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FULL DISCRETE TWO-LEVEL CORRECTION SCHEME FOR NAVIER-STOKES EQUATIONS*

Yanren Hou and Liquan Mei
College of Science, Xi'an Jiaotong University, Xi'an 710049, China
Email: yrhou@mail.xjtu.edu.cn, lqmei@mail.xjtu.edu.cn

Abstract

In this paper, a full discrete two-level scheme for the unsteady Navier-Stokes equations based on a time dependent projection approach is proposed. In the sense of the new projection and its related space splitting, non-linearity is treated only on the coarse level subspace at each time step by solving exactly the standard Galerkin equation while a linear equation has to be solved on the fine level subspace to get the final approximation at this time step. Thus, it is a two-level based correction scheme for the standard Galerkin approximation. Stability and error estimate for this scheme are investigated in the paper.

Mathematics subject classification: 65M55, 65M70.
Key words: Two-level method, Galerkin approximation, Correction, Navier-Stokes equation.

1. Introduction

We consider the two-dimensional Navier-Stokes equations

$$\frac{du}{dt} + \nu Au + B(u,u) = f, \quad u(0) = u_0, \tag{1.1}$$

in certain divergence-free Hilbert space H, where u_0 is the initial velocity field, A the Stokes operator, B the projection of the non-linearity on H, $\nu > 0$ the kinetic viscosity and f the external force.

To get efficient numerical schemes, the two-level (two-grid) strategy has been widely studied. In particular, a class of two-level method in connection with the approximate inertial manifolds (AIMs) initialized by Foias, Manley and Temam [5] has been extensively studied in the past decades, which is usually called the nonlinear Galerkin method (NLG). Let ϕ_i be the ith eigenvector of the Stokes operator A corresponding to the associated eigenvalue λ_i. For given $m, M \in \mathbf{N}$ $(m < M)$, let P_m (P_M) denote the spectral projection from H onto the space spanned by the first m (M) eigenvectors. And we also set

$$Q_m = I - P_m \ \ (Q_M = I - P_M).$$

The semi-discrete NLG reads: solve (1.1) up to a given time t_0 by a standard Galerkin method (SGM) in the fine-level subspace, and for $t > t_0$ find $v_m \in P_m H$ and $\hat{w}_m \in (P_M - P_m)H$ such that

$$\frac{dv_m}{dt} + \nu A v_m + P_m B(v_m + \hat{w}_m, v_m + \hat{w}_m) = P_m f, \tag{1.2}$$

$$\hat{w}_m = \Phi(v_m). \tag{1.3}$$

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Here Φ is the so-called AIM, a smooth mapping from P_mH onto $(P_M - P_m)H$ reflecting the approximate interactive relation between the lower and higher frequency components. For different choice of Φ we can get different NLG. For example, a frequently discussed AIM is expressed via the following generalized steady Stokes problem:

$$\nu A\hat{w}_m + (P_M - P_m)B(v_m, v_m) = (P_M - P_m)f. \tag{1.4}$$

This scheme is more efficient than SGM in P_MH. The convergence and error estimates for such NLG are obtained in the works of Marion and Temam [14, 15], Ammi and Marion [1], Marion and Xu [16] and Devulder et al. [4] in either finite element or spectral case. For example, in [4] they show that for $t > t_0$

$$\begin{aligned} |u(t) - (v_m(t) + \hat{w}_m(t))|_{L^2} &\le c(t)(L_m^3\lambda_{m+1}^{-\frac{3}{2}} + L_M\lambda_{M+1}^{-1}), \\ |u(t) - (v_m(t) + \hat{w}_m(t))|_{H^1} &\le c(t)(L_m^2\lambda_{m+1}^{-1} + L_M\lambda_{M+1}^{-\frac{1}{2}}), \end{aligned} \tag{1.5}$$

where $L_m = (1 + \ln \frac{\lambda_m}{\lambda_1})^{\frac{1}{2}}$. Lately, Garcia-Achilla et al. [6, 7] proposed a post-processing Galerkin scheme (PPG) based on the AIM defined by (1.4)

$$\frac{dv_m}{dt} + \nu Av_m + P_mB(v_m, v_m) = P_mf \quad \forall t \in [t_0, T], \tag{1.6}$$

$$\hat{w}_m(T) = \Phi(v_m(T)), \tag{1.7}$$

and obtained the similar error estimates. Since $\hat{w}_m$ is computed only once at $t = T$ and the lower and higher frequency components are fully dissociated, this is a very efficient scheme.

On the other hand, since the interaction of the lower and higher frequency components is reflected by a steady generalized Stokes equation, such schemes are only valid for $t > t_0$ when the time derivative of u possesses enough regularity. This is only acceptable for solutions slowly changed in higher frequency field. In fact, there have been few numerical experiments reported for such NLG and PPG to our knowledge, especially for highly oscillated solution in time field. Thus, to get reliable scheme for general cases, we should not neglect the self evolution of the higher frequency components. Another factor which affects the efficiency of the NLG or other two-level scheme is that they generally are coupled systems. When computing v_m we have to use $\hat{w}_m$ and vice versa. Such coupled systems are unavoidable if the space splitting are based on P_m and Q_m because such projections have nothing to do with the nonlinear system and the interaction of the different components has to be reflected by the coupled system they satisfied. Of course, PPG is an exception, in which the contribution of the higher frequency components $\hat{w}_m$ to itself and the lower frequency components v_m is neglected. This should be valid only for not very small viscosity case. Awaring of the reason of the generation of the coupled system, we alternate the way of thinking. If the decomposition of the lower and higher frequency components of the solution can reflect their interaction to some extent (for example, similar idea for steady state problems and certain scalar semi-linear evolutionary equations can be found in [11–13, 20]), it is reasonable to expect a two-level scheme in simpler form, at least in weakly coupled form, such that it is more efficient than usual two-level methods and can still preserve the order of convergence. These are the main motivation of this paper.

Based on the above considerations, we propose a full discrete two-level scheme for the Navier-Stokes equations: for given time step length $k > 0$, $u_M^n \in P_MH$ and $u_m^{n+1} \in P_mH$, find u_M^{n+1} in P_MH such that

$$(u_M^{n+1}, v) + k\nu(A^{\frac{1}{2}}u_M^{n+1}, A^{\frac{1}{2}}v) + k(B(u_m^{n+1}, u_M^{n+1}), v) = (u_M^n, v) + k(f^{n+1}, v), \tag{1.8}$$

where u_m^{n+1} is a rough approximation of u_M^{n+1} in P_mH. We will show in Section 3 that

$$u_m^{n+1} = P_m^{n+1} u_M^{n+1}, \quad P_m^{n+1} H = P_m H$$

for properly constructed system dependent projection P_m^{n+1}, where the supper-script $n+1$ means the projection is constructed according to the nonlinear system at current time step. Indeed, we can dissociate the equation of u_m^{n+1} from (1.8) by restricting (1.8) in P_mH (see Section 3):

$$(u_m^{n+1}, v) + k\nu(A^{\frac{1}{2}} u_m^{n+1}, A^{\frac{1}{2}} v) + k(B(u_m^{n+1}, u_m^{n+1}), v) = (u_M^n, v) + k(f^{n+1}, v). \tag{1.9}$$

This is the SGM equation in P_mH except for a successively updating of the approximation in the previous time step, that is

$$u_M^n = u_m^n + \hat{w}_m^n = P_m^n u_M^n + (I - P_m^n) u_M^n.$$

And the fine-level equation (1.8) is actually used to get the correction

$$\hat{w}_m^{n+1} = u_M^{n+1} - u_m^{n+1}$$

of u_m^{n+1} in the incremental subspace. Therefore we call this scheme the two-level correction scheme (TLC) in this paper. Formally, the computation of u_m^{n+1} does not depend on $\hat{w}_m^{n+1}$ and this will save a lot of "works" in computation. It seems that the contribution of the higher frequency components $\hat{w}_m^{n+1}$ to the lower frequency one u_m^{n+1} is neglected as PPG does. This is actually not true because such contribution is already reflected by the projection P_m^{n+1}.

Since the convection terms in both (1.9) and (1.8) preserve the antisymmetric property of the original convection term in (1.1), it should be a stable scheme. And we show later that there exists $k_0 > 0$ such that the scheme is unconditional numerical stable as long as $k > k_0$ and it shares the same error estimates (1.5) of the NLG.

We also point out that the above TLC scheme can be regarded as a full discrete version of the two-level scheme given by Girault and Lions [8], which is discussed in finite element case for 3-D Navier-Stokes equations. In [8], the authors obtained an error estimates for the velocity, but the error order is a half order lower than the estimates in this paper. We found the reason is the usage of the projection operator P_η they proposed in finite element case, which leads to the appearance of an interpolation term of the first order space derivative in the error equation. This will not be a problem in the spectral case because the usual L^2 orthogonal projection P_m is also an H^1 orthogonal projection. And in our way, we find some interesting property, for example $u_m^{n+1} = P_m^{n+1} u_M^{n+1}$ and

$$|(I - P_m^{n+1})v|_{L^2} \leq c\lambda_{m+1}^{-\frac{1}{2}} |v|_{H^1} \quad \forall v \in H \cap H^1(\Omega).$$

This will makes the analysis simple. Another reason is the usage of classical energy method for error analysis. To get more rigorous error estimates, we will use the discrete semi-group method for the error analysis.

Although we only consider the 2-D spectral TLC in this paper, all the analysis can be extended to the 3-D case by demanding more regularity of the solution. Furthermore, we believe the idea of the construction of the system dependent projection can be applied to finite element case and this might be able to improve the error estimates in [8] in finite element case. And we will discuss this question elsewhere.

2. Functional Settings

Let $\Omega \in \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. We consider the following incompressible Navier-Stokes equations

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega\times\mathbf{R}^+, \\ \nabla\cdot u = 0 & \text{in } \Omega\times\mathbf{R}^+, \end{cases}$$

with homogeneous Dirichlet or periodic boundary condition and initial velocity $u(x,t) = u_0(x)$. We introduce

$$\begin{aligned} &H = \{v \in L^2(\Omega)^2 : \nabla\cdot v = 0,\ (v\cdot n)|_{\partial\Omega} = 0\} && \text{Dirichlet boundary condition case,} \\ &H = \{v \in L^2(\Omega)^2 : \nabla\cdot v = 0,\ \int_\Omega v dx = 0\} && \text{periodic boundary condition case,} \end{aligned}$$

and the classical divergence-free projection P from $L^2(\Omega)^2$ onto H. For convenience, we introduce the Stokes operator $A = -P\Delta$ and a bi-linear operator $B(u,v) = P[(u\cdot\nabla)v]$. Then we can get the functional Navier-Stokes equations (1.1).

It is classical (see [19]) that A is a self-adjoint, unbounded linear operator in H with compact inverse whose domain $D(A)$ is dense in H. Thus A has discrete spectrum $0 < \lambda_1 \le \lambda_2 \le \cdots \to +\infty$ and its associated eigenvectors $\{\phi_j\}_{i=1}^\infty$ form an orthonormal basis of H.

As usual, we define the powers A^s of A for $s \in \mathbf{R}$; A^s maps $D(A^s)$ into H. And $D(A^s)$ is a Hilbert space when equipped with the scalar product $(A^s\cdot, A^s\cdot)$. In the rest, we set $V = D(A^{\frac{1}{2}})$ and denote by $\|\cdot\| = |A^{\frac{1}{2}}\cdot|$, the norm on V.

For convenience, we denote the following continuous bi-linear form on $V\times V$

$$a(u,v) = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v), \quad \forall u,v \in V,$$

which is obvious $V-$coercive, and the following continuous trilinear form on $V\times V\times V$

$$b(u,v,w) =< B(u,v), w >_{V'} \quad \forall u,v,w \in V.$$

Thanks to [19], we have

$$b(u,v,v) = 0, \quad \forall u,v \in V, \tag{2.1}$$

and

$$|b(u,v,w)| \le c|A^{s_1}u|\,|A^{\frac{1}{2}+s_2}v|\,|A^{s_3}w|, \quad \forall u \in D(A^{s_1}), v \in D(A^{\frac{1}{2}+s_2}), w \in D(A^{s_3}), \tag{2.2}$$

$$|b(u,v,w)|, |b(w,v,u)| \le c|u|_{L^\infty}|A^{\frac{1}{2}}v|\,|w|, \quad \forall u \in L^\infty(\Omega)^2, v \in V, w \in H, \tag{2.3}$$

where $s_i \ge 0$, $i = 1,2,3$ satisfying

$$2(s_1+s_2+s_3) \ge 1, \quad 2(s_1,s_2,s_3) \ne (1,0,0), (0,1,0), (0,0,1).$$

For given $M \in \mathbf{N}$, let us recall the definition of the orthogonal spectral projector P_M, Q_M and the following classical properties (see [3]):

$$|A^{s_1}P_M v| \le \lambda_M^{s_1-s_2}|A^{s_2}v|,\ |A^{s_2}Q_M v| \le \lambda_{M+1}^{s_2-s_1}|A^{s_1}v|, \quad \forall v \in D(A^{s_1}), s_1 \ge s_2. \tag{2.4}$$

Furthermore, if we denote

$$L_M = (1 + \ln\frac{\lambda_M}{\lambda_1})^{\frac{1}{2}},$$

we have the following Brezis-Gallouet inequality (see [2]) in finite dimensional case

$$|u_M|_{L^\infty} \le cL_M\|u_M\| \quad \forall u_M \in H_M, \tag{2.5}$$

where

$$H_M = P_M H = \text{span}\{\phi_1, \phi_2, \cdots, \phi_M\}.$$

Here and hereafter we always denote by c a generic positive constant independent of the data of (1.1) and time.

Furthermore, thanks to [6] we have: for any $w \in H_M$ and suitable v, u

$$|b(w,u,v)|,\ |b(v,u,w)| \le cL_M\|w\|\,\|u\|\,|v|, \tag{2.6}$$

$$|b(u,v,A^{-1}w)|,\ |b(v,u,A^{-1}w)| \le cL_M\|v\|\,|A^{-\frac{1}{2}}u|\,|w|. \tag{2.7}$$

Now let us recall some classical properties of the solution $u(t)$ of (1.1). For $u(0) = u_0 \in V$, the Navier-Stokes equations have a unique solution $u = u(t)$ defined for all $t > 0$ such that

$$|u(t)| \le M_0, \quad \|u(t)\| \le M_1, \quad \forall t \ge 0, \tag{2.8}$$

where M_0 and M_1 are two positive constants independent of t. By using the similar methods in [5, 7, 10], one can easily verify that there exist positive constants T_0, K and κ such that

$$|Q_M u(t)| \le K\frac{L_M}{\lambda_{M+1}}, \quad \|Q_M u(t)\| \le K\frac{L_M}{\sqrt{\lambda_{M+1}}}, \quad |u_{tt}(t)| \le \kappa, \quad \forall t \ge T_0. \tag{2.9}$$

At last, for given time step length $k > 0$ we give a full discrete SGM to conclude this section. For $U_M^0 = P_M u_0$, find $U_M^{n+1} \in H_M$ such that

$$(U_M^{n+1}, v) + k\nu a(U_M^{n+1}, v) + kb(U_M^{n+1}, U_M^{n+1}, v) = (U_M^n, v) + k(f^{n+1}, v), \quad \forall v \in H_M, \tag{2.10}$$

where $f^{n+1} = f(t_{n+1})$, $t_{n+1} = (n+1)k$.

3. New Projection and its Related Two-Level Scheme

For given two positive integers $M > m$, we will propose a new projection from H_M (or V) onto H_m and give their corresponding two-level algorithm in this section.

First of all, we assume that $u_m^{n+1} \in H_m$ is a certain approximation of the true solution $u(t_{n+1}) \in V$ and the SGM approximation $U_M^{n+1} \in H_M$ within H_m.

To construct the new projection, we introduce the following bi-linear form: $\forall w, v \in V$

$$\mathcal{L}_m^{n+1}(w,v) = (w,v) + k\nu a(w,v) + kb(u_m^{n+1}, w, v). \tag{3.1}$$

By Lax-Milgram theorem and (2.1) we can easily verify that the following linear problem

$$\mathcal{L}_m^{n+1}(w,v) =< g, v >_{V'}, \quad \forall v \in V,$$

is well-posed in V for any given $g \in V'$. Then we can define a projection P_m^{n+1} from V (or H_M) onto H_m: for any given $w \in V$, find $P_m^{n+1}w \in H_m$ such that

$$\mathcal{L}_m^{n+1}(w - P_m^{n+1}w, v) = 0, \quad \forall v \in H_m. \tag{3.2}$$

The superscript $n+1$ means that the new projection P_m^{n+1} is time dependent. Obviously,

$$H_m = P_m H_M = P_m^{n+1} H_M.$$

Denoting $Q_m^{n+1} = I - P_m^{n+1}$, we split H_M and V as

$$H_M = H_m + \hat{H}_{mM}^{n+1}, \quad V = H_m + \hat{V}_m^{n+1}, \tag{3.3}$$

where $\hat{H}_{mM}^{n+1} = Q_m^{n+1} H_M$ and $\hat{V}_m^{n+1} = Q_m^{n+1} V$. In the rest of this paper, we always denote for any $w_M \in H_M$ and $w \in V$ that

$$w_M = w_m + \hat{w}_{mM}^{n+1}, \quad w = w_m + \hat{w}_m^{n+1},$$

where $w_m \in H_m$, $\hat{w}_{mM}^{n+1} \in \hat{H}_{mM}^{n+1}$ and $\hat{w}_m^{n+1} \in \hat{V}_m^{n+1}$. In addition, the following lemma tells us that the space $\hat{V}_m^{n+1}$ (of course $\hat{H}_{mM}^{n+1}$) corresponds to the small scale components subspace, which only carries a little part of the entire energy.

Lemma 3.1. *For any given $w \in V$ and $\hat{w}_m^{n+1} = Q_m^{n+1} w \in \hat{V}_m^{n+1}$, we have*

$$|P_m A^{-\frac{1}{2}} \hat{w}_m^{n+1}| \le |Q_m A^{-\frac{1}{2}} \hat{w}_m^{n+1}|,$$

providing

$$\frac{ckL_m^2 \|u_m^{n+1}\|^2}{\nu} \le \frac{1}{2}.$$

Proof. Taking $v = P_m A^{-1} \hat{w}_m^{n+1}$ in (3.2), we have

$$|P_m A^{-\frac{1}{2}} \hat{w}_m^{n+1}|^2 + k\nu |P_m \hat{w}_m^{n+1}|^2 \le k|b(u_m^{n+1}, \hat{w}_m^{n+1}, P_m A^{-1} \hat{w}_m^{n+1})|.$$

Thanks to (2.1) and (2.7), we have

$$\begin{aligned}
&k|b(u_m^{n+1}, \hat{w}_m^{n+1}, P_m A^{-1} \hat{w}_m^{n+1})| \le ckL_m \|u_m^{n+1}\| \, |P_m \hat{w}_m^{n+1}| \, |A^{-\frac{1}{2}} \hat{w}_m^{n+1}| \\
&\le ckL_m \|u_m^{n+1}\| \, |P_m \hat{w}_m^{n+1}| (|P_m A^{-\frac{1}{2}} \hat{w}_m^{n+1}| + |Q_m A^{-\frac{1}{2}} \hat{w}_m^{n+1}|) \\
&\le k\nu |P_m \hat{w}_m^{n+1}|^2 + \frac{ckL_m^2 \|u_m^{n+1}\|^2}{\nu} \{ |P_m A^{-\frac{1}{2}} \hat{w}_m^{n+1}|^2 + |Q_m A^{-\frac{1}{2}} \hat{w}_m^{n+1}|^2 \}.
\end{aligned}$$

This completes the proof of this lemma. $\square$

Corollary 3.1. *Under the condition of Lemma 3.1, for any $w \in V$ and $\hat{w}_m^{n+1} = Q_m^{n+1} w \in \hat{V}_m^{n+1}$, we have*

$$|P_m \hat{w}_m^{n+1}| \le |Q_m \hat{w}_m^{n+1}|, \quad \|P_m \hat{w}_m^{n+1}\| \le \|Q_m \hat{w}_m^{n+1}\|.$$

Furthermore, the projection P_m^{n+1} is bounded in V and

$$\|P_m^{n+1} w\| \le 3\|w\|.$$

Proof. By using the inequality (2.4) and Lemma 3.1, we can easily obtain the first two inequalities. The last inequality is obvious if one notices $Q_m \hat{w}_m^{n+1} = Q_m w$.

For any given functions u^2, u^1 and v in V, let us denote

$$\begin{aligned}
g(u^2, u^1, v) &= < G(u^2, u^1), v >_{V'} \\
&= (u^2, v) + k\nu a(u^2, v) + kb(u^2, u^2, v) - k(f, v) - (u^1, v).
\end{aligned} \tag{3.4}$$

The SGM equation (2.10) is equivalent to

$$g(U_M^{n+1}, U_M^n, v) = 0 \quad \forall v \in H_M. \tag{3.5}$$

Now, we present the following two-level scheme for (2.10) (of course for (1.1)) corresponding to the projection P_m^{n+1}: for $u_m^0 = P_m u_0$, $\hat{w}_{mM}^0 = (P_M - P_m)u_0$

$$\begin{cases} g(u_m^{n+1}, u_M^n, v) = 0 \quad \forall v \in H_m, \\ \mathcal{L}_m^{n+1}(\hat{w}_{mM}^{n+1}, v) = -g(u_m^{n+1}, u_M^n, v) \quad \forall v \in \hat{H}_{mM}^{n+1}, \\ u_M^{n+1} = u_m^{n+1} + \hat{w}_{mM}^{n+1}. \end{cases} \tag{3.6}$$

The solvability of the first equation in scheme (3.6) is obvious since it is nothing but a SGM equation except for a successively updating of the approximation in the previous time step. For the solvability of the second equation, let us consider the bi-linear form $\mathcal{L}_m^{n+1}$ on $\hat{H}_{mM}^{n+1} \times \hat{H}_{mM}^{n+1}$. Thanks to (2.1) it is easy to verify that $\mathcal{L}_m^{n+1}$ is $\hat{H}_{mM}^{n+1}$-elliptic. Noting the first equation, we know that $G(u_m^{n+1}, u_M^n) \in H_m^\circ$, the polar set of H_m. Then the mapping $\hat{w}_{mM}^{n+1} \to G(u_m^{n+1}, u_M^n)$ defined by this equation is an isomorphism from $\hat{H}_{mM}^{n+1}$ onto H_m° (see Lemma 4.1 in [9]). This shows the solvability of the second equation.

Let us give a brief interpretation of the scheme (3.6) to complete this section. The first equation gives a relative rough prediction of the solution in a small subspace H_m. And the second equation corrects this prediction in an incremental subspace $\hat{H}_{mM}^{n+1}$ by solving a linearized residual equation. In fact, the bi-linear form $\mathcal{L}_m^{n+1}$ is an approximation of the Frechet derivative of the time discrete Navier-Stokes operator on time step $n+1$ by omitting $b(\hat{w}_{mM}^{n+1}, u_m^{n+1}, v)$ which does not obey the antisymmetric property. That is the second step in (3.6) correct the prediction u_m^{n+1} in certain direction around the tangent space at u_m^{n+1} rather than the system independent higher frequency subspace $Q_m H$ in usual NLG and PPG.

Concerning about the numerical implementation, we have to avoid operations in the incremental subspace $\hat{H}_{mM}^{n+1}$ since it is almost impossible to do computation in such a time dependent subspace whose construction is very complex from the point view of realistic computation. If we notice

$$B(u_m^{n+1}, u_m^{n+1}) + B(u_m^{n+1}, \hat{w}_{mM}^{n+1}) = B(u_m^{n+1}, u_M^{n+1}),$$

and the definition of P_m^{n+1}, we can rewrite the scheme (3.6) in the following equivalent form. For $u_M^0 = P_M u_0$ and $u_m^0 = P_m u_M^0$, solve the following equation in H_m and H_M sequentially:

$$(u_M^{n+1}, v) + k\nu a(u_M^{n+1}, v) + kb(u_m^{n+1}, u_M^{n+1}, v) = k(f, v) + (u_M^n, v), \tag{3.7}$$

which is equivalent to (1.9) and (1.8).

4. Error Analysis

In this section, we will establish the uniform numerical stability theorems for the TLC scheme (3.7) in $L^\infty(\mathbf{R}^+; H)$ and $L^\infty(\mathbf{R}^+; V)$ and give the error estimates of it in H and V, respectively. For convenience, here and hereafter, we use the same symbols M_0 and M_1 appeared in (2.8) to denote the H and V bounds for the TLC approximation u_M^n. For simplicity, we will use $|A^{-\frac{1}{2}} f|$ and $|f|$ to denote $\sup\limits_{t\geq 0} |A^{-\frac{1}{2}} f(t)|$ and $\sup\limits_{t\geq 0} |f(t)|$ in the rest.

4.1. Numerical Stability

Noticing the antisymmetric property of the trilinear form in (3.7), the $L^\infty(\mathbf{R}^+, H)$ stability of the scheme is straightforward, which we state directly in the following theorem.

Theorem 4.1. *There exists a positive constant*

$$M_0^2 = |u^0|^2 + \frac{c|A^{-\frac{1}{2}}f|^2}{\nu^2\lambda_1}$$

such that

$$|u_M^n|^2 \le M_0^2, \quad \forall n \ge 0.$$

Furthermore, for any given positive constant r and $N \in \mathbf{N}$ satisfying $kN \le r$, we have

$$k\nu \sum_{i=n_0}^{N+n_0} \|u_M^i\|^2 \le M_0^2 + \frac{cr|A^{-\frac{1}{2}}f|^2}{\nu}, \quad \forall n_0 \ge 1. \tag{4.1}$$

For $L^\infty(\mathbf{R}^+; V)$ stability, if we have $\|u_m^{n+1}\| \le c\|u_M^{n+1}\|$, the uniform stability can be obtained by usual method. But $u_m^{n+1} = P_m^{n+1}u_M^{n+1}$ in our case instead of $u_m^{n+1} = P_m u_M^{n+1}$, we have to show $\|u_m^{n+1}\| \le c\|u_M^{n+1}\|$ first. This depends on the following bounded result of $\|u_m^{n+1}\|$.

Lemma 4.1. *For given positive integer n, suppose that there exists a constant $M_1 > 0$ independent of n, k, m and M such that*

$$\|u_M^i\| \le M_1, \quad \forall i \le n.$$

Then

$$\|u_m^{n+1}\| \le \sqrt{2}M_1 + \frac{ck^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}|f|,$$

providing that

$$k \le \frac{1}{c\nu L_m^2 M_1^2}.$$

Proof. Taking $v = 2\delta = 2P_m(u_m^{n+1} - u_M^n)$ in (3.7) and using (2.1), we have

$$\begin{aligned}
&2|\delta|^2 + k\nu\|u_m^{n+1}\|^2 + k\nu\|\delta\|^2 = k\nu\|P_m u_M^n\|^2 - 2kb(u_m^{n+1}, u_m^{n+1}, \delta) + 2k(f^{n+1}, \delta)\\
&\le k\nu\|u_M^n\|^2 + 2k|b(u_m^{n+1}, P_m u_M^n, \delta)| + 2k|(f^{n+1}, \delta)|.
\end{aligned}$$

Thanks to (2.6), $u_m^{n+1} = P_m^{n+1}u_M^{n+1}$, corollary 3.1 and the assumption of this lemma, we have

$$\begin{aligned}
&2k|b(u_m^{n+1}, P_m u_M^n, \delta)| \le ckL_m M_1\|u_m^{n+1}\|\,|\delta| \le |\delta|^2 + ck^2L_m^2M_1^2\|u_m^{n+1}\|^2,\\
&2k|(f^{n+1}, \delta)| \le 2k|f^{n+1}|\,|\delta| \le |\delta|^2 + ck^2|f|^2.
\end{aligned}$$

If k is small enough such that $ckL_m^2M_1^2 \le \frac{\nu}{2}$, the combination of the above three inequalities leads to the result of this lemma. $\square$

Thanks to Corollary 3.1, we can claim $\|u_m^i\| \le c\|u_M^i\|$ for all $i \le n+1$. Then a discrete mimic of the procedure in [10] and the usage of some discrete Gronwall inequality (see, e.g., [18]) will lead to $\|u_M^i\| \le \tilde{M}_1$ for any $i \le n+1$, where $\tilde{M}_1$ is independent of M_1. Since this is a standard procedure, we omit it.

Theorem 4.2. *There exists a positive constant M_1 independent of m, M, n and k such that*

$$\|u_M^n\| \le M_1 \quad \forall n \ge 0,$$

providing that k and m satisfy the condition

$$k \le \frac{1}{c\nu L_m^2 M_1^2}.$$

Proof. Taking $M_1 = \max\{\tilde{M}_1, \|u_0\|\}$, we can get the result by mathematical induction. □

4.2. Error Estimates

In this subsection, we will give the L^2 and H^1 error estimates for the scheme (3.7). To get better estimates, we will use the discrete semigroup method for the analysis rather than the classical energy method. First of all, let us introduce some useful lemmas and we refer readers to the appendix for their proofs.

Lemma 4.2. *For any $r \in (0,1)$ and $n \in \mathbf{N}$, we have*

$$|A^r(I + k\nu A)^{-n}|_{\mathcal{L}(H)} \le r^r (k\nu)^{-r}(n-r)^{-r}.$$

Here and hereafter $\mathcal{L}(H)$ stands for the linear normed space of linear bounded operators on H and $|\cdot|_{\mathcal{L}(H)}$ denote the associated operator norm.

Lemma 4.3. *For $n, m \in \mathbf{N}$,*

$$\sum_{i=1}^{n} |A(I + k\nu A)^{-i} P_m|_{\mathcal{L}(H)} \le \frac{2L_m}{k\nu}.$$

Moreover, for $r \in (0,1)$, $n \in \mathbf{N}$ and $m \in \mathbf{N} \cup \{0\}$, we have

$$\sum_{i=1}^{n} |A^r(I + k\nu A)^{-i} Q_m|_{\mathcal{L}(H)} \le \frac{1}{1-r}(k\nu)^{-1}\lambda_{m+1}^{r-1}.$$

Furthermore, we need the following discrete Gronwall type inequality, which is an analogue of the continuous one proved by Okamoto [17].

Lemma 4.4. *For $n \in \mathbf{N}$, $\{d_n\}$ is a positive sequence and $r, \alpha, \beta > 0$ are three positive constants with $r \in (0,1)$. If it holds following inequality*

$$d_n \le \alpha + \beta \sum_{i=0}^{n-1} (n-i+1)^{-r} d_i, \qquad \text{for} \;\; n = 1, 2, \cdots, \tag{4.2}$$

there exists a constant $c' = c'(r) > 0$ such that

$$d_n \le c'\alpha \exp\{c'\beta^{\frac{1}{1-r}} n\}.$$

From now on, we assume that (2.8)-(2.9) hold for $T_0 = 0$. First of all, for any given non-negative integer n, time step length $k > 0$ and $m, M \in \mathbf{N}$ satisfying $m < M$, we denote

$$\begin{aligned} &t_n = nk, \quad u^n = u(t_n), \quad U_m^n = P_m^n u^n \in H_m, \quad \hat{W}_m^n = Q_m^n u^n \in \hat{V}_m^n, \\ &p^n = U_m^n + P_M \hat{W}_m^n, \quad q^n = Q_M \hat{W}_m^n. \end{aligned}$$

And we apply P_M to (1.1) and rewrite it at $t = t_{n+1}$:

$$(I + k\nu A)p^{n+1} + kP_M B(U_m^{n+1}, U_m^{n+1}) + kP_M B(U_m^{n+1}, \hat{W}_m^{n+1}) \\ +kP_M B(\hat{W}_m^{n+1}, U_m^{n+1}) + kP_M B(\hat{W}_m^{n+1}, \hat{W}_m^{n+1}) = p^n + kP_M f^{n+1} + kh^{n+1}, \quad (4.3)$$

where

$$h^{n+1} = \frac{1}{k}\int_{t_n}^{t_{n+1}} P_M(u_t(s) - u_t(t_{n+1}))ds.$$

Thanks to (2.9), we have

$$|h^j| \le \kappa k, \quad \forall j \ge 1. \quad (4.4)$$

Let us recall that $u_M^{n+1} = u_m^{n+1} + \hat{w}_{mM}^{n+1}$ and rewrite the scheme (3.7) in the following functional form:

$$(I + k\nu A)u_M^{n+1} + kP_M B(u_m^{n+1}, u_m^{n+1}) + kP_M B(u_m^{n+1}, \hat{w}_{mM}^{n+1}) = u_M^n + kP_M f^{n+1}, \quad (4.5)$$

If we denote

$$e_M^n = p^n - u_M^n, \quad e_m^n = U_m^n - u_m^n, \quad \hat{e}_{mM}^n = P_M \hat{W}_m^n - \hat{w}_{mM}^n,$$

we have

$$e_M^n = e_m^n + \hat{e}_{mM}^n.$$

Theorem 4.3. *Assume that $u_0 \in V$ and (2.8)-(2.9) are valid for $T_0 = 0$. If we suppose that the conditions of Theorem 4.2 are valid, in particular $k \le \frac{1}{c\nu L_m^2 M_1^2}$, we have for $s = 0$ and $s = \frac{1}{2}$*

$$|A^s(u^n - u_M^n)| \le C(n)(k + L_M \lambda_{M+1}^{s-1} + L_m^3 \lambda_{m+1}^{s-\frac{3}{2}}),$$

where

$$C(n) = cD_s \exp(cM_1^{\frac{1}{1-\sigma}} \nu^{\frac{\sigma}{1-\sigma}} nk),$$

D_s is a constant independent of n, k, m, M and σ is any constant in $(\frac{1}{2}, 1)$.

Proof. Subtracting (4.5) from (4.3) gives us

$$e_M^{n+1} = (I + k\nu A)^{-1} e_M^n - k(I + k\nu A)^{-1} P_M[E^{n+1} + Q^{n+1} + W^{n+1} - h^{n+1}], \quad (4.6)$$

where

$$E^{n+1} = B(e_m^{n+1}, u^{n+1}) + B(u_m^{n+1}, e_M^{n+1}),$$
$$Q^{n+1} = B(u_m^{n+1}, q^{n+1}), \quad W^{n+1} = B(\hat{W}_m^{n+1}, u^{n+1}).$$

Noticing $e_M^0 = 0$, we can derive from (4.6) that

$$e_M^{n+1} = -k\sum_{j=1}^{n+1}(I + k\nu A)^{-(n+2-j)} P_M[E^j + Q^j + W^j - h^j]. \quad (4.7)$$

For given s ($s = 0$ or $\frac{1}{2}$) and any $\sigma \in (\frac{1}{2}, 1)$, we have

$$|A^s e_M^{n+1}| \le I_1^n(E) + \tilde{I}_1^{n+1}(Q) + I_2^{n+1}(W) + I_3^{n+1}(W) + I_4^{n+1}(h) \\ +k|A^s(I + k\nu A)^{-1} P_M E^{n+1}|, \quad (4.8)$$

where

$$
\begin{aligned}
&I_1^n(E)=k\sum_{j=1}^{n}|A^\sigma(I+k\nu A)^{-(n+2-j)}P_M|_{\mathcal{L}(H)}|A^{s-\sigma}E^j|,\\
&\tilde{I}_1^{n+1}(Q)=k\sum_{j=1}^{n+1}|A^\sigma(I+k\nu A)^{-(n+2-j)}P_M|_{\mathcal{L}(H)}|A^{s-\sigma}Q^j|,\\
&I_2^{n+1}(W)=k\sum_{j=1}^{n+1}|A(I+k\nu A)^{-(n+2-j)}P_m|_{\mathcal{L}(H)}|P_mA^{s-1}W^j|,\\
&I_3^{n+1}(W)=k\sum_{j=1}^{n+1}|A^{\frac{s+1}{2}}(I+k\nu A)^{-(n+2-j)}Q_m|_{\mathcal{L}(H)}|Q_mP_MA^{\frac{s-1}{2}}W^j|,\\
&I_4^{n+1}(h)=k\sum_{j=1}^{n+1}|A^s(I+k\nu A)^{-(n+2-j)}P_M|_{\mathcal{L}(H)}|h^j|.
\end{aligned}
\tag{4.9}
$$

For the last term on the right-hand side of (4.8), we have

$$
k|A^s(I+k\nu A)^{-1}P_ME^{n+1}|\le k|A^{\frac12}(I+k\nu A)^{-1}|_{\mathcal{L}(H)}|A^{s-\frac12}P_ME^{n+1}|.
$$

For any $v\in H$, using (2.1), (2.2), (2.4), (2.6) and Corollary 3.1 gives

$$
\begin{aligned}
&|b(e_m^{n+1},u^{n+1},A^{s-\frac12}v)|\\
\le&\left\{\begin{array}{l}|b(e_m^{n+1},A^{0-\frac12}v,P_mu^{n+1})|+|b(e_m^{n+1},A^{0-\frac12}v,Q_mu^{n+1})|\\ |b(e_m^{n+1},u^{n+1},A^{\frac12-\frac12}v)|\end{array}\right.\\
\le&\left\{\begin{array}{l}c|e_m^{n+1}|\,|v|\,|P_mu^{n+1}|_{L^\infty}+c|A^{\frac14}e_m^{n+1}|\,|v|\,|A^{\frac14}Q_mu^{n+1}|\\ c|e_m^{n+1}|_{L^\infty}\|u^{n+1}\|\,|v|\end{array}\right.\\
\le& cL_m|A^se_m^{n+1}|\,\|u^{n+1}\|\,|v|\le cM_1L_m|A^se_M^{n+1}|\,|v|,
\end{aligned}
$$

and similarly

$$
|b(u_m^{n+1},e_M^{n+1},A^{s-\frac12}v)|\le cL_m\|u_m^{n+1}\|\,|A^se_M^{n+1}|\,|v|\le cM_1L_m|A^se_M^{n+1}|\,|v|.
$$

Then we obtain

$$
|A^{s-\frac12}P_ME^{n+1}|\le cM_1L_m|A^se_M^{n+1}|.
$$

Using Lemma 4.2 gives

$$
k|A^{\frac12}(I+k\nu A)^{-1}|_{\mathcal{L}(H)}\le\frac{k^{\frac12}}{\nu^{\frac12}}.
$$

Then we finally get

$$
k|A^s(I+k\nu A)^{-1}P_ME^{n+1}|\le\frac12|A^se_M^{n+1}|,\tag{4.10}
$$

provided k and m satisfy the condition in Theorem 4.2, that is $ckM_1^2L_m^2\le\nu$.

Now, let us estimate $|A^{s-\sigma}E^j|$, $|A^{s-\sigma}Q^j|$, $|P_mA^{s-1}W^j|$ and $|Q_mP_MA^{\frac{s-1}{2}}W^j|$ one by one. It follows from (2.1), (2.3), $\sigma\in(\frac12,1)$ and Corollary 3.1 that

$$
\begin{aligned}
b(e_m^j,u^j,A^{s-\sigma}v)&=\left\{\begin{array}{l}-b(e_m^j,A^{0-\sigma}v,u^j)\\ b(e_m^j,u^j,A^{\frac12-\sigma}v)\end{array}\right\}\\
&\le c|A^se_m^j|\,\|u^j\|\,|v|\le cM_1|A^se_M^j|\,|v|,\\
b(u_m^j,e_M^j,A^{s-\sigma}v)&=\left\{\begin{array}{l}-b(u_m^j,A^{0-\sigma}v,e_M^j)\\ b(u_m^j,e_M^j,A^{\frac12-\sigma}v)\end{array}\right\}\\
&\le c\|u_m^j\|\,|A^se_M^j|\,|v|\le cM_1|A^se_M^j|\,|v|.
\end{aligned}
$$

Similarly, using (2.9) gives

$$b(u_m^j, q^j, A^{s-\sigma}v) \leq cM_1|A^s q^j|\,|v| \leq cKM_1 \frac{L_M}{\lambda_{M+1}^{1-s}}|v|.$$

Then we can get

$$|A^{s-\sigma}E^j| \leq cM_1|A^s e_M^j|, \quad |A^{s-\sigma}Q^j| \leq cKM_1 \frac{L_M}{\lambda_{M+1}^{1-s}}. \tag{4.11}$$

For the term $|P_m A^{s-1} W^j|$, thanks to (2.7), (2.9) and Lemma 3.1, we have for $s = 0$

$$\begin{aligned}|(P_m A^{-1} W^j, v)| &= |b(\hat{W}_m^j, u^j, P_m A^{-1} v)| \\ &\leq cL_m|A^{-\frac{1}{2}}\hat{W}_m^j|\,\|u^j\|\,|v| \leq cKM_1 \frac{L_m^2}{\lambda_{m+1}^{\frac{3}{2}}}|v|.\end{aligned}$$

Moreover, for $s = \frac{1}{2}$, by using (2.6) and (2.9) we have

$$\begin{aligned}|(P_m A^{-\frac{1}{2}} W^j, v)| &= |b(\hat{W}_m^j, u^j, P_m A^{-\frac{1}{2}} v)| \\ &\leq cL_m|\hat{W}_m^j|\,\|u^j\|\,|v| \leq cKM_1 \frac{L_m^2}{\lambda_{m+1}}|v|,\end{aligned}$$

which gives

$$|P_m A^{s-1} W^j| \leq cKM_1 \frac{L_m^2}{\lambda_{m+1}^{\frac{3}{2}-s}}. \tag{4.12}$$

For the term $|Q_m P_M A^{\frac{s-1}{2}} W^j|$, by using (2.2), (2.4), (2.9) and Lemma 3.1 we obtain

$$b(\hat{W}_m^j, u^j, A^{\frac{s-1}{2}} P_M Q_m v) \leq cM_1|A^{\frac{1}{4}}\hat{W}_m^j|\,|A^{\frac{1}{4}} A^{\frac{s-1}{2}} Q_m v| \leq cKM_1 \frac{L_m}{\lambda_{m+1}^{1-\frac{s}{2}}}|v|.$$

Therefore,

$$|Q_m P_M A^{\frac{s-1}{2}} W_m^j| \leq cKM_1 \frac{L_m}{\lambda_{m+1}^{1-\frac{s}{2}}}. \tag{4.13}$$

Thanks to Lemma 4.2 and (4.11), we have

$$I_1^n(E) \leq \frac{cM_1}{k^{\sigma-1}\nu^\sigma} \sum_{j=1}^{n} (n+2-j)^{-\sigma}|A^s e_M^j|.$$

The usage of Lemma 4.3 and (4.11) yields

$$\tilde{I}_1^{n+1}(Q) \leq \frac{cKM_1}{\nu(1-\sigma)\lambda_1^{1-\sigma}} L_M \lambda_{M+1}^{s-1}.$$

The usage of Lemma 4.3, (4.12) and (4.13) admits

$$I_2^{n+1}(W) \leq \frac{cKM_1}{\nu} L_m^3 \lambda_{m+1}^{s-\frac{3}{2}}, \quad I_3^{n+1}(W) \leq \frac{cKM_1}{\nu(1-s)} L_m \lambda_{m+1}^{s-\frac{3}{2}}.$$

Moreover, using Lemma 4.3 and (4.4) leads to

$$I_4^{n+1}(h) \leq \frac{\kappa k}{\nu \lambda_1^{1-s}}.$$

Combining the above five estimates and (4.10) with (4.8) gives us

$$|A^s e_M^{n+1}| \le D_s(k + L_M \lambda_{M+1}^{s-1} + L_m^3 \lambda_{m+1}^{s-\frac{3}{2}}) + \frac{cM_1}{k^{\sigma-1}\nu^\sigma} \sum_{j=1}^{n} (n+2-j)^{-\sigma} |A^s e_M^j|,$$

where

$$D_s = D_s(nk, M_1, K, \kappa, \nu, \sigma) = c\max\{\frac{KM_1}{\nu(1-\sigma)\lambda_1^{1-\sigma}}, \frac{KM_1}{\nu}, \frac{KM_1}{\nu(1-s)}, \frac{\kappa}{\nu\lambda_1^{1-s}}\}.$$

Finally, the usage of Lemma 4.4 leads to the result of this theorem. □

From the proof of this theorem, we know that L_M and most power of L_m in the result come from the assumption $u_0 \in V$. If we further assume that $u_0 \in D(A)$, L_M and most power of L_m can be removed from the result. The only one which can not be removed is the one comes from the usage of Lemma 4.3. We summarize this fact in the following corollary.

Corollary 4.1. *Assume that $u_0 \in D(A)$ and there exists a time independent positive constant $M_2 > 0$ such that $|Au(t)| \le M_2$ for all $t > 0$. Then we have, for $s = 0$ and $s = \frac{1}{2}$*

$$|A^s(u^n - u_M^n)| \le C(n)(k + \lambda_{M+1}^{s-1} + L_m \lambda_{m+1}^{s-\frac{3}{2}}),$$

where $C(n) = C(nk, M_1, M_2, \nu, \sigma)$.

Remark 4.1. Thanks to the results of Theorem 4.3 and Corollary 4.1, to get an approximation of the same accuracy of the SGM in H_M, we only have to solve a SGM equation of fully nonlinear type within a relative small subspace H_m and then solve a linear equation to get the correction in the H_M. Thus TLC may save a lot of CPU time compared with SGM. For example, noticing the asymptotic property of the eigenvalues of the Stokes operator and assuming $u_0 \in D(A)$, Corollary 4.1 indicates that we should choose m such that $mL_m^{\frac{1}{3}} \sim M^{\frac{2}{3}}$ or $mL_m^{\frac{1}{2}} \sim M^{\frac{1}{2}}$ to reach the same L^2- or H^1-accuracy of the SGM approximation in H_M, respectively.

5. Numerical Example

In this section, we present some numerical results to support our previous analysis. We take example of 2-D Navier-Stokes equations confined in a rectangular domain $\Omega = [0, 2\pi]^2$ with periodic boundary conditions.

We chose an exact solution $u(x,t)$ and then computed the "highly oscillatory" time dependent forcing term from the solution. In this way we can check errors without computing a large scale Galerkin approximation as an "exact" solution. We choose $u(x,t) = u_1(x,t) + \overline{u_1(x,t)}$, where

$$u_1(x,t) = \sum_{|k|>0, k_1>0; k_2>0, k_1=0} \alpha_k(t) \begin{pmatrix} k_2 \\ -k_1 \end{pmatrix} e^{-i(k_1x_1+k_2x_2)},$$

$$\alpha_k(t) = \frac{1}{10|k|^4} \sin(\frac{|k_1|}{|k_2|+1} t + \omega),$$

$x = (x_1, x_2) \in \Omega$, $k = (k_1, k_2) \in Z^2$, $|k| = \sqrt{k_1^2 + k_2^2}$, $i = \sqrt{-1}$ and $\omega \in \mathbf{R}$ is a constant. It is easy to verify that $\nabla \cdot u = 0$. Such configuration almost ensures that $u \in D(A)$ and the result

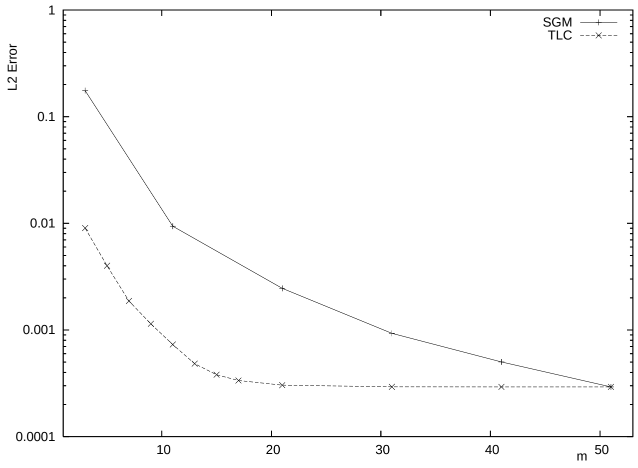

Fig. 5.1. L^2-error: $\nu = 0.01$.

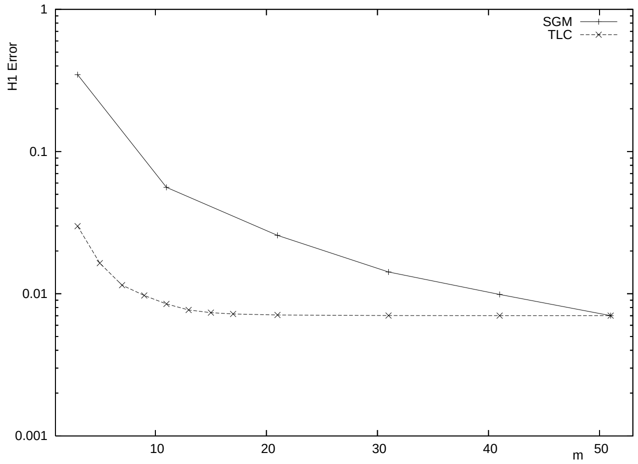

Fig. 5.2. H^1-error: $\nu = 0.01$.

of Corollary 4.1 applies. For given $m, M \in \mathbf{N}$ $(m < M)$, we look for approximate solution in the following form,

$$u_M^{n+1} = \sum_{-\frac{M}{2} \le k_1, k_2 \le \frac{M}{2}} \beta_k^{n+1} e^{-i(k_1x_1+k_2x_2)}, \quad \beta_k^{n+1} \in \mathbf{C}^2.$$

In the following we will compare the error of the SGM and TLC approximations at $t = 2$. For TLC, we fix $M = 51$ and let m change between $3 \sim 51$. And for SGM, we change M between $3 \sim 51$. All the implicit time stepping, both in the TLC (coarse-level computation) and SGM schemes, are achieved by the standard Newton iterative method with the same tolerance 10^{-9}. And all the algebraic equations arising in both schemes are solved by Gauss-Seidel iterative method with tolerance 10^{-9}. Since we care more about the impact of the spatial discretization to the entire error (especially the impact of different m in TLC), we choose the time step length small enough such that the entire error will not improve a lot when k becomes more smaller. Here we choose $k = 0.0001$ in the following two simulations.

In Figs. 5.1 and 5.2, we compare the total L^2- and H^1-relative errors of the TLC and SGM

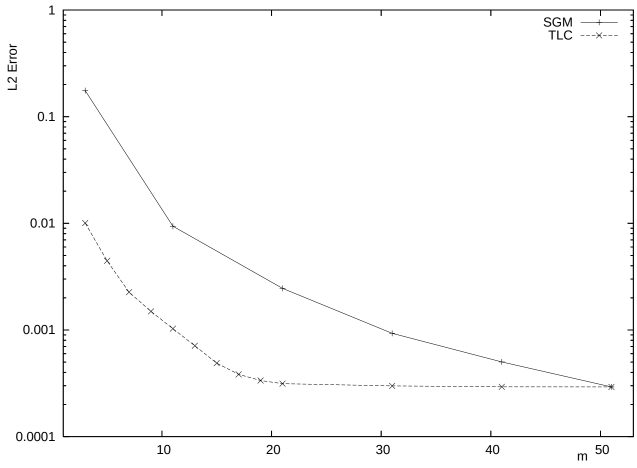

Fig. 5.3. L^2-error: $\nu = 0.001$.

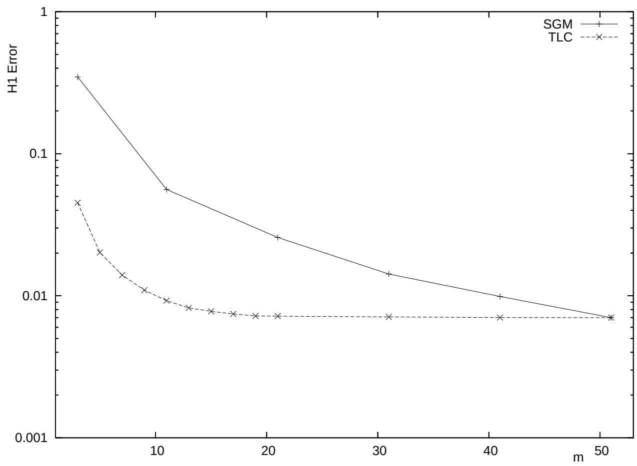

Fig. 5.4. H^1-error: $\nu = 0.001$.

approximations in the case of $\nu = 0.01$ at $t = 2$. We find that the TLC approximation can reach almost the same L^2- and H^1-accuracy as that of the SGM approximation with $M = 51$ when $m = 17$ and $m = 13$, respectively.

The results presented in Figs. 5.3 and 5.4 are related to the relative error comparison of the TLC and SGM approximations at $t = 2$ when $\nu = 0.001$. We also find that when $m = 19$ and $m = 15$ the TLC approximation can reach almost the same L^2- and H^1-accuracy as that of the SGM approximation with $M = 51$. These numerical results show a good agreement with our error analysis results and Remark 4.1 in Section 4.2.

To show the effectiveness of the TLC scheme, we compare the CPU time used by the TLC and SGM for obtaining approximate solutions with similar accuracy in Table 5.1. In this table, all the errors are relative errors, m is the coarse-level scale in the TLC and 'CPU' means

$$\frac{\text{CPU time used by TLC with such } m}{\text{CPU time used by SGM}}.$$

Table 5.1: CPU time comparison.

$\nu = 0.01$							
SGM (L^2-error)	.3122E-03	TLC(L^2-error)	.3357E-03	m	17	CPU	28.7%
SGM (H^1-error)	.7311E-02	TLC(H^1-error)	.7685E-02	m	13	CPU	20.9%
$\nu = 0.001$							
SGM (L^2-error)	.3124E-03	TLC(L^2-error)	.3552E-03	m	19	CPU	32.4%
SGM (H^1-error)	.7311E-02	TLC(H^1-error)	.7775E-02	m	15	CPU	23.1%

6. Appendix

In this appendix we give the proofs of Lemmas 4.2 and 4.3.

Proof of Lemma 4.2

Proof. It can be verified that

$$A^r(I + k\nu A)^{-n}\phi_i = \lambda_i^r(1 + k\nu\lambda_i)^{-n}\phi_i,$$

where λ_i is the eigenvalue of A and ϕ_i is its associated eigenfunction. It is also known that

$$|A^r(I + k\nu A)^{-n}|_{\mathcal{L}(H)} = \max_{i\geq 1} \lambda_i^r(1 + k\nu\lambda_i)^{-n} \leq \sup_{\lambda\geq\lambda_1} \lambda^r(1 + k\nu\lambda)^{-n}.$$

Denote $y(\lambda) = \lambda^r(1 + k\nu\lambda)^{-n}$. Then

$$y'(\lambda) = \lambda^r(1 + k\nu\lambda)^{-n-1}[r\lambda^{-1} - (n - r)k\nu].$$

From $y'(\lambda_0) = 0$, we can get

$$\lambda_0 = \frac{r}{(n - r)k\nu}, \quad y''(\lambda_0) < 0.$$

Consequently,

$$\begin{aligned}&|A^r(I + k\nu A)^{-n}|_{\mathcal{L}(H)}\\ \leq\ & y(\lambda_0) = r^r(\frac{n}{n - r})^{r-n}(k\nu)^{-r}n^{-r} \leq r^r(k\nu)^{-r}(n - r)^r.\end{aligned}$$

Since $r - n < 0$, we have

$$|A^r(I + k\nu A)^{-n}|_{\mathcal{L}(H)} \leq r^r(k\nu)^{-r}(n - r)^{-r}.$$

To prove Lemma 4.3, we need the following lemma.

Lemma 6.1. *For $n, m \in \mathbf{N}$, we have*

$$|A(I + k\nu A)^{-n}P_m|_{\mathcal{L}(H)} \leq \begin{cases} \lambda_m(1 + k\nu\lambda_m)^{-n}, & \textit{for}\ \ n - 1 \leq \frac{1}{k\nu\lambda_m}, \\ \frac{1}{k\nu(n-1)}, & \textit{for}\ \ \frac{1}{k\nu\lambda_m} \leq n - 1 \leq \frac{1}{k\nu\lambda_1}, \\ \lambda_1(1 + k\nu\lambda_1)^{-n}, & \textit{for}\ \ n - 1 \geq \frac{1}{k\nu\lambda_1}. \end{cases}$$

And for $r \in (0, 1)$, $n \in \mathbf{N}$ and $m \in \mathbf{N} \cup \{0\}$ it yields

$$|A^r(I + k\nu A)^{-n}Q_m|_{\mathcal{L}(H)} \leq \begin{cases} r^r(n - r)^{-r}(k\nu)^{-r}, & \textit{for}\ \ (n - r)k \leq \frac{r}{\nu\lambda_{m+1}}, \\ \lambda_{m+1}^r(1 + k\nu\lambda_{m+1})^{-n}, & \textit{for}\ \ (n - r)k \geq \frac{r}{\nu\lambda_{m+1}}. \end{cases}$$

Proof. First, let us prove the first inequality. Similar to the proof of Lemma 4.2, we know that

$$|A(I+k\nu A)^{-n}P_m|_{\mathcal{L}(H)} = \max_{1\le i\le m} \lambda_i(1+k\nu\lambda_i)^{-n} \le \sup_{\lambda_1\le\lambda\le\lambda_m} \lambda(1+k\nu\lambda)^{-n}.$$

Denote $y(\lambda)=\lambda(1+k\nu\lambda)^{-n}$. Then

$$y'(\lambda)=(1+k\nu\lambda)^{-n-1}[1-(n-1)k\nu\lambda].$$

Set $y'(\lambda_0)=0$, we can get

$$\lambda_0=\frac{1}{(n-1)k\nu}, \quad y''(\lambda_0)<0$$

when $n>1$. That is $y(\lambda)$ obtains its maximum value when $\lambda=\lambda_0$. So for $n>1$

$$|A(I+k\nu A)^{-n}P_m|_{\mathcal{L}(H)} \le \begin{cases} \lambda_m(1+k\nu\lambda_m)^{-n}, & \text{for } n-1\le \frac{1}{k\nu\lambda_m}, \\ \frac{1}{k\nu(n-1)}, & \text{for } \frac{1}{k\nu\lambda_m}\le n-1\le\frac{1}{k\nu\lambda_1}, \\ \lambda_1(1+k\nu\lambda_1)^{-n}, & \text{for } n-1\ge\frac{1}{k\nu\lambda_1}. \end{cases}$$

When $n=1$, we easily know that $y(\lambda)$ is an increasing function of $\lambda\in\mathbf{R}^+$, so the above result is still valid. This proves the first inequality in the lemma.

The proof of the second inequality is completely the same as that of Lemma 4.2 and we leave its proof to readers.

Proof of Lemma 4.3

Proof. The first inequality can be obtained from the first inequality in Lemma 6.1. In fact, if we take $i_1=[\frac{1}{k\nu\lambda_m}+1]$, $i_2=[\frac{1}{k\nu\lambda_1}+1]$, we have

$$\begin{aligned} &\sum_{i=1}^{n}|A(I+k\nu A)^{-i}P_m|_{\mathcal{L}(H)} \\ &\le \sum_{i=1}^{i_1}\lambda_m(1+k\nu\lambda_m)^{-i}+\sum_{i=i_1+1}^{i_2}\frac{1}{k\nu(i-1)}+\sum_{i=i_2+1}^{n}\lambda_1(1+k\nu\lambda_1)^{-i} \\ &\le \frac{1}{k\nu}+\frac{1}{k\nu}\ln\frac{\lambda_m}{\lambda_1}+\frac{1}{k\nu}\le\frac{2L_m}{k\nu}. \end{aligned}$$

To prove the second inequality, we have to use the second inequality in Lemma 6.1. By taking $i_1=[\frac{r}{k\nu\lambda_{m+1}}+r]$, we can obtain

$$\begin{aligned} &\sum_{i=1}^{n}|A^r(I+k\nu A)^{-i}Q_m|_{\mathcal{L}(H)} \\ &\le \sum_{i=1}^{i_1} r^r(k\nu)^{-r}i^{-r}+\sum_{i=i_1+1}^{n}\lambda_{m+1}^r(1+k\nu\lambda_{m+1})^{-i} \\ &\le r(1-r)^{-1}(k\nu)^{-1}\lambda_{m+1}^{r-1}+(k\nu)^{-1}\lambda_{m+1}^{r-1}\le\frac{1}{1-r}(k\nu)^{-1}\lambda_{m+1}^{r-1}. \end{aligned}$$

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