^2 u + \frac{k^2h^2}{12}(u_{x^4} + u_{y^4}) + \frac{h^2}{6}(u_{x^2y^4} + u_{y^2x^4}) + \mathcal{O}(h^4). \end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4) yields the following fourth order accurate finite difference scheme

$$\left(1+\frac{k^2h^2}{12}\right)(\delta_x^2+\delta_y^2)u+\frac{h^2}{6}\delta_x^2\delta_y^2u+k^2u=f+\frac{h^2}{12}(\delta_x^2+\delta_y^2)f, \tag{2.5}$$

where the truncation error is

$$\begin{aligned}T^{(1)}=&\Big[\frac{k^2}{144}(u_{x^4}+u_{y^4})+\frac{1}{360}(u_{x^6}+u_{y^6})\\&+\frac{1}{72}(u_{x^2y^4}+u_{y^2x^4})-\frac{1}{144}(f_{x^4}+f_{y^4})\Big]h^4.\end{aligned} \tag{2.6}$$

The scheme (2.5) is a standard fourth-order accurate formula. However, in many physical problems, the source function $f(x,y)$ usually have no relationship with the wave number $k$. So in order to make the scheme more efficient for large $k$, we naturally wish that the truncation error $T^{(1)}$ has less relevance to $k$. Hence we try to drop the terms dependent on $k$ in the truncation error $T^{(1)}$. To this end, we need to provide better second order approximations to the terms $u_{x^6}+u_{y^6}$ and $u_{x^2y^4}+u_{y^2x^4}$. Observe

$$\begin{aligned}&u_{x^2y^4}+u_{y^2x^4}=(\Delta u)_{x^2y^2}=(f-k^2u)_{x^2y^2}\\&=f_{x^2y^2}-k^2u_{x^2y^2}=f_{x^2y^2}-k^2\delta_x^2\delta_y^2u+\mathcal{O}(h^2),\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&u_{x^6}+u_{y^6}=\Delta^3u-3(u_{x^2}+u_{y^2})_{x^2y^2}\\&=\Delta^2(f-k^2u)-3(u_{x^2}+u_{y^2})_{x^2y^2}=\Delta^2f-k^2\Delta^2u-3(u_{x^2}+u_{y^2})_{x^2y^2}\\&=\Delta^2f-k^2(\Delta f-k^2\Delta u)-3(u_{x^2}+u_{y^2})_{x^2y^2}\\&=-k^2(\delta_x^2+\delta_y^2)f+k^4(\delta_x^2+\delta_y^2)u+3k^2\delta_x^2\delta_y^2u+\Delta^2f-3f_{x^2y^2}+\mathcal{O}(h^2).\end{aligned} \tag{2.8}$$

Inserting (2.4), (2.7) and (2.8) into the truncation error $T^{(1)}$ yields

$$\begin{aligned}T^{(1)}=&\Big\{-\frac{k^4}{240}(\delta_x^2+\delta_y^2)u-\frac{7k^2}{360}\delta_x^2\delta_y^2u+\frac{k^2}{240}(\delta_x^2+\delta_y^2)f\\&+\frac{1}{360}\Delta^2f+\frac{1}{180}f_{x^2y^2}-\frac{1}{144}(f_{x^4}+f_{y^4})\Big\}h^4+\mathcal{O}(h^6).\end{aligned} \tag{2.9}$$

Adding the first three terms on the right side of (2.9), which are all relevant to $k$, to the finite difference scheme (2.5) lead to another fourth-order accuracy compact difference scheme

$$\begin{aligned}&\left(1+\frac{k^2h^2}{12}+\frac{k^4h^4}{240}\right)(\delta_x^2+\delta_y^2)u+\left(\frac{h^2}{6}+\frac{7k^4h^4}{360}\right)\delta_x^2\delta_y^2u+k^2u\\&=f+\left(\frac{h^2}{12}+\frac{k^2h^4}{240}\right)(\delta_x^2+\delta_y^2)f,\end{aligned} \tag{2.10}$$

and the corresponding truncation error is

$$T^{(2)}=\left[\frac{1}{360}\Delta^2f+\frac{1}{180}f_{x^2y^2}-\frac{1}{144}(f_{x^4}+f_{y^4})\right]h^4+\mathcal{O}(h^6). \tag{2.11}$$

In general, the source term $f$ is independent of the wave number $k$ — in this sense the leading truncation error for the scheme (2.10) is independent of the wave number $k$. When $h$ is sufficient small, the term $\mathcal{O}(h^4)$ will play the leading role in the truncation errors. Hence, it is expected that the accuracy of the scheme (2.10) may be better than that of the scheme (2.5). If $\Delta f$ is known analytically, it can be used directly in (2.5) and (2.10).

### 2.1. Neumann boundary conditions

The schemes (2.5) and (2.10) are of fourth-order accuracy in interior grid points $(x_i, y_j)$. The compact setting gives a straightforward treatment for the Dirichlet boundary conditions. However, for Neumann boundary conditions, it is uncertain that the overall truncation errors on $\overline{\Omega}$ remain $\mathcal{O}(h^4)$; this certainly depends on the approximations of the Neumann boundary conditions. Here, we introduce a method for Eq. (1.1) with Neumann boundary condition such that the schemes (2.5) and (2.10) remain fourth-order accuracy in the sense of truncation errors on the whole domain $\overline{\Omega}$. For simplicity (and without loss of generality), we consider the boundary conditions

$$u\Big|_{x=0,1} = 0, \quad u\Big|_{y=0} = 0, \quad \frac{\partial u}{\partial y}\Big|_{y=1} = g(x).$$

In order to approximate Neumann boundary condition in a manner consistent with our fourth-order difference schemes, we need to derive a fourth-order approximation for the boundary condition $\frac{\partial u}{\partial y}|_{y=1} = g(x)$. For this purpose, we place a row of *ghost* points: $(x_i, y_{J+1}), 0 \le i \le J, y_{J+1} = (J+1)h$, outside of the region $\overline{\Omega}$ and express $\frac{\partial u}{\partial y}|_{y=1}$ by

$$\frac{\partial u}{\partial y}\Big|_{y=1} \approx \frac{u_{i,J+1} - u_{i,J-1}}{2h}.$$

Using the Taylor expansion at $(x_i, y_J)$ gives

$$\frac{u_{i,J+1} - u_{i,J-1}}{2h} = (u_y)_{i,J} + \frac{h^2}{6}(u_{yyy})_{i,J} + \mathcal{O}(h^4).$$

Assuming $f$ is sufficiently smooth on $\overline{\Omega}$ and adding Eq. (1.1) on the boundary $0 \le x \le 1, y = 1$, as we did before, we have

$$\begin{aligned}
&\frac{u_{i,J+1} - u_{i,J-1}}{2h} \\
&= g_i + \frac{h^2}{6}(f_y - k^2 u_y - u_{xxy})_{i,J} + \mathcal{O}(h^4) \\
&= g_i + \frac{h^2}{6}(f_y)_{i,J} - \frac{k^2h^2}{6}\frac{u_{i,J+1} - u_{i,J-1}}{2h} - \frac{h^2}{6}\frac{\delta_x^2 u_{i,J+1} - \delta_x^2 u_{i,J-1}}{2h} + \mathcal{O}(h^4),
\end{aligned}$$

which leads to a fourth-order approximation expression for the boundary condition $\frac{\partial u}{\partial y}|_{y=1} = g(x)$:

$$\left(1 + \frac{k^2h^2}{6}\right)\frac{u_{i,J+1} - u_{i,J-1}}{2h} + \frac{h^2}{6}\frac{\delta_x^2 u_{i,J+1} - \delta_x^2 u_{i,J-1}}{2h} = g_i + \frac{h^2}{6}(f_y)_{i,J}. \tag{2.12}$$

To eliminate the values of $u$ at the ghost points, we add the difference equation (2.5) at the boundary points $(x_i, y_J)(1 \le i \le J-1)$,

$$\left(1 + \frac{k^2h^2}{12}\right)(\delta_x^2 + \delta_y^2)u_{i,J} + \frac{h^2}{6}\delta_x^2\delta_y^2 u_{i,J} + k^2 u_{i,J} = f_{i,J} + \frac{h^2}{12}\Delta f_{i,J}. \tag{2.13}$$

By eliminating the terms $u_{i,J+1}$ from (2.12) with the use of (2.13), we obtained a fourth-order boundary condition consistent with the compact scheme (2.5). Similar treatment can be made to provide a fourth-order boundary condition consistent with the compact scheme (2.10). This can be done by replacing (2.13) with (2.10) at $j = J$.

## 3. Convergence Analysis

In this section, we give a convergence analysis for the schemes (2.5) and (2.10) with Dirichlet boundary condition. For Neumann boundary condition, the treatment is basically similar to the former.

To begin, we first give the following norm notations which will be used in the later contexts. For the vector $u = \{u_i\}_{i=1}^{J-1} \in \mathbf{R}^{J-1}$,

$$\|u\| = \left(\sum_{i=1}^{J-1} |u_i|^2 h\right)^{\frac{1}{2}}, \qquad \|u\|_\infty = \sup_{1\le i\le J-1} |u_i|,$$

and for the vector $u = \{u_{i,j}\}_{i,j=1}^{J-1} \in \mathbf{R}^{(J-1)\times(J-1)}$,

$$|||u||| = \left(\sum_{i=1}^{J-1}\sum_{j=1}^{J-1} |u_{i,j}|^2 h^2\right)^{\frac{1}{2}}, \qquad |||u|||_\infty = \sup_{1\le i,j\le J-1} |u_{i,j}|.$$

For $u_j = \{u_{1,j}, u_{2,j}, \cdots, u_{J-1,j}\}^T$, with $1 \le j \le J-1$, we have

$$|||u||| = \left(\sum_{j=1}^{J-1} \|u_j\|^2 h\right)^{\frac{1}{2}}.$$

### 3.1. One dimensional case

Before we give the proof of the fourth-order convergence for the schemes (2.5) and (2.10), we first consider the same problem for the one dimensional Helmholtz equation. It will be demonstrated in next section that our two-dimensional schemes can be splitted to several one-dimensional problems, so a good understanding of the 1D convergence property will be very useful. Moreover, one-dimensional case itself is also interesting.

Similar to the derivation of the previous section, the fourth order compact finite difference for the one dimensional Helmholtz equation

$$u_{xx} + k^2 u = f, \quad 0 < x < 1, \tag{3.1}$$

is given by

$$\left(1 + \frac{k^2h^2}{12}\right)\delta_x^2 u + k^2 u = f + \frac{h^2}{12}\delta_x^2 f, \tag{3.2}$$

and the corresponding truncation error is $T = \mathcal{O}(h^4)$.

We rewrite (3.2) as follows

$$\left(-1 - \frac{k^2h^2}{12}\right)u_{i-1} + \left(2 - k^2h^2 + \frac{k^2h^2}{6}\right)u_i + \left(-1 - \frac{k^2h^2}{12}\right)u_{i+1} = h^2(f_i + \frac{h^2}{12}\delta_x^2 f_i), \tag{3.3}$$

or the simple matrix form

$$DU = \vec{b}, \tag{3.4}$$

where $D$ is a symmetric tri-diagonal matrix. When $h \to 0$ and $kh \to 0$, the coefficient matrix $D$ tends to the matrix

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}_{(J-1)\times(J-1)}. \tag{3.5}$$

It is well known that the matrix (3.5) is positive definite. Hence when $kh$ is sufficiently small, the coefficient matrix $D$ is also positive definite. So under the condition that $kh$ is sufficiently small, (3.2) has a unique solution.

Assume that $v_i$ is the value of the exact solution of (3.1) at the grid points $x_i = ih, (i = 0, 1, \cdots, J)$, and let $e_i = u_i - v_i$. Then the error vector $E = \{e_i\}$ satisfies

$$\left(1 + \frac{k^2h^2}{12}\right) \delta_x^2 e_i + k^2 e_i = -T_i. \tag{3.6}$$

We rewrite (3.6) as follows

$$\left(-1 - \frac{k^2h^2}{12}\right) e_{i-1} + \left(2 - k^2h^2 + \frac{k^2h^2}{6}\right) e_i + \left(-1 - \frac{k^2h^2}{12}\right) e_{i+1} = h^2 T_i, \tag{3.7}$$

or the simple matrix form

$$DE = h^2 T,$$

It is well known that the matrix (3.5) has the eigenvalues

$$\lambda_j = 2 - 2\cos\frac{j\pi}{J} = 4\sin^2\frac{j\pi h}{2}$$

and the corresponding eigenvectors

$$\eta_j = (\eta_{1,j}, \eta_{2,j}, \cdots, \eta_{J-1,j})^T, \quad \eta_{m,j} = \sin\frac{mj\pi}{J}, \qquad m = 1, \cdots, J-1 \tag{3.8}$$

for $j = 1, \cdots, J-1$. When $h$ is sufficiently small, the smallest eigenvalue behaves like

$$\min_{1\le j\le J-1} \lambda_j = 4\sin^2\frac{\pi h}{2} \sim \pi^2 h^2. \tag{3.9}$$

Since the eigenvector set $\{\eta_j\}$ constitutes a full orthogonal basis of $\mathbf{R}^{J-1}$, we can write the error vector

$$E = \sum_{j=1}^{J-1} a_j \eta_j, \quad \|E\|^2 = \sum_{j=1}^{J-1} |a_j|^2 \|\eta_j\|^2.$$

Now consider the inner product $(DE, E) = (h^2T, E)$. When $kh \to 0$, we have

$$(DE, E) = (D\sum_{j=1}^{J-1} a_j\eta_j, \sum_{j=1}^{J-1} a_j\eta_j) \to (\sum_{j=1}^{J-1} a_j\lambda_j\eta_j, \sum_{j=1}^{J-1} a_j\eta_j)$$
$$= \sum_{j=1}^{J-1} \lambda_j |a_j|^2 \|\eta_j\|^2 \ge C\pi^2h^2 \sum_{j=1}^{J-1} |a_j|^2 \|\eta_j\|^2 = C\pi^2h^2\|E\|^2, \tag{3.10}$$

where (and hereafter) $C$ denotes a generic constant which may have different value at different place. On the other hand, we have

$$(h^2T, E) \leq h^2\|T\|\|E\|. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\|E\| \leq C\|T\|$$

which leads to the following result.

**Theorem 3.1.** *Suppose $kh$ is sufficiently small, where $k$ is the wave number and $h$ is the grid length. Then the scheme* (3.2) *has a unique solution. Moreover, for the sufficiently smooth solution $u$ of* (3.1) *and the approximate solution $u_h$ of the scheme* (3.2)*, with Dirichlet or Neumann boundary condition, there exists the following error estimate*

$$\|u - u_h\| \leq Ch^4,$$

*where $C$ is a constant dependent of $k$, $u$ and $f$.*

Note that if the boundary condition is given by a mixed boundary condition $u_x|_{x=0} = a, u|_{x=1} = b$, then the limiting matrix corresponding to (3.5) is of the same form but with the size $J \times J$ which has the eigenvalues

$$\lambda_j = 2 - 2\cos\frac{(2j-1)\pi}{2J}$$

and the corresponding eigenvectors

$$\eta_j = (\eta_{1,j}, \eta_{2,j}, \cdots, \eta_{J,j})^T, \quad \eta_{m,j} = \cos\frac{(m-1)(2j-1)\pi}{2J}, \qquad m = 1, \cdots, J,$$

for $j = 1, \cdots, J$.

If the boundary condition is given by the Neumann boundary condition $u_x|_{x=0} = a, u_x|_{x=1} = b$, then the limiting matrix corresponding to (3.5) is of the form same to (3.5) but with size $(J+1) \times (J+1)$ which has the eigenvalues

$$\lambda_j = 2 - 2\cos\frac{(2j-1)\pi}{J}$$

and the corresponding eigenvectors

$$\eta_j = (\eta_{1,j}, \eta_{2,j}, \cdots, \eta_{J+1,j})^T, \quad \eta_{m,j} = \cos\frac{(m-1)(2j-1)\pi}{J}, \qquad m = 1, \cdots, J+1,$$

for $j = 1, \cdots, J+1$.

In the above two cases, the results of Theorem 3.1 can be established similar to the Dirichlet boundary condition case.

### 3.2. Two dimensional case

In the same manner, we may obtain that when $kh$ is sufficiently small, there is a unique solution for (2.5) or (2.10).

Assume $v_{i,j}$ is the value of the exact solution $v(x, y)$ of Eq. (1.1) at the grid point $(x_i, y_j)$, and let $e_{i,j} = u_{i,j} - v_{i,j}$. Then the error $E = \{e_{i,j}\}$ satisfies

$$\left(1 + \frac{k^2h^2}{12}\right)(\delta_x^2 + \delta_y^2)e_{i,j} + \frac{h^2}{6}\delta_x^2\delta_y^2 e_{i,j} + k^2 e_{i,j} = -T_{i,j}^{(1)}, \tag{3.12}$$

and

$$\left(1 + \frac{k^2h^2}{12} + \frac{k^4h^4}{240}\right)(\delta_x^2 + \delta_y^2)e_{i,j} + \left(\frac{h^2}{6} + \frac{7k^4h^4}{360}\right)\delta_x^2\delta_y^2 e_{i,j} + k^2 e_{i,j} = -T_{i,j}^{(2)} \tag{3.13}$$

for the schemes (2.5) and (2.10), respectively.

For $1 \le j \le J-1$, denote $E_j = (e_{1,j}, \cdots, e_{J-1,j})^T$ and $T_j^{(l)} = (T_{1,j}^{(l)}, \cdots, T_{J-1,j}^{(l)})^T$. Let

$$E_j = \sum_{m=1}^{J-1} a_{mj}\eta_m, \qquad T_j^{(l)} = \sum_{m=1}^{J-1} \widehat{T}_{mj}^{(l)}\eta_m, \qquad l = 1, 2. \tag{3.14}$$

Then we have

$$\begin{aligned} |||E|||^2 &= \sum_{j=1}^{J-1} \|E_j\|^2 h = \sum_{j=1}^{J-1}\sum_{m=1}^{J-1} |a_{m,j}|^2 \|\eta_m\|^2 h \\ &= \sum_{m=1}^{J-1}\Big(\sum_{j=1}^{J-1} |a_{m,j}|^2 h\Big)\|\eta_m\|^2 = \sum_{m=1}^{J-1} \|a_m\|^2\|\eta_m\|^2, \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} |||T^{(l)}|||^2 &= \sum_{j=1}^{J-1} \|T_j^{(l)}\|^2 h = \sum_{j=1}^{J-1}\sum_{m=1}^{J-1} |\widehat{T}_{m,j}^{(l)}|^2 \|\eta_m\|^2 h \\ &= \sum_{m=1}^{J-1}\Big(\sum_{j=1}^{J-1} |\widehat{T}_{m,j}^{(l)}|^2 h\Big)\|\eta_m\|^2 = \sum_{m=1}^{J-1} \|\widehat{T}_m^{(l)}\|^2\|\eta_m\|^2, \end{aligned} \tag{3.16}$$

where $a_m = (a_{m,1}, \cdots, a_{m,J-1})$, $\widehat{T}_m = (\widehat{T}_{m,1}, \cdots, \widehat{T}_{m,J-1})$. Hence, in order to estimate $|||E|||$, it is only necessary to estimate $\|a_m\|$ for each $m$.

With the expressions (3.8) for the eigenvectors, (3.14) can be rewritten as

$$e_{i,j} = \sum_{m=1}^{J-1} a_{m,j}\sin im\pi h, \qquad T_{i,j}^{(l)} = \sum_{m=1}^{J-1} \widehat{T}_{m,j}^{(l)}\sin im\pi h, \qquad l = 1, 2. \tag{3.17}$$

Inserting the above relations into Eqs. (2.5) and (2.10), respectively, we get

$$A_m a_{m,j-1} + B_m a_{m,j} + A_m a_{m,j+1} = h^2\widehat{T}_{m,j}^{(l)}, \qquad l = 1, 2 \tag{3.18}$$

where, for scheme (2.5)

$$\begin{aligned} A_m &= -\left(1 + \frac{k^2h^2}{12} - \frac{2}{3}\sin^2\frac{m\pi h}{2}\right), \\ B_m &= \left(2 + 4\sin^2\frac{m\pi h}{2}\right)\left(1 + \frac{k^2h^2}{12}\right) - \frac{4}{3}\sin^2\frac{m\pi h}{2} - k^2h^2, \end{aligned} \tag{3.19}$$

and for scheme (2.10)

$$A_m = -\left[1+\frac{k^2h^2}{12}+\frac{k^4h^4}{240}-4\left(\frac{1}{6}+\frac{7k^2h^2}{360}\right)\sin^2\frac{m\pi h}{2}\right],$$
$$B_m = \left(2+4\sin^2\frac{m\pi h}{2}\right)\left(1+\frac{k^2h^2}{12}+\frac{k^4h^4}{240}\right)-\left(\frac{4}{3}+\frac{7k^2h^2}{45}\right)\sin^2\frac{m\pi h}{2}-k^2h^2. \quad (3.20)$$

Assume $h \to 0$, $kh \to 0$. Then the coefficient matrix of Eq. (3.18) also tends to the matrix (3.5). Thanks to the analysis of subsection 3.1, we know that for each $m$, $1 \le m \le J-1$, the following inequality holds:

$$\|a_m\| \le C\|\widehat{T}_m^{(l)}\|, \qquad l = 1, 2.$$

Hence

$$|||E|||^2 = \sum_{m=1}^{J-1} \|a_m\|^2 \|\eta_m\|^2 \le C \sum_{m=1}^{J-1} \|\widehat{T}_m^{(l)}\|^2 \|\eta_m\|^2 = C|||T|||^2. \quad (3.21)$$

So the following result can be obtained.

**Theorem 3.2.** *Suppose $kh$ is sufficiently small, where $k$ is the wave number and $h$ is the grid length. Then schemes* (2.5) *or* (2.10) *have a unique solution. Moreover, for sufficiently smooth solution $u$ of* (1.1) *and the approximate solution $u_h$ of schemes* (2.5) *or* (2.10)*, with Dirichlet or Neumann boundary condition, there exists the following error estimate*

$$|||u - u_h||| \le Ch^4,$$

*where $C$ is a constant dependent of $k$, $u$ and $f$.*

In fact, by the expressions of the truncation errors (2.6) and (2.11), the relations of the error estimates in Theorems 3.1 and 3.2 can be expressed as

$$\|u - u_h\| \le C\left(k^2\|u\|_{C^4(\overline{\Omega})} + \|u\|_{C^6(\overline{\Omega})}\right)h^4 \quad (3.22)$$

for the scheme (2.5), and

$$\|u - u_h\| \le Ch^4 + C\left(k^2\|u\|_{C^6(\overline{\Omega})} + \|u\|_{C^8(\overline{\Omega})}\right)h^6 \quad (3.23)$$

for the scheme (2.10), where the constant $C$ is independent of $k$ and $u$.

## 4. A Fast Algorithm Based on FFT

In this section, we assume that the boundary condition satisfies $u(0, y) = u(1, y) = 0$, i.e.,

$$u_{0,j} = u_{J,j} = 0, \qquad 0 \le j \le J.$$

If this assumption is not satisfied, then a simple linear transformation will make this assumption true. We will seek the solution of scheme (2.5) or (2.10) in the following form:

$$u_{i,j} = \sum_{m=1}^{J-1} a_{m,j} \sin im\theta, \qquad (0 \le i, j \le J) \quad (4.1)$$

where $\theta = \pi/J$. Eq. (4.1) is also called the fast Fourier sine transform (see, e.g., [9]). Here the number $a_{m,j}$ are unknowns that we wish to determine. Once the $a_{m,j}$ are determined, the fast Fourier sine transform can be used to compute $u_{i,j}$ efficiently.

Inserting (4.1) into the second-order central difference $\delta_x^2 u_{i,j}$ and $\delta_y^2 u_{i,j}$ yields

$$\delta_x^2 u_{i,j} = 4\sum_{m=1}^{J-1} a_{m,j} \sin^2 \frac{m\theta}{2} \sin m\theta,$$

$$\delta_y^2 u_{i,j} = \sum_{m=1}^{J-1} (a_{m,j+1} - 2a_{m,j} + a_{m,j-1}) \sin im\theta. \tag{4.2}$$

Substituting the above two expressions into the schemes (2.5) and (2.10) respectively, we can deduce the following trigonometric system

$$A_m a_{m,j-1} + B_m a_{m,j} + A_m a_{m,j+1} = D_{m,j}^{(l)}, \qquad 1 \le j \le J-1, \tag{4.3}$$

where $A_m, B_m$ are of the same sense as in the previous section, namely, (3.19) or (3.20), and $D_{m,j}^{(l)}$, $(l = 1, 2)$ are determined by the following sine transformations

$$f_{i,j} + \frac{h^2}{12}(\delta_x^2 + \delta_y^2) f_{i,j} = \sum_{m=1}^{J-1} D_{m,j}^{(1)} \sin im\theta, \qquad 1 \le j \le J-1,$$

$$f_{i,j} + (\frac{h^2}{12} + \frac{k^2 h^4}{240})(\delta_x^2 + \delta_y^2) f_{i,j} = \sum_{m=1}^{J-1} D_{m,j}^{(2)} \sin im\theta, \qquad 1 \le j \le J-1.$$

System (4.3) can be easily and directly solved since it is tri-diagonal. A tri-diagonal system of $J$ equations can be solved in $\mathcal{O}(J)$ operations, then $J$ tri-diagonal systems are of a cost of $\mathcal{O}(J^2)$. The fast Fourier sine transform uses $\mathcal{O}(J \log J)$ operations on a vector with $J$ components. Thus, the total computational cost in the schemes (2.5) or (2.10) is $\mathcal{O}(J^2 \log J)$.

## 5. Numerical Tests

In the following, we use three examples to illustrate the accuracy and efficiency of the schemes (2.5) and (2.10) with Dirichlet and Neumann boundary conditions. The test problems used here are chosen such that the analytic solutions are available, so rigorous comparisons can be made. In Example 5.1, we consider the test problem with Dirichlet boundary condition. The test problem with Neumann boundary condition is considered in Example 5.2. Example 5.3 is presented to make comparisons of the schemes (2.5) and (2.10). In all the computations, the error is measured in the $L_\infty$ norm and the convergence order is taken as $r = \log(e_{h_1}/e_{h_2})/\log(h_1/h_2)$.

**Example 5.1.** *Consider the problem*

$$\begin{cases} \nabla^2 u + k^2 u = (k^2 - \pi^2 - k^2\pi^2) \sin \pi x \sin k\pi y, \quad (x, y) \in \Omega = (0, 1)^2, \\ u \mid_{\partial\Omega} = 0. \end{cases}$$

*Its exact solution is given by* $u(x, y) = \sin \pi x \sin k\pi y$.

The errors and the convergence rates with respect to the $L_\infty$ norm for the schemes (2.5) and (2.10) are given in Table 5.1. It is shown that the convergence orders for the schemes (2.5)

Table 5.1: Example 5.1: order of convergence of computed solution with Dirichlet BC. ($k = 30$).

| | 2nd-order difference | | scheme (2.5) | | scheme (2.10) | |
|---|---|---|---|---|---|---|
| $J$ | error | order | error | order | error | order |
| 32 | 1.52 | | 3.61e-1 | | 4.37e-1 | |
| 64 | 2.30e-1 | 2.73 | 1.28e-2 | 4.82 | 1.54e-2 | 4.83 |
| 128 | 5.19e-2 | 2.15 | 7.10e-4 | 4.17 | 8.53e-4 | 4.17 |
| 256 | 1.27e-2 | 2.03 | 4.31e-5 | 4.04 | 5.18e-5 | 4.04 |
| 512 | 3.10e-3 | 2.03 | 2.68e-6 | 4.01 | 3.21e-6 | 4.01 |
| 1024 | 7.85e-4 | 1.98 | 1.67e-7 | 4.00 | 2.00e-7 | 4.00 |
| 2048 | 1.96e-4 | 2.00 | 1.04e-8 | 4.27 | 1.25e-8 | 4.00 |

Table 5.2: Example 5.1: order of convergence of computed solution with Dirichlet BC. ($k = 300$).

| | scheme (2.5) | | scheme (2.10) | |
|---|---|---|---|---|
| $J$ | error | order | error | order |
| 128 | 42.38 | | 59.73 | |
| 256 | 1.57 | | 1.90 | |
| 512 | 3.42e-2 | | 4.12e-2 | |
| 1024 | 1.80e-3 | 4.10 | 2.10e-3 | 4.29 |
| 2048 | 1.05e-4 | 4.25 | 1.27e-4 | 4.05 |

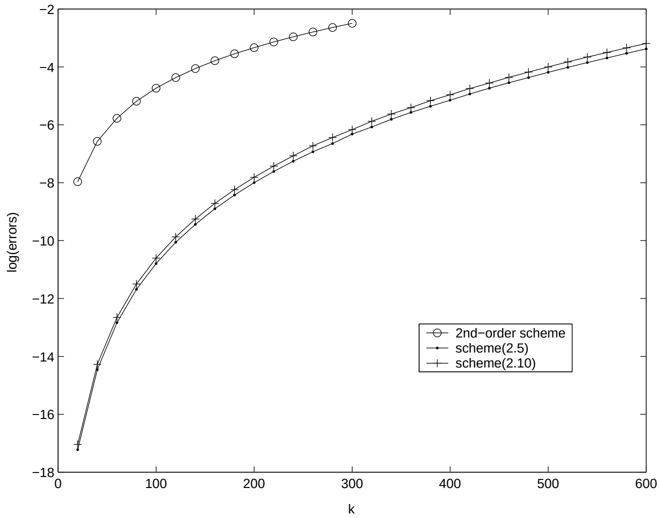


Fig. 5.1. Example 5.1: the errors for fixed $h$ and increasing wave number $k$.

and (2.10) are four. Moreover, it is also seen that higher accurate approximate solutions are provided for the fourth-order compact difference schemes (2.5) and (2.10). For comparison, we also list the errors obtained by using the 2nd-order central difference scheme. It is observed that the numerical solutions obtained by the 4th-order compact schemes are much more accurate than those obtained by using the second-order central difference scheme.

In Table 5.2, we list the numerical errors for a larger wave number, $k = 300$. As expected, for large $k$, the fourth order convergence rates are maintained when $J$ is large enough such that $kh$ is sufficiently small.

In Fig. 5.1, we set $h = 2^{-10}$. It is observed that the errors are developed with the increasing wave number $k$ for both the schemes (2.5) and (2.10). As a comparison, we also plot the errors obtained by using the second-order central difference scheme. For higher-order schemes, it is seen that for small grid length $h$, the high resolutions are obtained even for large $k$.

**Example 5.2.** *We consider the equation in Example* 5.1 *with the same exact solution but with different boundary conditions*

$$u|_{x=0,1} = 0, \qquad u|_{y=0} = 0, \qquad \frac{\partial u}{\partial y}|_{y=1} = k\pi \cos k\pi \sin \pi x.$$

In this numerical test, we use the scheme (2.5), together with the four-order boundary approximation for handling the Neumann boundary condition (as described in Section 2.1) to compute the numerical solutions. For comparison, a second-order treatment for the Neumann condition is also employed. From Table 5.3, we see that the convergence orders for the numerical solutions, obtained by the scheme (2.5) and two different boundary value approximations, are two and four respectively. This suggests that it is important to use higher order boundary value approximation if higher order difference schemes are used.

Table 5.3: Example 5.2: order of convergence used scheme (2.5). ($k = 5$).

| | second-order | | fourth-order | |
|---|---|---|---|---|
| $J$ | error | order | error | order |
| 64 | 5.52e-2 | | 8.33e-5 | |
| 128 | 1.38e-2 | 2.00 | 5.20e-6 | 4.00 |
| 256 | 3.50e-3 | 1.98 | 3.25e-7 | 4.00 |
| 512 | 8.64e-4 | 2.02 | 2.03e-8 | 4.00 |
| 1024 | 2.16e-4 | 2.00 | 1.27e-9 | 4.00 |

Table 5.4: Example 5.3: the critical wave number $k$ required for a given accuracy.

| $J$ | $k \approx 10$ | $k \approx 800$ |
|---|---|---|
| 512 | 1.0005 | 0.3596 |
| 1024 | 1.0000 | 0.0232 |
| 2048 | 1.0001 | 0.0061 |

Table 5.5: Example 5.3: $\|E\|_{(2.10)}/\|E\|_{(2.5)}$ for small $k$ and large $k$.

| $J$ | $\|E\|$ | $k \le k_c$ |
|---|---|---|
| 256 | $\|E\| \le 10^{-2}$ | $k \le 235$ |
| 512 | $\|E\| \le 10^{-2}$ | $k \le 455$ |
| 1024 | $\|E\| \le 10^{-2}$ | $k \le 800$ |
| 2048 | $\|E\| \le 10^{-4}$ | $k \le 730$ |

**Example 5.3.** *Consider the problem*

$$\begin{cases} \nabla^2 u + k^2 u = \pi^2 \sin \pi x \sin \pi y, & (x, y) \in (0,1) \times (0, 1/2), \\ u|_{x=0,1} = 0, \; u|_{y=0} = 0, \; u|_{y=1/2} = \sin \pi x (\sin \frac{l\pi}{2} + \frac{1}{l^2 - 1}), \end{cases}$$

*where* $k^2 = \pi^2(1 + l^2)$ *with odd number* $l$. *Note that here* $f$ *is independent of* $k$. *The exact solution is*

$$u(x, y) = \sin \pi x \sin l\pi y + \frac{\sin \pi x \sin \pi y}{l^2 - 1}.$$

For this exact solution, the error estimates in (3.22)-(3.23) suggest that

$$\|E\| \le Ck^6h^4 \tag{5.1}$$

for the scheme (2.5); and

$$\|E\| \le Ch^4 + Ck^8h^6 \tag{5.2}$$

for the scheme (2.10). Here the constant $C$ is independent of $k$.

We compare the scheme (2.5) and the scheme (2.10) by observing the error bounds of (5.1) and (5.2),

$$\frac{h^4 + k^8h^6}{k^6h^4} = \frac{1}{k^6} + k^2h^2, \tag{5.3}$$

where the constants $C$ in (5.1) and (5.2) are omitted. From the above relation we see that the scheme (2.10) is more useful than the scheme (2.5) for large $k$ and small $h$ ($kh$ small). While for small and moderate values of $k$ the accuracy of the schemes (2.5) and (2.10) is almost the same.

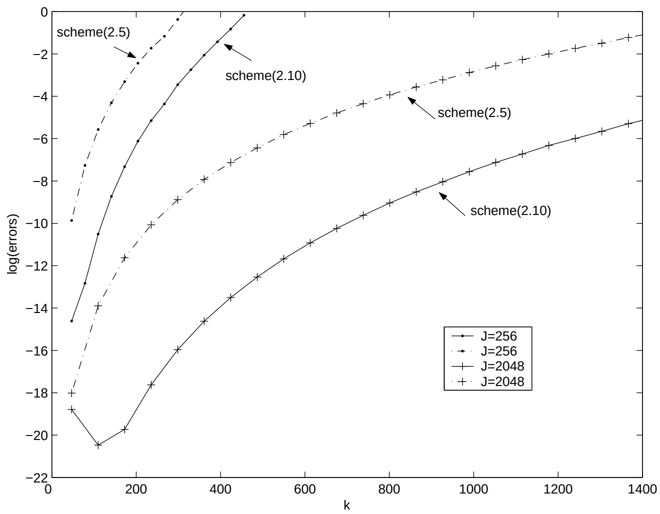


Fig. 5.2. Example 5.3: numerical errors against the wave number $k$.

In Fig. 5.2, the error behaviors for the schemes 2.5 and (2.10) are examined with fixed $J$ and increasing $k$. In Table 5.4, we set $k = \pi\sqrt{1+3^2} \approx 10$ and $k = \pi\sqrt{1+255^2} \approx 800$ respectively, and investigate the error ratio $\|E\|_{(2.10)}/\|E\|_{(2.5)}$ for different values of $h$, where $\|E\|_{(2.10)}$ and $\|E\|_{(2.5)}$ denote the errors of the scheme (2.10) and the scheme (2.5), respectively. The numerical results shown in Fig. 5.2 and Table 5.4 confirm our theoretical prediction that the scheme (2.10) is more efficient than the scheme (2.5) when the wave number $k$ is large.

It is seen from Fig. 5.2 that for a grid size, there exists a critical value $k_c$ such that the corresponding computations will not be convergent if $k \ge k_c$. In Table 5.5, we list the critical values of the wave number $k$ with some required accuracy, i.e., $\|E\| \le 10^{-2}$ (or $10^{-4}$). The numerical results are obtained using the scheme (2.10).

## 6. Conclusions

In this work we developed two fourth-order compact schemes for solving the Helmholtz equation. We also make some convergence analysis under the conditions that $kh$ is sufficiently small. A higher-order approximation for the Dirichlet and the Neumann boundary conditions is designed. Moreover, an efficient implementation of the finite difference approximation, based on a FFT approach, is proposed. Numerical experiments are carried out to confirm the theoretical predictions.

In a future work, we will extent the present work to handle some more complicated situations. One issue arises when the boundary condition involved a global (integral) condition, as seen in [6]. Another issue is to consider the case when the wave number $k$ is a piecewise constant, see also [6]. In this case, maintaining fourth-order accuracy in the compact setting seems quite difficult.

**Acknowledgments.** The author thanks Prof. Tao Tang for many helpful discussions. This research is supported by Natural Science Foundation of China under grant number 10471047. Part of this work was carried out during the author's visit to the Joint Research Institute of Applied Mathematics between Peking University and Hong Kong Baptist University.

## References

[1] I.M. Babuska and S.A. Sauter, Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers? *SIAM Rev.*, **42**:3 (2000), 451-484.

[2] G. Bao and W. Sun, A fast algorithm for the electromagnetic scattering from a large cavity, *SIAM J. Sci. Comput.*, **27**:2 (2005), 553-574.

[3] E. Erturk and C. Gokcol, Fourth-order compact formulation of Navier-Stokes equations and driven cavity flow at high Reynolds numbers, *Int. J. Numer. Meth. Fl.,* **50** (2006), 421-436.

[4] I. Harari and E. Turkel, Accurate finite difference methods for time-harmonic wave propagation, *J. Comput. Phys.*, **119** (1995), 252-270.

[5] M. Li, T. Tang and B. Fornberg, A compact fourth-order finite difference scheme for the steady incompressible Navier-Stokes equations, *Int. J. Numer. Meth. Fl.*. **20** (1995), 1137-1151.

[6] M. Li and T. Tang, A compact fourth-order finite difference scheme for unsteady Navier-Stokes equations, *J. Sci. Comput.*, **16** (2001), 29-46.

[7] Y.V.S.S. Sanyasiraju and V. Manjula, Higher order semi compact scheme to solve transient incompressible Navier-Stokes equations, *Comput. Mech.*, **35** (2005), 441-448.

[8] J. Shen and L. Wang, Spectral approximation of the Helmholtz equation with high wave numbers, *SIAM J. Numer. Anal.*, **43**:2 (2005), 623-644.

[9] J. Shen and T. Tang, Spectral and High-Order Methods with Applications, Science Press, Beijing, 2007.

[10] W.F. Spotz, High-Order Compact Finite Difference Schemes for Computational Mechanics. PhD thesis, University of Texas at Austin, December, 1995.

[11] Z.F. Tian and Y. B. Ge, A fourth-order compact finite difference scheme for the steady stream function-vorticity formulation of the Navier-Stokes/Boussinesq equations, *Int. J. Numer. Meth. Fl.*, **41** (2003), 495-518.

[12] D. Xiu and J. Shen, An efficient spectral method for acoustic scattering from rough surfaces, *Commun. Comput. Phys.*, **2** (2007), 54-72.

[13] K.Q. Zhang, B. Shotorban, W.J. Minkowycz and F. Mashayek, A compact finite difference method on staggered grid for Navier-Stokes flows, *Int. J. Numer. Meth. Fl.*, **52** (2006), 867-881.

[14] S. Kim, Compact schemes for acoustics in the frequence domain, *Math. Comput. Model.*, **37** (2003), 1335-1341.