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AN ANISOTROPIC NONCONFORMING FINITE ELEMENT METHOD FOR APPROXIMATING A CLASS OF NONLINEAR SOBOLEV EQUATIONS*

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Abstract

An anisotropic nonconforming finite element method is presented for a class of nonlinear Sobolev equations. The optimal error estimates and supercloseness are obtained for both semi-discrete and fully-discrete approximate schemes, which are the same as the traditional finite element methods. In addition, the global superconvergence is derived through the postprocessing technique. Numerical experiments are included to illustrate the feasibility of the proposed method.

Mathematics subject classification: 65N30, 65N15.
Key words: Nonlinear Sobolev equations, Anisotropic, Nonconforming finite element, Supercloseness, Global superconvergence.

1. Introduction

Consider the following nonlinear Sobolev equations [1]

$$\begin{cases} -\nabla\cdot\big(a(u)\nabla u_t\big)-\nabla\cdot\big(b(u)\nabla u\big)=f(\mathrm{x},t), & \mathrm{x}\in\Omega, t\in(0,T],\\ u(\mathrm{x},t)=0, & \mathrm{x}\in\partial\Omega, t\in[0,T],\\ u(\mathrm{x},0)=u_0(\mathrm{x}), & \mathrm{x}\in\Omega, \end{cases} \tag{1.1}$$

where $\mathrm{x}=(x,y)$, Ω is a bounded convex domain in R^2, ∇ and $\nabla\cdot$ denote the gradient and the divergence operators, respectively; $a(u)=a(\mathrm{x},t,u)$ and $b(u)=b(\mathrm{x},t,u)$ depend on x, t and u. In (1.1) and below, for notational convenience, we drop the dependence of these coefficients on x and t. Furthermore, we assume that $a(u)$ and $b(u)$ satisfy the following properties as [2]

(i) There exist constants a_0, a_1, b_0 and b_1, such that

$$0<a_0\le a(u)\le a_1,\quad 0<b_0\le b(u)\le b_1. \tag{1.2}$$

(ii) Both $a(u)$ and $b(u)$ are globally Lipschitz continuous in u, i.e., for some constants C_ξ, they satisfy

$$|\xi(u_1)-\xi(u_2)|\le C_\xi|u_1-u_2|,\quad u_1,u_2\in R,\quad \xi=a,b. \tag{1.3}$$

In addition, $a(u)$ and $b(u)$ are twicely continuously differentiable with respective to u.

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It is known that Sobolev equations have important applications including the flow of fluids through fissured rock, the transport problems of humidity in soil, thermodynamics etc. Many studies have been devoted to conforming finite elements. For example, for linear case, [3] considered the first-order generalized difference scheme and gave L^p-norm and $W^{1,p}$-norm error estimates by means of the Ritz-Volterra projection; [4] studied two least-squares Galerkin finite element schemes, which yielded the approximate solutions with optimal accuracy in $(L^2)^2 \times L^2$ norm and the first-order and second-order accuracy in time, respectively; [5] proposed an H^1-Galerkin mixed finite element method and established optimal error estimates for the semi-discrete scheme and fully-discrete scheme. For nonlinear case, [6] gave finite difference streamline diffusion schemes with convection dominated term, and derived the stability and optimal error estimates; [7] considered the time stepping along characteristic finite element methods, and demonstrated optimal convergence rate in the sense of H^1 and L^2; [2] presented discontinuous Galerkin method with penalties and derived $L^\infty(H^1)$ error estimate for the semi-discrete scheme and $L^\infty(H^1)$ and $L^2(H^1)$ for the fully-discrete scheme.

However, there are still some defects in the work mentioned above. On the one hand, although the detailed and systematic theoretical analysis were given in [2-7], there were no numerical tests except [4] in one-dimension. On the other hand, to the best of our knowledge, all the known results in the literature are based on the classical regularity assumption or quasi-uniform assumption on the meshes, i.e., there exists a constant $C > 0$, such that for all element K, $h_K/\rho_K \le C$ or $h/h_{\min} \le C$, where $h = \max\limits_K h_K, h_{\min} = \min\limits_K h_K$, h_K and ρ_K are the diameter and the superior diameter of all circles contained in K, respectively (see [8] for details). However, in some cases, the solutions of some elliptic problems may have anisotropic behavior in some parts of the solution domain. This means that the solutions only vary significantly in certain directions. An obvious idea to reflect this anisotropy is to use anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution and a coarser mesh size in the perpendicular direction. Besides, some problems may be defined in narrow domain, for example, in modeling a gap between rotator and stator in an electrical machine, the cost of calculation will be very high when the regular partition is employed. Therefore, it is a better choice to employ anisotropic meshes with few degrees of freedom to overcome the above difficulties. Because the anisotropic elements K are characterized by $h_K/\rho_K \to \infty$ when the limit is considered as $h \to 0$, the well-known Bramble-Hilbert lemma can not be used directly in estimating the interpolation error. At the same time, the consistency error estimate, the key of the nonconforming finite element analysis, will become very difficult to be dealt with, for there will appear a factor $|F|/|K| \to \infty$ when the estimate is made on the longer sides F of the element K. It means that the traditional techniques for finite element analysis are no longer valid.

Recently, there have appeared some studies focusing on the study of convergence, supercloseness and superconvergence of anisotropic finite element methods. Both conforming and nonconforming finite elements have been applied to some linear problems, we refer to Acosta [9-10], Apel [11-13], Duran [14] and Shi [15-25]. Whether the results of the above literature are valid for nonlinear problems with anisotropic nonconforming elements remains open.

The purpose of this paper is to apply an anisotropic nonconforming finite element method to (1.1). Firstly, we consider both semi-discrete and backward Euler fully-discrete schemes and obtain the optimal convergence estimates. By virtue of the special property of the element and the postprocessing technique, the supercloseness and superconvergence are obtained. Secondly, we carry out some numerical tests to examine the numerical performance of the element with anisotropic rectangular meshes.

The outline of this paper is as follows. In Section 2, we briefly introduce the construction of a Crouzeix-Raviart type nonconforming element possessing the anisotropic property [21]. In Sections 3 and 4, the existence and uniqueness of the approximate solution, the optimal error estimates and superclose result are derived for the semi-discrete scheme and fully-discrete scheme, respectively. In Section 5, we present the superconvergence results. In the last section, some numerical examples supporting our theoretical results are given.

2. Construction of Nonconforming Finite Element

Let $\hat{K} = [-1,1]\times[-1,1]$ be the reference element in the $\hat{x}$-$\hat{y}$ plane with vertices $\hat{a}_1 = (-1,-1)$, $\hat{a}_2 = (1,-1)$, $\hat{a}_3 = (1,1)$ and $\hat{a}_4 = (-1,1)$. Let $\hat{l}_1 = \overline{\hat{a}_1\hat{a}_2}$, $\hat{l}_2 = \overline{\hat{a}_2\hat{a}_3}$, $\hat{l}_3 = \overline{\hat{a}_3\hat{a}_4}$ and $\hat{l}_4 = \overline{\hat{a}_4\hat{a}_1}$ be the four edges of $\hat{K}$. We define the finite element $(\hat{K},\hat{P},\hat{\Sigma})$ (see [25-26]) as

$$\hat{\Sigma} = \{\hat{v}_0,\hat{v}_1,\hat{v}_2,\hat{v}_3,\hat{v}_4\},\quad \hat{P} = span\{1,\hat{x},\hat{y},\varphi(\hat{x}),\varphi(\hat{y})\},$$

where

$$\varphi(t) = \frac{1}{2}(3t^2-1),\quad \hat{v}_0 = \frac{1}{|\hat{K}|}\int_{\hat{K}}\hat{v}\,d\hat{x}\,d\hat{y},\quad \hat{v}_i = \frac{1}{|\hat{l}_i|}\int_{\hat{l}_i}\hat{v}\,d\hat{s},\quad i=1,2,3,4.$$

The interpolation defined above is properly posed and the interpolation function can be expressed as

$$\hat{I}\hat{v} = \hat{v}_0 + \frac{1}{2}(\hat{v}_2-\hat{v}_4)\hat{x} + \frac{1}{2}(\hat{v}_3-\hat{v}_1)\hat{y} + \frac{1}{2}(\hat{v}_2+\hat{v}_4-2\hat{v}_0)\varphi(\hat{x}) + \frac{1}{2}(\hat{v}_3+\hat{v}_1-2\hat{v}_0)\varphi(\hat{y}).$$

It has been proved that the above interpolation operator has the anisotropic property [19], *i.e.*, for multi-index $\alpha = (\alpha_1,\alpha_2)$, when $|\alpha| = 1$, there holds

$$\|\hat{D}^\alpha(\hat{v}-\hat{I}^1\hat{v})\|_{0,\hat{K}} \le C|\hat{D}^\alpha\hat{v}|_{1,\hat{K}},\quad \forall\hat{v}\in H^2(\hat{K}). \tag{2.1}$$

For the sake of simplicity, let $\Omega\subset R^2$ be a polygon domain with edges parallel to the coordinate axes, $\mathcal{J}_h$ be a rectangular subdivision of Ω, which does not need to satisfy the above regularity assumption or quasi-uniform assumption. Given $K\in\mathcal{J}_h$, denote the barycenter of element K by (x_K,y_K), the length of edges parallel to x-axis and y-axis by $2h_x$, $2h_y$ respectively. Then there exists an affine mapping $F_K:\hat{K}\to K$

$$\begin{cases} x = x_K + h_x\hat{x}, \\ y = y_K + h_y\hat{y}. \end{cases} \tag{2.2}$$

The associated finite element space $V_h \not\subset H^1(\Omega)$ is defined as

$$V_h = \Big\{v; v\,|_K = \hat{v}\circ F_K^{-1}, \hat{v}\in\hat{P}, \int_{K\cap\partial K} v ds = \int_{K'\cap\partial K'} v\,ds,$$
$$\text{if } K, K' \text{ are adjacent; and } \int_{\partial K\cap\partial\Omega} v\,ds = 0\Big\}.$$

The interpolation operator $I_h: H^1(\Omega)\to V_h$ is defined as

$$I_h|_K = I_K,\quad I_K v = (\hat{I}\hat{v})\circ F_K^{-1},\quad \forall v\in H^1(\Omega). \tag{2.3}$$

3. Anisotropic Error Estimates for the Semi-Discrete Scheme

In this section, we discuss the error estimates and superclose of the semi-discrete scheme for (1.1) on anisotropic meshes.

For our subsequent use, we employ the classical Hilbert Sobolev spaces $W^{m,p}(\Omega)$ with a norm $||\cdot||_{m,p}$. $L^2(\Omega)$ denotes the set of square integrable functions on Ω with its norm $||\cdot||$.

Then the corresponding weak formulation of (1.1) is: Find u: $(0,T]\rightarrow H_0^1(\Omega)$, such that

$$\begin{cases} \big(a(u)\nabla u_t,\nabla v\big)+\big(b(u)\nabla u,\nabla v\big)=(f,v), & \forall v\in H_0^1(\Omega),\\ u(\mathrm{x},0)=u_0(\mathrm{x}), & \mathrm{x}\in\Omega. \end{cases} \tag{3.1}$$

The Galerkin approximation to (3.1) reads as follows: Find u_h: $(0,T]\rightarrow V_h$, such that

$$\begin{cases} \big(a(u_h)\nabla u_{ht},\nabla v_h\big)_h+\big(b(u_h)\nabla u_h,\nabla v_h\big)_h=(f,v_h), & \forall v_h\in V_h,\\ u_h(\mathrm{x},0)=I_hu_0(\mathrm{x}), & \mathrm{x}\in\Omega, \end{cases} \tag{3.2}$$

where $(\cdot,\cdot)_h=\sum_{K\in\mathcal{J}_h}(\cdot,\cdot)$.

Theorem 3.1. *Problem* (3.2) *has a unique solution.*

Proof. Let the basis functions in V_h be denoted by $\phi_i(\mathrm{x}), i=1,\cdots,r$. Then u_h can be expressed as

$$u_h=\sum_{i=1}^{r}h_i(t)\phi_i(\mathrm{x}),(\mathrm{x},t)\in\Omega\times(0,T]. \tag{3.3}$$

For $j=1,\cdots,r$, we take $v_h=\phi_j(\mathrm{x})$ in (3.2) and utilize (3.3) to see that, for $t\in(0,T]$,

$$\begin{cases} A\dfrac{dH(t)}{dt}+BH(t)=F,\\ H(0)=H_0, \end{cases} \tag{3.4}$$

where H_0 is given, and

$$H(t)=\big(h_1(t),\cdots,h_r(t)\big)^T,\quad A=\Big(\Big(a\big(\sum_{i=1}^{r}h_i(t)\phi_i\big)\nabla\phi_i,\nabla\phi_j\Big)\Big)_{r\times r},$$
$$B=\Big(\Big(b\big(\sum_{i=1}^{r}h_i(t)\phi_i\big)\nabla\phi_i,\nabla\phi_j\Big)_h\Big)_{r\times r},\quad F=\big((f,\phi_j)\big)_{r\times 1}.$$

Since (3.4) gives a system of nonlinear ordinary differential equations (ODEs) for the vector function $H(t)$, by the assumptions on a,b and the theory of ODEs, it follows that $H(t)$ exists and is unique for $t>0$ (see [27]). Therefore the proof is complete. □

The following lemma on anisotropic meshes will play an essential role in our analysis and can be found in [25].

Lemma 3.1. *Suppose $u,u_t\in H^2(\Omega),u_{tt}\in H^1(\Omega)$ and I_h is the interpolation operator defined in* (2.2) *of u. Then there hold*

$$\big(\nabla(u-I_hu),\nabla v_h\big)_h=0,\quad ||v_h||\le C||v_h||_h,\quad \forall v_h\in V_h, \tag{3.5}$$

and

$$\begin{aligned}&||u - I_h u|| \le Ch^2|u|_2, \quad ||u_t - I_h u_t|| \le Ch^2|u_t|_2, \quad ||u_{tt} - I_h u_{tt}|| \le Ch|u_{tt}|_1,\\ &||u - I_h u||_h \le Ch|u|_2, \quad ||u_t - I_h u_t||_h \le Ch|u_t|_2,\end{aligned} \tag{3.6}$$

where

$$||\cdot||_h = \left(\sum_{K\in\mathcal{J}_h} |\cdot|^2_{1,K}\right)^{\frac{1}{2}}, \quad ||\cdot|| = \left(\int_\Omega (\cdot,\cdot)^2\,dx\,dy\right)^{\frac{1}{2}}.$$

Moreover, we have

Lemma 3.2. *If $u, u_t \in H^2(\Omega)$, then for all $v_h \in V_h$, there hold*

$$\left|\sum_{K\in\mathcal{J}_h}\int_{\partial K} a(u)\frac{\partial u_t}{\partial n} v_h\,ds\right| \le Ch||u_t||_2||v_h||_h, \tag{3.7}$$

$$\left|\sum_{K\in\mathcal{J}_h}\int_{\partial K} b(u)\frac{\partial u}{\partial n} v_h\,ds\right| \le Ch||u||_2||v_h||_h. \tag{3.8}$$

Furthermore, if $u, u_t \in H^3(\Omega)$, then for all $v_h \in V_h$, there hold

$$\left|\sum_{K\in\mathcal{J}_h}\int_{\partial K} a(u)\frac{\partial u_t}{\partial n} v_h\,ds\right| \le Ch^2||u_t||_3||v_h||_h, \tag{3.9}$$

$$\left|\sum_{K\in\mathcal{J}_h}\int_{\partial K} b(u)\frac{\partial u}{\partial n} v_h\,ds\right| \le Ch^2||u||_3||v_h||_h. \tag{3.10}$$

Proof. Here we only give the proof of (3.10); and (3.7)-(3.9) can be proved similarly. For two adjacent $K, K' \in J_h$, we have

$$P_{0,i}b(u)\frac{\partial u}{\partial n_K} = -P_{0,i}b(u)\frac{\partial u}{\partial n_{K'}} = \textit{constant},$$

where $P_{0,i}\omega = \frac{1}{|l_i|}\int_{l_i}\omega\,ds$, $1 \le i \le 4$. Therefore

$$\begin{aligned}&\sum_{K\in J_h}\sum_{i=1}^4\int_{l_i} P_{0,i}b(u)\frac{\partial u}{\partial n}(v_h - P_{0,i}v_h)ds\\ =&\sum_{K\in J_h}\sum_{i=1}^4 P_{0,i}b(u)\frac{\partial u}{\partial n}\int_{l_i}(v_h - P_{0,i}v_h)ds = 0.\end{aligned}$$

Similarly, since

$$P_{0,i}v_h|_K = P_{0,i}v_h|_{K'} = \textit{constant}, \quad P_{0,i}v_h|_{\partial\Omega} = 0, \quad b(u)\frac{\partial u}{\partial n_K} = -b(u)\frac{\partial u}{\partial n_{K'}},$$

we have

$$\sum_{K\in J_h}\sum_{i=1}^4\int_{l_i} b(u)\frac{\partial u}{\partial n}P_{0,i}v_h\,ds = 0.$$

Consequently,

$$
\begin{aligned}
&\sum_{K\in J_h}\sum_{i=1}^{4}\int_{l_i} b(u)\frac{\partial u}{\partial n}v_h\,ds\\
&=-\sum_{K\in J_h}\sum_{i=1}^{4}\Big(\int_{l_i} P_{0,i}b(u)\frac{\partial u}{\partial n}(v_h-P_{0,i}v_h)ds+\int_{l_i} b(u)\frac{\partial u}{\partial n}v_h\,ds-\int_{l_i} b(u)\frac{\partial u}{\partial n}P_{0,i}v_h\,ds\Big)\\
&=\sum_{K\in J_h}\sum_{i=1}^{4}\int_{l_i}\Big(b(u)\frac{\partial u}{\partial n}-P_{0,i}b(u)\frac{\partial u}{\partial n}\Big)(v_h-P_{0,i}v_h)\,ds=:\sum_{K\in J_h}\sum_{i=1}^{4}I_i,
\end{aligned}
$$

where

$$
I_i=\int_{l_i}\Big(b(u)\frac{\partial u}{\partial n}-P_{0,i}b(u)\frac{\partial u}{\partial n}\Big)\big(v_h-P_{0,i}v_h\big)\,ds,\quad 1\le i\le 4.
$$

Therefore I_2+I_4 can be expressed as

$$
\begin{aligned}
I_2+I_4=&\int_{y_K-h_y}^{y_K+h_y}\Big[\big(b(u)\frac{\partial u}{\partial x}\big)(x_K+h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}\big(b(u)\frac{\partial u}{\partial x}\big)(x_K+h_x,y)\,dy\Big]\\
&\cdot\Big[v_h(x_K+h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}v_h(x_K+h_x,y)\Big]\,dy\\
&-\int_{y_K-h_y}^{y_K+h_y}\Big[\big(b(u)\frac{\partial u}{\partial x}\big)(x_K-h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}\big(b(u)\frac{\partial u}{\partial x}\big)(x_K-h_x,y)\,dy\Big]\\
&\cdot\Big[v_h(x_K-h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}v_h(x_K-h_x,y)\Big]\,dy. \qquad (3.11)
\end{aligned}
$$

Since

$$
\frac{\partial v_h}{\partial z}(x_K+h_x,z)=\frac{\partial v_h}{\partial z}(x_K-h_x,z),
$$

we have

$$
\begin{aligned}
&v_h(x_K+h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}v_h(x_K+h_x,y)\,dy\\
&=\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}dt\int_t^y\frac{\partial v_h}{\partial z}(x_K+h_x,z)\,dz\\
&=v_h(x_K-h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}v_h(x_K-h_x,y)\,dy. \qquad (3.12)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\big(b(u)\frac{\partial u}{\partial x}\big)(x_K+h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}\big(b(u)\frac{\partial u}{\partial x}\big)(x_K+h_x,y)\,dy\\
&\quad-\big(b(u)\frac{\partial u}{\partial x}\big)(x_K-h_x,y)+\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}\big(b(u)\frac{\partial u}{\partial x}\big)(x_K-h_x,y)\,dy\\
&=\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}dt\int_t^y dz\int_{x_K-h_x}^{x_K+h_x}\Big(\big(b(u)\big)_u\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial z}+b(u)\frac{\partial^3 u}{\partial x^2\partial z}\\
&\quad+\big(b(u)\big)_{uu}(\frac{\partial u}{\partial x})^2\frac{\partial u}{\partial z}+2\big(b(u)\big)_u\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial z}\Big)(x,z)\,dx. \qquad (3.13)
\end{aligned}
$$

Using similar estimates of [25], we have

$$\int_{y_K-h_y}^{y_K+h_y}\Big[v_h(x_K+h_x,y)-\frac{1}{2h_y}\int_{y_K-h_y}^{y_K+h_y}v_h(x_K+h_x,y)\,dy\Big]^2dy\le\frac{2h_y^2}{3h_x}\|\frac{\partial v_h}{\partial y}\|_{0,K}^2. \tag{3.14}$$

For notational convenience, let

$$\beta=\big(b(u)\big)_u\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial z}+a(u)\frac{\partial^3 u}{\partial x^2\partial z}+\big(b(u)\big)_{uu}(\frac{\partial u}{\partial x})^2\frac{\partial u}{\partial z}+2\big(b(u)\big)_u\frac{\partial^2 u}{\partial x\partial z}\frac{\partial u}{\partial z}. \tag{3.15}$$

Applying Hölder inequality yields

$$\int_{y_K-h_y}^{y_K+h_y}\Big[\int_{y_K-h_y}^{y_K+h_y}dt\int_t^y dz\int_{x_K-h_x}^{x_K+h_x}\beta\,dx\Big]^2dy\le Ch_xh_y^4||\beta||_{0,K}^2. \tag{3.16}$$

Substituting (3.12)-(3.16) into (3.11), and applying Cauchy-Schwartz inequality, we have

$$\begin{aligned}|I_2+I_4|\le Ch_y^2\|\frac{\partial v_h}{\partial y}\|_{0,K}\Big[&\|\frac{\partial^3 u}{\partial x^2\partial y}\|_{0,K}+\|\frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial y}\|_{0,K}\\&+\|(\frac{\partial u}{\partial x})^2\frac{\partial u}{\partial y}\|_{0,K}+\|\frac{\partial^2 u}{\partial x\partial y}\frac{\partial u}{\partial y}\|_{0,K}\Big].\end{aligned} \tag{3.17}$$

Similarly, we obtain

$$\begin{aligned}|I_1+I_3|\le Ch_x^2\|\frac{\partial v_h}{\partial x}\|_{0,K}\Big[&\|\frac{\partial^3 u}{\partial y^2\partial x}\|_{0,K}+\|\frac{\partial^2 u}{\partial y^2}\frac{\partial u}{\partial x}\|_{0,K}\\&+\|(\frac{\partial u}{\partial y})^2\frac{\partial u}{\partial x}\|_{0,K}+\|\frac{\partial^2 u}{\partial y\partial x}\frac{\partial u}{\partial x}\|_{0,K}\Big].\end{aligned} \tag{3.18}$$

It follows from (3.17)-(3.18) that

$$\begin{aligned}&\Big|\sum_{K\in\mathcal{J}_h}\int_{\partial K}b(u)\frac{\partial u}{\partial n}v_h\,ds\Big|\\&\le C\sum_K(h_x^2+h_y^2)\big(|u|_1+|u|_2+|u|_3\big)||v_h||_h\le Ch^2||u||_3||v_h||_h,\end{aligned}$$

which completes the proof. □

Based on Lemmas 3.1-3.2, we have

Theorem 3.2. *Assume that u and u_h are the solutions of* (3.1) *and* (3.2), *respectively. If $u,u_t\in H^2(\Omega)$, then*

$$||u-u_h||_h\le Ch\bigg(|u|_2+\Big[\int_0^t\big(||u_t(\tau)||_2^2+||u(\tau)||_2^2\big)\,d\tau\Big]^{\frac12}\bigg). \tag{3.19}$$

Moreover, if $u,u_t\in H^3(\Omega)$, then

$$||I_hu-u_h||_h\le Ch^2\bigg(\int_0^t(||u(\tau)||_3^2+||u_t(\tau)||_3^2)\,d\tau\bigg)^{\frac12}. \tag{3.20}$$

Proof. Let $u-u_h=(u-I_hu)+(I_hu-u_h)=:\eta+\theta$. It is easy to see that for all $v_h\in V_h$, there holds the following error equation

$$\begin{aligned}&\big(a(u_h)\nabla\theta_t,\nabla v_h\big)_h+\big(b(u_h)\nabla\theta,\nabla v_h\big)_h\\&=-\Big(\big(a(u)-a(u_h)\big)\nabla u_t,\nabla v_h\Big)_h-\Big(\big(b(u)-b(u_h)\big)\nabla u,\nabla v_h\Big)_h-\big(a(u_h)\nabla\eta_t,\nabla v_h\big)_h\\&\quad-\big(b(u_h)\nabla\eta,\nabla v_h\big)_h+\sum_{K\in\mathcal{J}_h}\int_{\partial K}a(u)\frac{\partial u_t}{\partial n}v_h\,ds+\sum_{K\in\mathcal{J}_h}\int_{\partial K}b(u)\frac{\partial u}{\partial n}v_h\,ds.\end{aligned}\tag{3.21}$$

Firstly, by (1.2), we have

$$\big(a(u_h)\nabla\theta_t,\nabla\theta_t\big)_h+\big(b(u_h)\nabla\theta,\nabla\theta_t\big)_h\ge a_0||\nabla\theta_t||^2+\frac{b_0}{2}\frac{d}{dt}||\nabla\theta||^2.\tag{3.22}$$

Secondly, Lemma 3.1, (1.2), ε-Young inequality and Cauchy-Schwartz inequality imply that

$$\begin{aligned}&\Big|\Big(\big(a(u)-a(u_h)\big)\nabla u_t,\nabla\theta_t\Big)_h\Big|\\&\le C_a||u_t||_{W^{1,\infty}(\Omega)}(||\eta||+||\theta||)||\nabla\theta_t||\le C\left(||\eta||^2+||\nabla\theta||^2\right)+\varepsilon||\nabla\theta_t||^2.\end{aligned}\tag{3.23}$$

Similarly,

$$\Big|\Big(\big(b(u)-b(u_h)\big)\nabla u,\nabla\theta_t\Big)_h\Big|\le C\left(||\eta||^2+||\nabla\theta||^2\right)+\varepsilon||\nabla\theta_t||^2.\tag{3.24}$$

Applying (3.7), (3.8) and ε-Young inequality to yield

$$\begin{aligned}&\Big|\sum_{K\in\mathcal{J}_h}\int_{\partial K}b(u)\frac{\partial u}{\partial n}\theta_t ds+\sum_{K\in\mathcal{J}_h}\int_{\partial K}a(u)\frac{\partial u_t}{\partial n}\theta_t ds\Big|\\&\le Ch^2\left(||u_t||_2^2+||u||_2^2\right)+\varepsilon||\nabla\theta_t||^2.\end{aligned}\tag{3.25}$$

With properly small ε, substituting (3.22)-(3.25) into (3.21) with $v_h=\theta_t$ gives

$$\begin{aligned}&||\nabla\theta_t||^2+\frac{d}{dt}||\nabla\theta||^2\\&\le C\Big(h^2||u_t||_2^2+h^2||u||_2^2+||\eta||^2+||\nabla\eta||^2+||\nabla\eta_t||^2+||\nabla\theta||^2\Big).\end{aligned}\tag{3.26}$$

Integrating both sides of (3.26) from 0 to t, and noticing that $\theta(0)=0$, we obtain

$$||\nabla\theta||^2\le C\int_0^t\Big(h^2||u_t||_2^2+h^2||u||_2^2+||\eta||^2+||\nabla\eta||^2+||\nabla\eta_t||^2+||\nabla\theta||^2\Big)d\tau.\tag{3.27}$$

Here, we have omitted a positive term $||\nabla\theta_t||^2$ on the left hand of (3.27) in order to coincide with the following analysis. Then applying Gronwall lemma and (3.6), we have

$$||\nabla\theta||^2\le C\int_0^t\Big(h^2||u_t(\tau)||_2^2+h^2||u(\tau)||_2^2\Big)d\tau.\tag{3.28}$$

By the triangle inequality and Lemma 3.1, we get (3.19).

On the other hand, by Lemma 3.1, we have

$$\begin{aligned}&\big|\big(a(u_h)\nabla\eta_t,\nabla\theta_t\big)_h-\big(b(u_h)\nabla\eta,\nabla\theta_t\big)_h\big|\\&=\Big|\Big(\big(a(u_h)-\overline{a(u_h)}\big)\nabla\eta_t,\nabla\theta_t\Big)_h-\Big(\big(b(u_h)-\overline{b(u_h)}\big)\nabla\eta,\nabla\theta_t\Big)_h\Big|\\&\le Ch^2\left(||\nabla\eta_t||^2+||\nabla\eta||^2\right)+\varepsilon||\nabla\theta_t||^2,\end{aligned}\tag{3.29}$$

where

$$\overline{a(u_h)} = \frac{1}{|K|}\int_K a(u_h)\,dx\,dy, \qquad \overline{b(u_h)} = \frac{1}{|K|}\int_K b(u_h)\,dx\,dy.$$

Similarly, with sufficiently small ε, we have

$$||\nabla\theta||^2 \le C\int_0^t \Big(h^4||u||_3^2 + h^4||u_t||_3^2 + h^2||\nabla\eta||^2 + ||\eta||^2 + ||\nabla\theta||^2 + h^2||\nabla\eta_t||^2\Big)\,d\tau.$$

The estimate (3.20) follows by the triangle inequality and Lemma 3.1. □

Now we give the following estimate in L^2-norm.

Theorem 3.3. *Let u and u_h be the solutions of* (3.1) *and* (3.2)*, respectively. For $u, u_t \in H^3(\Omega)$, we have*

$$\|u - u_h\| \le Ch^2\Big\{|u|_2 + \Big(\int_0^t \big(||u_t(\tau)||_3^2 + ||u(\tau)||_3^2\big)\,d\tau\Big)^{\frac{1}{2}}\Big\}. \tag{3.30}$$

Proof. Similarly to the proof of Theorem 3.2, we choose $v_h = \theta \in V_h$ in (3.21). Using Lemmas 3.1-3.2 and ε-Young inequality gives

$$\frac{d}{dt}\|\nabla\theta\|^2 \le C\Big(||\eta||^2 + h^2||\nabla\eta||^2 + ||\nabla\theta||^2 + h^2||\nabla\eta_t||^2 + h^4||u||_3^2 + h^4||u_t||_3^2\Big). \tag{3.31}$$

Integrating both sides of (3.31) from 0 to t and noticing that $\theta(0) = 0$ yield

$$||\nabla\theta||^2 \le C\int_0^t \Big(||\eta||^2 + ||\nabla\theta||^2 + h^2||\nabla\eta||^2 + h^2||\nabla\eta_t||^2 + h^4||u||_3^2 + h^4||u_t||_3^2\Big)\,d\tau.$$

By Gronwall lemma and Lemma 3.1, we obtain

$$||\theta|| \le C\|\nabla\theta\|^2 \le Ch^4\int_0^t \Big(||u_t(\tau)||_3^2 + ||u(\tau)||_3^2\Big)d\tau. \tag{3.32}$$

Finally, applying the triangle inequality and Lemma 3.1, we complete the proof. □

4. Backward Euler-Galerkin Scheme

In this section, we consider a backward Euler-Galerkin scheme for (1.1) and present the corresponding optimal error analysis.

Let $\triangle t$ and $u_h(t_n) \in V_h$ be the time step and the approximation of $u(t)$ at time $t = t_n$, respectively. The time discretization scheme will be established with the backward difference quotient

$$\bar{\partial}_t u_h(t_n) = \frac{u_h(t_n) - u_h(t_{n-1})}{\triangle t}.$$

On time level $t = t_n$, we can rewrite (3.2) as

$$\begin{cases} \big(a(u_h(t_n))\bar{\partial}_t\nabla u_h(t_n), \nabla v_h\big)_h + \big(b(u_h(t_n))\nabla u_h(t_n), \nabla v_h\big)_h = (f(t_n), v_h), \quad \forall v_h \in V_h, \\ u_h(0) = I_h u(0). \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $u(t_n)$ and $u_h(t_n)$ be the solutions of* (3.1) *and* (4.1)*, respectively. Suppose that $u, u_t, u_{tt} \in H^3(\Omega)$, then the following error estimates hold*

$$\begin{aligned}
&\|I_h u(t_n) - u_h(t_n)\|_h \\
&\le Ch^2\Big(\int_0^{t_n} (||u_t(\tau)||_3^2 + ||u_{tt}(\tau)||_3^2 + ||u(\tau)||_3^2)\, d\tau\Big)^{\frac{1}{2}} + C(\triangle t)\Big(\int_0^{t_n} |u_{tt}(\tau)|_1^2\, d\tau\Big)^{\frac{1}{2}}, \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
\|u(t_n) - u_h(t_n)\| \le Ch^2\Big\{\int_0^{t_n} |u_t(\tau)|_2\, d\tau + \Big(\int_0^{t_n} (||u_t(\tau)||_3^2 + ||u_{tt}(\tau)||_3^2 + ||u(\tau)||_3^2)\, d\tau\Big)^{\frac{1}{2}} \\
+ |u_0|_2\Big\} + C(\triangle t)\Big(\int_0^{t_n} |u_{tt}(\tau)|_1^2\, d\tau\Big)^{\frac{1}{2}}, \qquad (4.3)
\end{aligned}$$

$$\begin{aligned}
\|u(t_n) - u_h(t_n)\|_h \le Ch\Big\{\Big(\int_0^{t_n} (||u_t(\tau)||_2^2 + ||u_{tt}(\tau)||_2^2 + ||u(\tau)||_2^2)\, d\tau\Big)^{\frac{1}{2}} \\
+ |u_0|_2 + \int_0^{t_n} |u_t(\tau)|_2\, d\tau\Big\} + C(\triangle t)\Big(\int_0^{t_n} |u_{tt}(\tau)|_1^2\, d\tau\Big)^{\frac{1}{2}}. \qquad (4.4)
\end{aligned}$$

Proof. Set $u(t_n) - u_h(t_n) = \big(u(t_n) - I_h u(t_n)\big) + \big(I_h u(t_n) - u_h(t_n)\big) = \eta^n + \theta^n$. There holds the following error equation for all $v_h \in V_h$

$$\begin{aligned}
&\big(a(u_h)\bar{\partial}_t \nabla\theta^n, \nabla v_h\big)_h + \big(b(u_h)\nabla\theta^n, \nabla v_h\big)_h \\
&= \big(a(u_h)R^n, \nabla v_h\big)_h - \Big(\big(a(u) - a(u_h)\big)\nabla u_t(t_n), \nabla v_h\Big)_h - \Big(\big(b(u) - b(u_h)\big)\nabla u(t_n), \nabla v_h\Big)_h \\
&\quad - \big(b(u_h)\nabla\eta^n, \nabla v_h\big)_h + \sum_{K\in\mathcal{J}_h}\int_{\partial K} a(u)\frac{\partial u_t(t_n)}{\partial n} v_h\, ds + \sum_{K\in\mathcal{J}_h}\int_{\partial K} b(u)\frac{\partial u(t_n)}{\partial n} v_h\, ds, \qquad (4.5)
\end{aligned}$$

where

$$\begin{aligned}
R^n &= \bar{\partial}_t I_h \nabla u(t_n) - \nabla u_t(t_n) = \bar{\partial}_t \nabla\eta^n + \bar{\partial}_t \nabla u(t_n) - \nabla u_t(t_n) \\
&= -\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n} \nabla\eta_t(\tau)\, d\tau - \frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\int_{\tau}^{t_n} \nabla u_{tt}(\sigma)\, d\sigma\, d\tau. \qquad (4.6)
\end{aligned}$$

By the property (1.2), we get

$$\begin{aligned}
&\big(a(u_h)\bar{\partial}_t \nabla\theta^n, \nabla\theta^n\big)_h + \big(b(u_h)\nabla\theta^n, \nabla\theta^n\big)_h \\
&\ge a_0(2\triangle t)^{-1}\Big(||\nabla\theta^n||^2 - ||\nabla\theta^{n-1}||^2 + ||\nabla\theta^n - \nabla\theta^{n-1}||^2\Big) + b_0||\nabla\theta^n||^2. \qquad (4.7)
\end{aligned}$$

Applying the Cauchy-Schwartz inequality and ε-Young inequality yields

$$\begin{aligned}
&|\big(a(u_h)R^n, \nabla\theta^n\big)_h| \\
&\le C\Big[\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n} ||\nabla\eta_t(\tau)||^2 d\tau + \triangle t\int_{t_{n-1}}^{t_n} ||\nabla u_{tt}(\tau)||^2 d\tau\Big] + \varepsilon||\nabla\theta^n||^2. \qquad (4.8)
\end{aligned}$$

It follows from the assumption (1.3) and ε-Young inequality that

$$\begin{aligned}
&\Big|\Big(\big(a(u) - a(u_h)\big)\nabla u_t, \nabla\theta^n\Big)_h - \Big(\big(b(u) - b(u_h)\big)\nabla u, \nabla\theta^n\Big)_h\Big| \\
&\le C\Big(||\eta^n||^2 + ||\nabla\theta^n||^2\Big) + \varepsilon||\nabla\theta^n||^2. \qquad (4.9)
\end{aligned}$$

Similarly, by (1.2) and ε-Young inequality, we obtain

$$|\big(b(u_h)\nabla\eta^n,\nabla\theta^n\big)_h| \le C||\nabla\eta^n||^2+\varepsilon||\nabla\theta^n||. \tag{4.10}$$

By using (3.7) and ε-Young inequality, we get

$$\begin{aligned}&\Big|\sum_{K\in\mathcal{J}_h}\int_{\partial K} a(u)\frac{\partial u_t(t_n)}{\partial n}\theta^n\,ds+\sum_{K\in\mathcal{J}_h}\int_{\partial K} b(u)\frac{\partial u(t_n)}{\partial n}\theta^n\,ds\Big|\\ &\le Ch^2||u_t(t_n)||_2^2+Ch^2||u(t_n)||_2^2+\varepsilon||\nabla\theta^n||^2.\end{aligned} \tag{4.11}$$

Combining the above inequalities from (4.7) to (4.11) with $v_h=\theta^n$ in (4.5) , and choosing ε small enough, we can derive

$$\begin{aligned}||\nabla\theta^n||^2-||\nabla\theta^{n-1}||^2 \le & C(\triangle t)\Big[\int_{t_{n-1}}^{t_n}\big(\triangle t^{-1}||\nabla\eta_t(\tau)||^2+\triangle t||\nabla u_{tt}(\tau)||^2\big)\,d\tau+||\eta^n||^2\\ &+||\nabla\eta^n||^2+||\nabla\theta^n||^2+h^2||u_t(t_n)||_2^2+h^2||u(t_n)||_2^2\Big].\end{aligned} \tag{4.12}$$

Summing up form $i=1$ to n, applying Gronwall lemma and noticing that $\theta(0)=0$, we obtain

$$\begin{aligned}||\nabla\theta^n||^2 \le & C(\triangle t)\Big[\sum_{i=1}^{n}\big(||\eta^i||^2+||\nabla\eta^i||+h^2||u_t(t_i)||_2^2+h^2||u(t_i)||_2^2\big)\Big]\\ &+C\Big[\int_0^{t_n}\big(||\nabla\eta_t(\tau)||^2+\triangle t^2||\nabla u_{tt}(\tau)||^2\big)\,d\tau\Big].\end{aligned} \tag{4.13}$$

By the integral technique, we have

$$\begin{aligned}&\sum_{i=1}^{n}\big(||u_t(t_i)||_2^2+||u(t_i)||_2^2\big)\\ &\le \triangle t\Big[\int_0^{t_n}\big(||u_{tt}(\tau)||_2^2+||u_t(\tau)||_2^2+||u(\tau)||_2^2\big)\,d\tau\Big].\end{aligned} \tag{4.14}$$

The result (4.2) follows from triangle inequality and Lemma 3.1.

Now we begin to analyze (4.3) and (4.4) by using the technique of [1]. Since

$$\begin{aligned}&|\big(a(u_h)R^n,\nabla\theta^n\big)_h|\\ =&\Big|-\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\Big(\big(a(u_h)-\overline{a(u_h)}\big)\nabla\eta_t(\tau),\nabla\theta^n\Big)_h\,d\tau\\ &-\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\int_{\tau}^{t_n}\Big(\big(a(u_h)-\overline{a(u_h)}\big)\nabla u_{tt}(s),\nabla\theta^n\Big)_h\,ds\,d\tau\\ &-\frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\int_{\tau}^{t_n}\big(\overline{a(u_h)}\nabla u_{tt}(s),\nabla\theta^n\big)_h\,ds\,d\tau\Big|\\ \le & C\int_{t_{n-1}}^{t_n}\Big(\frac{h^2}{\triangle t}||\nabla\eta_t(\tau)||^2+\triangle t||\nabla u_{tt}(\tau)||^2\Big)\,d\tau+\varepsilon||\nabla\theta^n||^2,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\big|\big(b(u_h)\nabla\eta^n,\nabla\theta^n\big)_h\big| &= \big|\Big(\big(b(u_h)-\overline{b(u_h)}\big)\nabla\eta^n,\nabla\theta^n\Big)_h\big|\\ &\le Ch^2||\nabla\eta^n||^2+\varepsilon||\nabla\theta^n||^2.\end{aligned} \tag{4.16}$$

By (3.9) and (3.10), we get

$$\begin{aligned}&\Big|\sum_{K\in\mathcal{J}_h}\int_{\partial K} a(u)\frac{\partial u_t}{\partial n}\theta^n\,ds+\sum_{K\in\mathcal{J}_h}\int_{\partial K} b(u)\frac{\partial u}{\partial n}\theta^n\,ds\Big|\\ &\le Ch^4||u_t||_3^2+Ch^4||u||_3^2+\varepsilon||\nabla\theta^n||^2. \end{aligned}\tag{4.17}$$

Substituting (4.7),(4.9) and (4.15)-(4.17) with $v_h=\theta^n$ in (4.5), and choosing properly small ε to yield

$$\begin{aligned}||\nabla\theta^n||^2-||\nabla\theta^{n-1}||^2\le &C(\triangle t)\Big[\int_{t_{n-1}}^{t_n}\Big(\frac{h^2}{\triangle t}||\nabla\eta_t(\tau)||^2+\triangle t||\nabla u_{tt}(\tau)||^2\Big)\,d\tau+||\eta||^2\\ &+h^2||\nabla\eta||^2+||\nabla\theta^n||^2+h^4||u_t||_3^2+h^4||u||_3^2\Big]. \end{aligned}\tag{4.18}$$

Similarly, by summing up n and using Gronwall lemma, we have

$$\begin{aligned}||\nabla\theta^n||^2\le C(\triangle t)\Big[&\sum_{i=1}^n||\eta^i||^2+h^2||\nabla\eta^i||+h^4||u_t(t_i)||_3^2+h^4||u(t_i)||_3^2\Big]\\ &+C\Big(\int_0^{t_n}||\nabla\eta_t(\tau)||^2+\triangle t^2||\nabla u_{tt}(\tau)||^2\,d\tau\Big). \end{aligned}\tag{4.19}$$

Noticing that

$$\sum_{i=1}^n\Big(||u_t(t_i)||_3^2+||u(t_i)||_3^2\Big)\le C(\triangle t)\int_0^{t_n}\Big(||u_{tt}||_3^2+||u_t||_3^2+||u||_3^2\Big)\,d\tau,$$

by Lemma 3.1, we obtain

$$\begin{aligned}&||\theta^n||\le C||\nabla\theta^n||\\ &\le C\Big[h^2\Big(\int_0^{t_n}(||u(\tau)||_3^2+||u_t(\tau)||_3^2+||u_{tt}(\tau)||_3^2)d\tau\Big)^{\frac12}+\triangle t\Big(\int_0^{t_n}|u_{tt}(\tau)|_1^2\,d\tau\Big)^{\frac12}\Big]. \end{aligned}\tag{4.20}$$

Using the triangle inequality together with Lemma 3.1 leads to the desired results. □

5. Global Superconvergence

Let $\mathcal{J}_{2h}=\{\tilde{K}\}$ be an rectangular partition of Ω parallel with axis. Dividing each $\tilde{K}$ into four equal rectangles yields the new rectangular anisotropic partition $\mathcal{J}_h$ of Ω. That is to say, $\tilde{K}=\bigcup_{i=1}^4 K_i, K_i\in\mathcal{J}_h(i=1,2,3,4)$. Let L_1,L_2,L_3 and L_4 be the four edges of $\tilde{K}$. In order to get the superconvergence result, we construct the following post-processing interpolation operator I_{2h}^2 on $\tilde{K}$ as follows (see [23])

$$\begin{cases} I_{2h}^2u|_{\tilde{K}}\in P_2(\tilde{K}),\quad\forall\tilde{K}\in\mathcal{J}_{2h},\\ \displaystyle\int_{L_i}(I_{2h}^2u-u)\,ds=0,\quad i=1,2,3,4,\\ \displaystyle\int_{K_1\cup K_3}(I_{2h}^2u-u)\,dx\,dy=0,\quad\int_{K_2\cup K_4}(I_{2h}^2u-u)\,dx\,dy=0,\quad\forall\tilde{K}\in\mathcal{J}_{2h}, \end{cases}\tag{5.1}$$

where $P_2(\tilde{K})$ denotes the set of polynomials of degree 2.

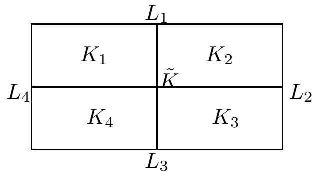

Fig. 5.1. Illustration of an element $\tilde{K}$ which consists of four small elements.

Lemma 5.1. ([23]) *On the anisotropic meshes, for all* $u \in H^3(\Omega)$*, the interpolation operator* I_{2h}^2 *satisfies*

$$I_{2h}^2 I_h u = I_{2h}^2 u, \quad \|I_{2h}^2 u - u\|_h \le Ch^2 |u|_3, \tag{5.2}$$

$$\|I_{2h}^2 v\|_h \le C\|v\|_h, \quad \forall v \in V_h. \tag{5.3}$$

Theorem 5.1. *Under the assumptions of Theorems* 3.2 *and* 4.1*, we can get the following global superconvergence result*

$$\|u - I_{2h}^2 u_h\|_h \le Ch^2 \Bigg[||u||_3 + \bigg(\int_0^t (||u_t(\tau)||_3^2 + ||u(\tau)||_3^2)\, d\tau \bigg)^{\frac{1}{2}} \Bigg]. \tag{5.4}$$

Proof. Note that

$$I_{2h}^2 u_h - u = I_{2h}^2 u_h - I_{2h}^2 I_h u + I_{2h}^2 I_h u - u. \tag{5.5}$$

By (5.2), we obtain

$$\|I_{2h}^2 I_h u - u\|_h = \|I_{2h}^2 u - u\|_h \le Ch^2 |u|_3. \tag{5.6}$$

Consequently, it follows from (5.3) that

$$\begin{aligned} &\|I_{2h}^2 u_h - I_{2h}^2 I_h u\|_h \\ =& \|I_{2h}^2 (u_h - I_h u)\|_h \le C\|u_h - I_h u\|_h \\ \le& Ch^2 \Bigg[||u||_3 + \bigg(\int_0^t (||u_t(\tau)||_3^2 + ||u(\tau)||_3^2)\, d\tau \bigg)^{\frac{1}{2}} \Bigg]. \end{aligned} \tag{5.7}$$

Thus we can get the desired result via (5.5)-(5.7). □

6. Numerical Experiments

In order to illustrate our theoretical analysis in previous sections, we carry out two numerical simulations using the nonconforming finite element for the nonlinear Sobolev equations (1.1).

Experiment 1. Given $a(u) = \sin(u) + 1.01$, $b(u) = \sin(u) + 1.01$. Then the exact solution is $u = e^t xy(1-x)(1-y)$.

We consider two meshes on Ω: Mesh 1 and Mesh 2 as shown in Fig. 6.1. Mesh 1 are square meshes and Mesh 2 are rectangular meshes with n divisions along the x-axis and m divisions along the y-axis, respectively, where $n/m = 1/10$.

The numerical solutions on Mesh 1 and Mesh 2 are plotted in Fig. 6.2, which are found in good agreement with the exact solution.

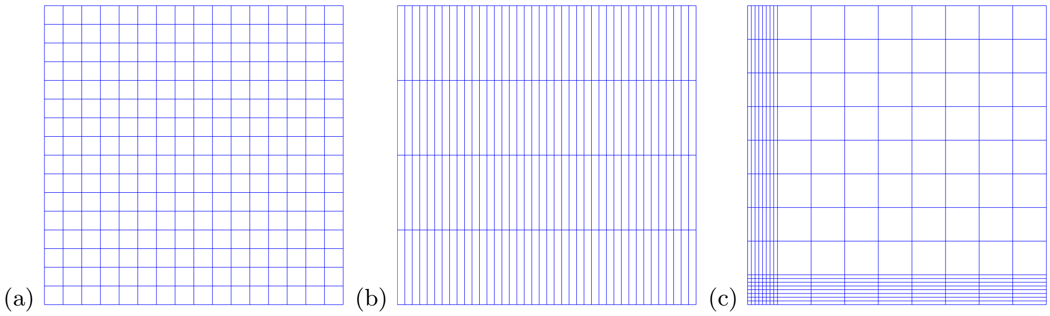

Fig. 6.1. (a) Mesh 1; (b) Mesh 2; (c) Mesh 3.

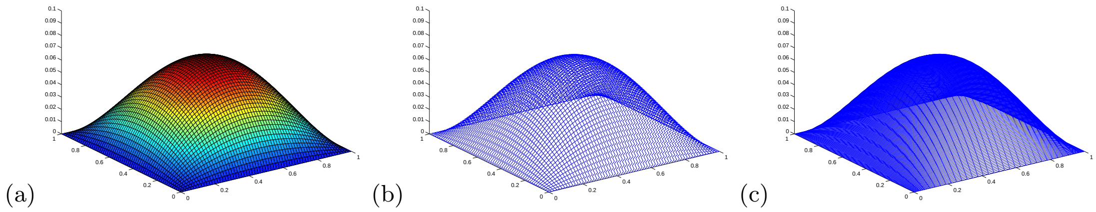

Fig. 6.2. Experiment 1 at $t = 0.1$: (a) exact solution; (b) FEM solution on Mesh 1; (c) FEM solution on Mesh 2.

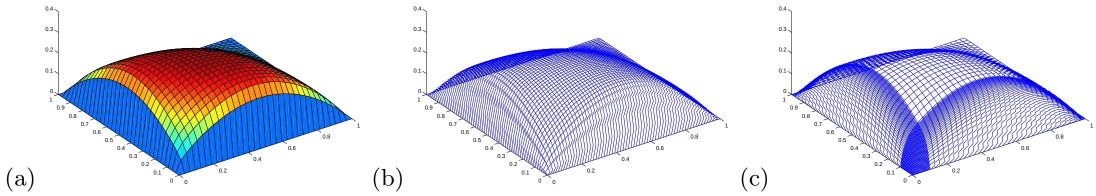

Fig. 6.3. Experiment 2 at $t = 0.1$ and $\mu = 0.02$: (a) exact solution; (b) FEM solution on Mesh 1; (c) FEM solution on Mesh 3.

Experiment 2. Given $a(u) = \sin(u) + 1.01$, $b(u) = \sin(u) + 1.01$. Then the exact solution is

$$u = e^t[y(1-x)(1-y)(1-e^{-\frac{x}{\mu}}) + x(1-x)(1-y)(1-e^{-\frac{y}{\mu}})].$$

When μ is small enough, the exact solution varies significantly near the boundary of the domain. We denote the boundary layer of the two edges of Ω ($x = 0$ and $y = 0$) by $(0, 0.1) \times (0, 1)$ and $(0, 1) \times (0, 0.1)$), respectively. Each boundary layer is divided into n segments with $0.5n$ segments in $[0, 0.1]$ and $0.5n$ segments in $[0.1, 1.0]$. Here we consider $\mu = 0.02$.

The numerical solutions on Mesh 1 and Mesh 3 are shown in Fig. 6.3, where Mesh 3 gives refined mesh near the layers. It is clear that Mesh 3 gives better numerical solution.

In Tables 6.1-6.3, u_h and $I_h u$ denote the finite element solution of problem (3.2) and the interpolation of u; α and $I_{2h}^2 u_h$ represent the average convergence order and the post-processing interpolation of u_h, respectively.

Tables 6.1 and 6.2 give the numerical errors obtained for Experiment 1 using Mesh 1 and Mesh 2. As the exact solutions are smooth, as expected that both meshes yield similar accuracy.

In order to show the good performances of anisotropic meshes, we give a comparison of the numerical results of Experiment 2 on Mesh 1 and Mesh 3 at $t = 0.1$. Tables 6.3 and 6.4 present

Table 6.1: Numerical results of Experiment 1 on Mesh 1 at $t = 1$.

$n \times m$	4×4	8×8	16×16	32×32	α
$\|\|u - u_h\|\|_h$	0.0929514859	0.0462722597	0.0231269933	0.0115600372	1.0024439624
$\|\|u - u_h\|\|$	0.0068196986	0.0016830983	0.0004311137	0.0001044366	2.0096695597
$\|\|I_h u - u_h\|\|_h$	0.0227032573	0.0054598915	0.0014208785	0.0003339775	2.0290015259
$\|\|u - I_{2h}^2 u_h\|\|_h$	0.0928848664	0.0252562833	0.0064488990	0.0016152231	1.9485460686

Table 6.2: Numerical results of Experiment 1 on Mesh 2 at $t = 1$.

$n \times m$	2×20	4×40	8×80	16×160	α
$\|\|u - u_h\|\|_h$	0.1231389904	0.0647267605	0.0323633802	0.0161816901	0.9759511473
$\|\|u - u_h\|\|$	0.0080782671	0.0020553902	0.0005138475	0.0001284618	1.9915448421
$\|\|I_h u - u_h\|\|_h$	0.0361811121	0.0091947673	0.0022986918	0.0005746729	1.9921173084
$\|\|u - I_{2h}^2 u_h\|\|_h$	0.1239241398	0.0317862127	0.0079465531	0.0019866382	1.9876614255

Table 6.3: Numerical results of Experiment 2 on Mesh 3 at $t = 0.1$.

$n \times m$	8×8	16×16	32×32	64×64	α
$\|\|u - u_h\|\|_h$	0.2717014445	0.1334134968	0.0663179095	0.0331020769	1.0123428636
$\|\|u - u_h\|\|$	0.0038727749	0.0009551080	0.0002389055	0.0000597034	2.0064700461
$\|\|I_h u - u_h\|\|_h$	0.0333197803	0.0066401811	0.0014492098	0.0003398491	2.2051135645
$\|\|u - I_{2h}^2 u_h\|\|_h$	0.3666523703	0.1195242932	0.0324563756	0.0084233402	1.8146255371

Table 6.4: Numerical results of Experiment 2 on Mesh 1 at $t = 0.1$.

$n \times m$	8×8	16×16	32×32	64×64	α
$\|\|u - u_h\|\|_h$	0.5486807867	0.3335207890	0.1289363686	0.0440594955	1.2128140581
$\|\|u - u_h\|\|$	0.0127328763	0.0030821459	0.0005954116	0.0001152333	2.2626188387
$\|\|I_h u - u_h\|\|_h$	0.0127759516	0.0020869701	0.0006337161	0.0001467605	2.1479409198
$\|\|u - I_{2h}^2 u_h\|\|_h$	0.5021471271	0.5368144150	0.3222638558	0.1170999050	0.7001234081

the numerical results at $t = 0.1$ on Mesh 3 and Mesh 1, respectively. It is observed that the numerical errors in Table 6.3 are smaller than those in Table 6.4; and the superconvergence result in Table 6.4 is very poor. We conclude that when the solution varies significantly only in certain directions, to use anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the perpendicular direction is indeed a good choice. In other words, the quasi-uniform assumption in the traditional finite element analysis is not appropriate.

It can be seen from the Tables 6.1-6.4 that on the anisotropic meshes, when $h \to 0$, $||u-u_h||_h$, $||u - u_h||$, $||I_h u - u_h||_h$ and $||u - I_{2h}^2 u_h||_h$ converge at optimal rates of $\mathcal{O}(h)$, $\mathcal{O}(h^2)$, $\mathcal{O}(h^2)$ and $\mathcal{O}(h^2)$, respectively, which coincide with our theoretical predictions.

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References

[1] Q. Lin and N.N. Yan, Construction and Analysis for Effective Finite Element Methods, Baoding, Hebei University Press, 1996.

[2] T.J. Sun and D.P. Yang, A priori error estimates for symmetric interior penalty discontinuous Galerkin method applied to nonlinear Sobolev equations, *Appl. Math. Comput.* **200** (2008), 147-

159.
[3] Y.H. Cao, The generalized difference scheme for linear Sobolev equationin in two dimensions, *Math. Numer. Sini.*, **27** (2005), 243-256.
[4] H. Guo and H.X. Rui, Least-squares Galerkin procedures for Sobolev equations, *Math. Appl. Sini.*, **29** (2006), 610-618.
[5] L. Guo and H.Z. Chen, H^1-Galerkin mixed finite element method for the Sobolev equation, *J. Syst. Scie. and Math. Scie.*, **26** (2006), 301-314.
[6] T.J. Sun and D.P. Yang, The finite difference streamline diffusion methods for Sobolev equations with convection-dominated term, *Appl. Math. Comput.*, **125** (2002), 325-345.
[7] H.M. Gu, Charcteristic finite element methods for nonlinear Sobolev equations, *Appl. Math. Comput.*, **102** (1999), 51-62.
[8] P.G. Ciarlet, The Finite Element Method for Elliptic Problem, Amsterdam, North-Holland, 1978.
[9] G. Acosta and R.G. Duran, The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equations, *SIAM J. Numer. Anal.*, **37** (1999), 18-36.
[10] G. Acosta, Langrange and average interpolation over 3D anisotropic meshes, *J. Comput. Appl. Math.*, **135** (2001), 91-109.
[11] T. Apel and M. Dobrowolski, Anisotropic interpolation with applications to the finite element method, *Computing*, **47** (1992), 277-293.
[12] T. Apel, Anisotropic Finite Element: Local Estimates and Applications, Stuttgart Teubner, 1999.
[13] T. Apel, S. Nicaise and J. Schӧberl, Crouzeix-Raviart type finite elements on anisotropic meshes, *Numer. Math.*, **89** (2001), 193-223.
[14] R.G. Duran and A.L. Lombardi, Error estimates on anisotropic Q_1 elements for functions in weighted sobolev spaces, *Math. Comput.,* **74** (2005), 1679-1706.
[15] D.Y. Shi, S.P. Mao and S.C. Chen, A locking-free anisotropic nonconforming finite element for planar linear elasticity problem, *Acta. Math. Sci.,* **27B** (2007), 193-202.
[16] D.Y. Shi and Y.R. Zhang, Rectangular Crouzeix-raviart anisotropic finite element method for nonstationary Stokes problem with moving grids, *Acta. Math. Sci.,* **26A** (2006), 659-670.
[17] D.Y. Shi, S.P. Mao and S.C. Chen, A class of anisotropic Crouzeix-Raviart type finite element approximations to signorini variational inequality problem, *Math. Numer. Sini.*, **27** (2005), 45-54.
[18] D.Y. Shi and H.B. Guan, A class of Crouzeix-Raviart type nonconforming finite element methods for parabolic variational inequality problem with moving grid on anisotropic meshes, *Hokkaido Math. J.,* **36** (2007), 687-709.
[19] S.C. Chen, D.Y. Shi and Y.C. Zhao, Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes, *IMA J. Numer. Anal.*, **24** (2004), 77-95.
[20] D.Y. Shi and H. Liang, superconvergence analysis of Wilson's element on anisotropic meshes, *Appl. Math. and Meth.*, **28** (2007), 119-125.
[21] D.Y. Shi and H. Liang, Superconvergence analysis and extrapolation of a new unconventional Hermite-type anisotropic rectangular element, *Math. Numer. Sini.*, **27** (2005), 369-382.
[22] D.Y. Shi, S.P. Mao and S. C. Chen, Superconvergence analysis of ACM plate element on anisotropic meshes, *J. Comput. Math.*, **23** (2005), 635-646.
[23] D.Y. Shi, P.L. Xei and S.C. Chen, The nonconforming finite element approximation to hyperbolic integro-differential equations on anisotropic meshes, *Math. Appl. Sini.*, **30** (2007), 654-666.
[24] D.Y. Shi and L.F. Pei, Low order Crouzeix-Raviart type nonconforming finite element methods for approximating Maxwell's equations, *Int. J. Numer. Anal. Mod.,* **5** (2008), 373-385.
[25] D.Y. Shi, S.P. Mao, and S.C. Chen, An anisotropic noncomforming finite element with some superconvergence results, *J. Comput. Math.*, **23** (2005), 261-274.
[26] Q. Lin, L. Tobiska and A.H. Zhou, Superconvergence and extrapolation of nonconforming low order finite elements applied to the Poisson equation, *IMA, J. Numer. Anal.*, **25** (2005), 160-181.
[27] J.K. Hale, Ordinary Differential Equations, New York, Willey-Interscience, 1969.