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THE HP-VERSION OF BEM – FAST CONVERGENCE, ADAPTIVITY AND EFFICIENT PRECONDITIONING*

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Abstract

In this survey paper we report on recent developments of the *hp*-version of the boundary element method (BEM). As model problems we consider weakly singular and hypersingular integral equations of the first kind on a planar, open surface. We show that the Galerkin solutions (computed with the *hp*-version on geometric meshes) converge exponentially fast towards the exact solutions of the integral equations. An *hp*-adaptive algorithm is given and the implementation of the *hp*-version BEM is discussed together with the choice of efficient preconditioners for the ill-conditioned boundary element stiffness matrices. We also comment on the use of the *hp*-version BEM for solving Signorini contact problems in linear elasticity where the contact conditions are enforced only on the discrete set of Gauss-Lobatto points. Numerical results are presented which underline the theoretical results.

Mathematics subject classification: 65N55.
Key words: *hp*-version of the boundary element method, Adaptive refinement, Preconditioning, Signorini contact.

1. Exponential Convergence

In this paper we consider the *hp*-version of the boundary element method (BEM) for Dirichlet and Neumann screen problems of the Laplacian in $\mathbb{R}^3\backslash\overline{\Gamma}$, where Γ is a planar surface piece with polygonal boundary (for details see also the survey paper [18]). That is, given f or g on Γ find $u \in \mathbb{R}^3\backslash\overline{\Gamma}$ satisfying

$$\begin{aligned}
&\Delta u = 0 \text{ in } \mathbb{R}^3\backslash\bar{\Gamma},\\
&u = f \in H^{1/2}(\Gamma) \text{ (Dirichlet)} \quad \text{or} \quad \frac{\partial u}{\partial n} = g \in H^{-1/2}(\Gamma)\text{(Neumann)},\\
&u = \mathcal{O}(|x|^{-1}) \quad \text{as } |x| \to \infty.
\end{aligned}$$

These exterior boundary value problems are called screen problems and can be formulated equivalently as first kind integral equations with weakly singular and hypersingular kernels, namely

$$V\psi(x) := \frac{1}{2\pi}\int_\Gamma \frac{1}{|x-y|}\psi(y)\,ds_y = 2f(x),\ \ x \in \Gamma \text{ (Dirichlet)}, \tag{1.1}$$

$$Wv(x) := -\frac{1}{2\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}\frac{1}{|x-y|}v(y)\,ds_y = 2g(x),\ \ x \in \Gamma \text{ (Neumann)}. \tag{1.2}$$

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As we have shown in [17] these integral equations have unique solutions $\psi \in \tilde{H}^{-1/2}(\Gamma)$, $v \in \tilde{H}^{1/2}(\Gamma) = H^{1/2}_{00}(\Gamma)$.

The Galerkin boundary element schemes for (1.1) and (1.2) read with the L^2-duality on Γ $\langle \cdot, \cdot \rangle$: Find $\psi_N \in S^0_{h,p}$

$$\langle V\psi_N, \phi_N \rangle = \langle 2f, \phi_N \rangle, \quad \forall \phi_N \in S^0_{h,p} \subset \tilde{H}^{-1/2}(\Gamma), \tag{1.3}$$

and find $v_N \in S^1_{h,p}$

$$\langle Wv_N, w_N \rangle = \langle 2g, w_N \rangle, \quad \forall w_N \in S^1_{h,p} \subset \tilde{H}^{1/2}(\Gamma). \tag{1.4}$$

Since the operators V and W define coercive, continuous bilinear forms we immediately have quasi-optimality of the Galerkin errors:

$$\|\psi - \psi_N\|_{\tilde{H}^{-1/2}(\Gamma)} \lesssim \operatorname{dist}\big(\psi, S^0_{h,p}(\Gamma)\big),$$
$$\|v - v_N\|_{\tilde{H}^{1/2}(\Gamma)} \lesssim \operatorname{dist}\big(v, S^1_{h,p}(\Gamma)\big).$$

For the screen problems above these estimates yield only very low order of convergence rate $\mathcal{O}(h^{1/2-\varepsilon}p^{-1+2\varepsilon})$ with arbitrary $\varepsilon > 0$ (see, e.g., [4, 15, 16]).

The indices h and p in the notation for the trial spaces $S^0_{h,p}(\Gamma)$ and $S^1_{h,p}(\Gamma)$ refer to h- and p-versions, respectively; where in the h-version a more accurate Galerkin solution is obtained by mesh refinement (and the polynomial degree p is kept fixed) whereas in the p-version a higher accuracy is obtained by increasing the polynomial degree on the same mesh. The implementation of the h-version is standard. In the p-version BEM for the weakly singular integral equation we use tensor products of Legendre polynomials on rectangular meshes and for the hypersingular integral equation we take instead antiderivatives of Legendre polynomials. On triangular meshes more sophisticated trial functions must be used, as we will show further below.

If one uses a geometric mesh refinement together with a properly chosen polynomial degree distribution one obtains even exponentially fast convergence rates for the Galerkin errors of the above integral equations. We have the following result proven in [1] for $d = 2$ and in [6, 9, 13] for $d = 3$ where d denotes the spatial dimension; i.e., Γ is polygon for $d = 2$, and Γ is a planar surface piece if $d = 3$.

Theorem 1.1. *For given piecewise analytic functions f, g in* (1.1) *and* (1.2) *and corresponding Galerkin solutions $\psi_N \in S^0_{h,p-1}(\Gamma^n_\sigma)$, $v_N \in S^1_{h,p}(\Gamma^n_\sigma)$ of* (1.3) *and* (1.4) *on the geometric mesh Γ^n_σ there holds*

$$\left.\begin{array}{l} \|\psi - \psi_N\|_{\tilde{H}^{-1/2}(\Gamma)} \\ \|v - v_N\|_{\tilde{H}^{1/2}(\Gamma)} \end{array}\right\} \le \left\{ \begin{array}{ll} C\exp(-b\sqrt{N}), & d = 2, \\ C\exp(-b\sqrt[4]{N}) + \mathcal{O}(N^{-\alpha}), & d = 3, \end{array}\right.$$

with constants C, $b > 0$ independent of the dimension N of the trial space and arbitrary $\alpha > 0$.

The local mesh at a right angle corner of Γ is given in Fig. 1.1. The proof of the theorem is based on analysing the error in countably normed spaces and is based on the following lemma in [13].

Lemma 1.1. *For $u \in B^2_\beta(Q)$, $0 < \beta < 1$, there exists a spline $u_N \in S^1_{h,p}(Q^n_\sigma)$ and constants C, $b > 0$ independent of N, but dependent on σ, μ, β such that*

$$\|u - u_N\|_{H^1(Q)} \le C\, e^{-b\sqrt[4]{N}}, \tag{1.5}$$

with $p_1 = 1$, $p_k = \max\big(2, [\mu(k-1)] + 1\big)$ $(k > 1)$ for $\mu > 0$.

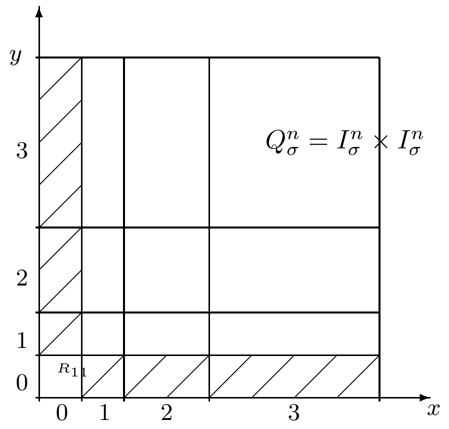

Fig. 1.1. Geometric mesh on the square Q^n_σ ($\sigma = 0.5$, $n = 4$).

In the above lemma we need the countably normed function space $B^2_\beta(Q)$ which we introduce now for the square $Q = [0,1]^2$ with the help of weighted Sobolev spaces $H^{k,2}_\beta(Q)$ as

$$B^2_\beta(Q) = \Big\{u : u \in H^{k,2}_\beta(Q), \quad \forall k \geq 2, \|\Phi_{\beta,\alpha,2} D^\alpha u\|_{L^2(Q)} \leq C\, d^{k-2}(k-2)! \\ \text{for } |\alpha| = k = 2,3,\ldots, \text{ with } C \geq 1, d \geq 1 \text{ indpt. of } k\Big\}.$$

$$\Phi_{\beta,(\alpha_1,\alpha_2),2}(x,y) = \begin{cases} x^{\beta+\alpha_1-2}, & \alpha_1 \geq 2, \alpha_2 = 0, \\ x^\beta + y^\beta, & \alpha_1 = 1, \alpha_2 = 1, \\ x^{\beta+\alpha_1-2}y + x^{\beta+\alpha_1-1} + y^\beta, & \alpha_1 \geq 2, \alpha_2 = 1, \\ x^{\beta+\alpha_1-2}y^{\alpha_2} + (x^\beta + y^\beta)x^{\alpha_1-1}y^{\alpha_2-1} + x^{\alpha_1}y^{\beta+\alpha_2-2}, & \alpha_1 \geq 2, \alpha_2 \geq 2, \\ x^\beta + xy^{\beta+\alpha_2-2} + y^{\beta+\alpha_2-1}, & \alpha_1 = 1, \alpha_2 \geq 2, \\ y^{\beta+\alpha_2-2}, & \alpha_1 = 0, \alpha_2 \geq 2, \end{cases}$$

where the weighted Sobolev spaces $H^{k,2}_\beta(Q)$ are given by

$$|u|^2_{H^{k,2}_\beta(Q)} = \sum_{|\alpha|=2}^{k} \int_Q \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2} u(x,y)\right|^2 \Phi^2_{\beta,\alpha,2}(x,y)\, dy\, dx,$$
$$\|u\|^2_{H^{k,2}_\beta(Q)} = \|u\|^2_{H^1(Q)} + |u|^2_{H^{k,2}_\beta(Q)}.$$

Proof. 1.) In element R_{11} at the origin: Due to $u \in H^{2,2}_\beta(Q)$ there exists a bilinear interpolant $\phi_{11} \in \mathcal{P}_{11}(R_{11})$ with $u(0,0) = \phi_{11}(0,0)$, $u(0,h_1) = \phi_{11}(0,h_1)$, $u(h_1,0) = \phi_{11}(h_1,0)$, $u(h_1,h_1) = \phi_{11}(h_1,h_1)$ ($h_1 = x_1 = \sigma^{n-1}$)

$$\|u - \phi_{11}\|^2_{H^1(R_{11})} \leq C\, h_1^{2(1-\beta)} \|u\|^2_{H^{2,2}_\beta(Q)}.$$

2.) On strips near edges $\big\{(x,y)\,|\, h_1 \leq x \leq 1, 0 \leq y \leq h_1\big\} \cup \big\{(x,y)\,|\, 0 \leq x \leq h_1, h_1 \leq y \leq 1\big\}$ there exist polynomials $\phi_{k1} \in \mathcal{P}_{p_k 1}(R_{k1})$ and $\phi_{1l} \in \mathcal{P}_{1p_l}(R_{1l})$, coinciding with u at vertices

$(0 < \beta < 1)$:

$$|u - \phi_{k1}\|^2_{H^1(R_{k1})} \leq C\, h_1^{2(1-\beta)} |u|^2_{H^{2,2}_\beta(Q)} + C\, x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2(s_k+1)} |u|^2_{H^{s_k+2,2}_\beta(Q)} \quad (k \geq 2), \tag{1.6a}$$

$$|u - \phi_{1l}\|^2_{H^1(R_{1l})} \leq C\, h_1^{2(1-\beta)} |u|^2_{H^{2,2}_\beta(Q)} + C\, x_{l-1}^{2(1-\beta)} \frac{\Gamma(p_l - s_l + 1)}{\Gamma(p_l + s_l + 1)} \left(\frac{\lambda}{2}\right)^{2(s_l+1)} |u|^2_{H^{s_l+2,2}_\beta(Q)} \quad (l \geq 2). \tag{1.6b}$$

Therefore (corresponding estimates hold away from the edges) on R_{kl} $(2 \leq k, l \leq n)$ with $1 \leq s_k \leq p_k$ for $0 \leq \alpha_1, \alpha_2 \leq 1$ there holds:

$$\|D^\alpha(u - \phi_{kl})\|^2_{L^2(R_{kl})} \leq C \Bigg\{ x_{k-1}^{2(2-\alpha_1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 3 - 2|\alpha|)} \left(\frac{\lambda}{2}\right)^{2s_k} |u|^2_{H^{s_k+3,2}_\beta(Q)} + x_{l-1}^{2(2-\alpha_2-\beta)} \frac{\Gamma(p_l - s_l + 1)}{\Gamma(p_l + s_l + 3 - 2|\alpha|)} \left(\frac{\lambda}{2}\right)^{2s_l} |u|^2_{H^{s_l+3,2}_\beta(Q)} \Bigg\}.$$

3.) Combining 1.) and 2.) we obtain $(1 \leq s_k \leq p_k)$

$$\sum_{k,l=1}^{n} \|u - \phi_{kl}\|^2_{H^1(R_{kl})} \leq C\, h_1^{2(1-\beta)} \|u\|^2_{H^{2,2}_\beta(Q)} + (2n-2)\, C\, h_1^{2(1-\beta)} |u|^2_{H^{2,2}_\beta(Q)} + 2nC \sum_{k=2}^{n} x_{k-1}^{2(1-\beta)} \frac{\Gamma(p_k - s_k + 1)}{\Gamma(p_k + s_k + 1)} \left(\frac{\lambda}{2}\right)^{2(s_k+1)} |u|^2_{H^{s_k+3,2}_\beta(Q)}.$$

Now with $h_1 = \sigma^{n-1}$ and

$$|u|_{H^{s_k+3,2}_\beta(Q)} \leq C d^{s_k+1} \Gamma(s_k + 2), \tag{1.7}$$

we obtain (1.5). Note: $u \in B^2_\beta(Q)$ implies (1.7). □

Figs. 1.2 and 1.3 show numerical experiments (cf. [14]) obtained with the integral equations for linear elasticity treating crack problems with the open surface piece Γ as crack surface. The operators are here given with the Green's function for the Lamé equation

$$G(x, y) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \left\{ \frac{1}{|x - y|} I + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x - y)(x - y)^t}{|x - y|^3} \right\}.$$

The following theorem, proven in [7], describes the regularity of the solutions of the integral equations. It shows that the solution can be written as a function in a countably normed space plus special singularity terms (and higher order terms THO) which reflect the crack singularity behaviour near the edges of the screen (crack surface).

Theorem 1.2. *For piecewise analytic data f and g the solutions of the integral equations* (1.1) *and* (1.2) *satisfy*

$$\psi - \psi_s \in B^1_\beta(\Gamma), \quad \psi_s = \theta^{-1/2} R(r) + THO,$$
$$v - v_s \in B^2_\beta(\Gamma), \quad v_s = \theta^{1/2} \tilde{R}(r) + THO,$$

with

$$R \in B^0_{\tilde\beta}((0,\tilde\varrho)), \quad \tilde R \in B^1_{\tilde\beta}((0,\tilde\varrho)), \quad \tilde\beta > \frac{1}{2} - \lambda_1,$$

where λ_1 depends on the smallest eigenvalue of a corresponding boundary value problem of the Laplace-Beltrami operator at the vertices. Here, θ and r denote polar coordinates describing for a point x on Γ its distance to the vertex and the angle from to nearest edge; $\tilde\varrho$ is the radius of a local cut-off-function at the vertex.

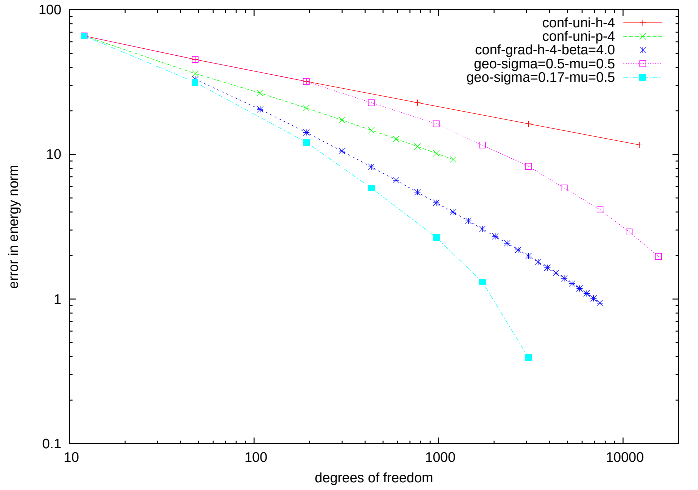

Fig. 1.2. Weakly singular integral equation (Lamé).

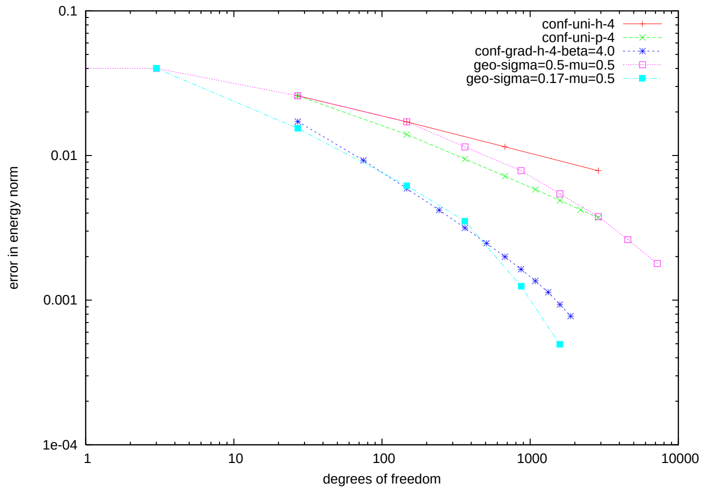

Fig. 1.3. Hypersingular integral equation (Lamé).

The legends for Figs. 1.2 and 1.3 have the following meanings: conf-uni-h-4 and conf-uni-p-4 mean conforming h-version of BEM and conforming uniform p-version of BEM on uniform rectangular meshes, respectively; conf-grad-h-4-beta=4.0 stands for conforming h-version of the BEM on graded meshes graded algebraically towards the edges of $\Gamma = [-1,1]^2$ with grading parameter $\beta = 4$; geo-sigma=0.5-mu=0.5 and geo-sigma=0.17-mu=0.5 stand for two hp-versions of the BEM with geometric mesh parameter geo-sigma and parameter mu for the polynomial degree distribution. Figs. 1.2 and 1.3 show clearly the exponentially fast convergence of the hp-version on the geometric mesh with optimal mesh grading parameter $\sigma = 0.17$. The paramter $\mu = 0.5$ describes the increase of the polynomial degree, namely (q,p), (q,p), $(q,p+1)$, $(q,p+1)$, $(q,p+2)$, $(q,p+2)$,... in the x_2-direction and correspondingly in the x_1-direction, for a geometric mesh consisting of rectangles only and refined towards the edges. Very good results are also obtained for the h-version on an algebraically graded mesh; this is in agreement with the theoretical results in [15]. Also Figs. 1.2 and 1.3 show that the uniform p-version converges twice as fast as the uniform h-version [4, 16].

2. Preconditioning

Next, we like to comment on the use of efficient preconditioners for the boundary element matrices resulting from the Galerkin equations (1.3) and (1.4) which we write for simplicity as: *Find $u \in S$ such that*

$$a(u,v) = \langle f, v \rangle$$

for any $v \in S$ where in case of the single layer equation the bilinear form is given by $a(\cdot,\cdot) = \langle V\cdot,\cdot\rangle$ whereas for the hypersingular equation we have $a(\cdot,\cdot) = \langle W\cdot,\cdot\rangle$. Note that S stands for the appropriately chosen boundary element space. Then the above Galerkin system corresponds to the linear system $Au = f$ with a dense and ill-conditioned matrix A with condition number cond $(A) = \mathcal{O}(p^3/h)$ for a quasi-uniform hp-version. When solving the system with a conjugate gradient method, the error reduction factor of the CG iteration behaves like

$$\delta = 1 - \mathcal{O}\big(h^{1/2} p^{-3/2}\big),$$

as $h \to 0, p \to \infty$. One can use the tool of the additive Schwarz operator to construct efficient preconditioners B so that the preconditioned equation $BA\,u = Bf$ can be solved iteratively with a bounded or only moderately growing number of iterations. The technique of the additive Schwarz method is based on a subspace decomposition $S = S_0 + S_1 + \cdots + S_N$ together with Galerkin projectors $P_j : S \to S_j$, $j = 0, \ldots, N$, defined via

$$a(P_j v, \phi) = a(v, \phi), \quad \forall v \in S, \quad \phi \in S_j,$$

where $a(\cdot,\cdot)$ is a symmetric and positive definite bilinear form on $S \times S$. Then the additive Schwarz operator is defined by

$$BA := P_{\mathrm{AS}} := \sum_{j=0}^{N} P_j,$$

and solving $Au = f$ is equivalent to solving $P_{\mathrm{AS}} u = g$ where $g = g_0 + g_1 + \ldots + g_N$, with

$$a(g_j, w) = \langle f, w \rangle, \quad \forall w \in V_j.$$

The author has analysed in a series of papers the use of the Schwarz method for the h- and the p-version of the boundary element method, see [19] and [20]. Here we present only in detail the case of the hypersingular operator W when using the p-version on quasi-uniform triangular meshes. In this case the space S coincides with the space

$$S_N^p(\Gamma) = \big\{u : u|_{\Gamma_i} \in P^p(\Gamma_i)\big\},$$

and we have a subspace splitting with $S_0 = \Psi_W(\Gamma)$ which are the wire basket functions (edge / vertex functions) and with the set of interior functions (bubbles) S_j on the triangle Γ_j $(j = 1, \dots, N)$. In [5] Heuer, Leydecker and Stephan prove that the condition number of the preconditioned stiffness matrix has a bound which is independent of the mesh size h and which grows only polylogarithmically in p, the maximum polynomial degree; we show in [5] that the condition number of the additive Schwarz operator behaves like

$$\text{cond}\ (P_{AS}) = \mathcal{O}\big((1 + \log p)^4\big).$$

This is supported by our numerical experiments presented in Fig. 2.1.

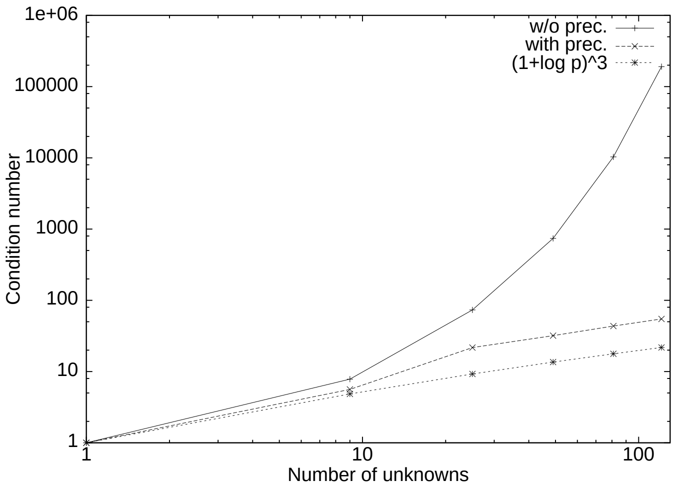

Fig. 2.1. Condition number for hypersingular stiffness matrix (of the p-version on triangles) with and without additive Schwarz preconditioner.

In the following we comment on the construction of piecewise polynomials appropriate for our subspace splitting. Our nodal and edge basis functions are constructed by extensions from edges onto elements. For this procedure we use the following specific extension operators

$$\begin{aligned} E_1^1(f)(x,y) :=& \frac{x}{y} \int_x^{x+y} \frac{f(t)}{t}\, dt \qquad \text{if } f(0) = 0, \\ E_2^1(f)(x,y) :=& \frac{1-x-y}{y} \int_x^{x+y} \frac{f(t)}{1-t}\, dt \qquad \text{if } f(1) = 0, \\ E^1(f)(x,y) :=& \frac{x(1-x-y)}{y} \int_x^{x+y} \frac{f(t)}{t(1-t)}\, dt \qquad \text{if } f(0) = f(1) = 0. \end{aligned}$$

For the other edges we can proceed similarly (see [5] for details).

For the construction of vertex basis functions we consider special low energy functions ϕ_0. Then a vertex basis function $\tilde{\phi}_{V_1}$, e.g. for vertex V_1, is defined as follows (cf. Fig. 2.2). Set $\tilde{\phi}_{V_1} = \phi_0$ on I_1 and I_3, and $\tilde{\phi}_{V_1} = 0$ on I_2. Extend $\tilde{\phi}_{V_1}$ from I_1 onto T by using the extension operator E_2^1,

$$\psi_1 := E_2^1 \tilde{\phi}_{V_1} = E_2^1 \phi_0.$$

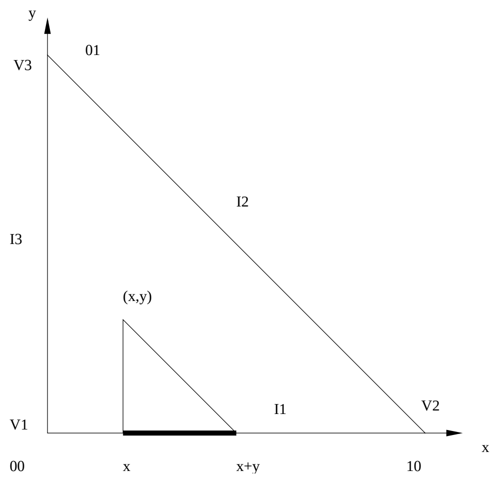

Fig. 2.2. Reference triangle T with vertices V_i and edges I_i.

Let g_3 be the trace of ψ_1 on I_3 and define

$$\psi_3 := E^3(g_3 - \tilde{\phi}_{V_1}),$$

the extension of $g_3 - \tilde{\phi}_{V_1}$ from I_3 onto T with $\psi_3 = 0$ on I_1 and I_2. Eventually we set $\tilde{\phi}_{V_1} := \psi_1 - \psi_3$. The other vertex functions are defined analogously.

For the construction of edge basis functions we use antiderivatives of Legendre polynomials together with extension operators E^i, $i = 1, 2, 3$. As interior (or bubble) functions we simply use tensor products of antiderivatives of Legendre polynomials $\mathcal{L}_k$ and take

$$\phi_{k,l}(x, y) = \frac{\mathcal{L}_{k+1}(2x-1)}{1-x}\frac{\mathcal{L}_l(2y-1)}{1-y}(1-x-y), \quad 1 \le k,\ 2 \le l,\ k+l \le p.$$

When ordering the basis functions appropriately the preconditioning matrix has block diagonal form with entries S_W and S_{Γ_j} denoting the discretisations with the wire basket and bubble functions respectively.

3. Adaptive Refinement with Hierarchical Error Indicator

The above introduced Galerkin projectors P_j can be further used to construct hierarchical error indicators [8]. For *hp*-adaptive BEM algorithms with error indicators of residual type see the paper by Carstensen, Funken and Stephan [2]. For example let us consider again the hypersingular integral operator W on an enriched space $\tilde{S}_{h,p}(\Gamma) \supset S_{h,p}(\Gamma)$ with a 2-level decomposition

$$\tilde{S}_{h,p}(\Gamma) = Z_0 + Z_1 + \cdots + Z_J.$$

With an improved Galerkin approximation $\tilde{\phi}_{h,p}$ (not computed) we define the error indicators

$$\theta_j := \|P_j(\tilde{\phi}_{h,p} - \phi_{h,p})\|_W, \quad j = 0, \ldots, J.$$

where $\|\cdot\|_W$ denotes the energy norm and the Galerkin projectors $P_j : \tilde{S}_{h,p}(\Gamma) \to Z_j$ are defined by

$$\langle WP_j\varphi, \psi\rangle_{L^2(\Gamma)} = \langle W\varphi, \psi\rangle_{L^2(\Gamma)}, \quad \forall \psi \in Z_j,$$
$$Z_0 = S_{h,p}(\Gamma), \quad Z_j = S_{h,p}(\Gamma_j) \cap H^1_0(\Gamma_j).$$

Hence the error indicators θ_j are calculated as follows: For $\psi \in Z_j$ there holds

$$\begin{aligned}&\langle WP_j(\tilde{\phi}_{h,p} - \phi_{h,p}), \psi\rangle_{L^2(\Gamma)}\\ &= \langle W(\tilde{\phi}_{h,p} - \phi_{h,p}), \psi\rangle_{L^2(\Gamma)} = \langle g, \psi\rangle_{L^2(\Gamma)} - \langle W\phi_{h,p}, \psi\rangle_{L^2(\Gamma)}\end{aligned}$$

Therefore, we have: Let A_j be the stiffness matrix of W for the subspace Z_j, $\vec{\vartheta}_j$ be the vector of coefficients representing $P_j(\tilde{\phi}_{h,p} - \phi_{h,p})$ then θ_j is given as

$$\theta_j^2 = \|P_j(\tilde{\phi}_{h,p} - \phi_{h,p})\|_W^2 = \vec{\vartheta}_j^T A_j \vec{\vartheta}_j.$$

Of course, $\theta_j = 0$ if $Z_j \subset S_{h,p}(\Gamma)$.

Now, when using a so-called saturation assumption we have the error estimate

$$\sum_{j=0}^{J} \theta_j^2 \simeq \|\tilde{\phi}_{h,p} - \phi_{h,p}\|_W^2 \simeq \|\phi - \phi_{h,p}\|_W^2.$$

We now compare the local indicators θ_j with $\theta_{\max}$ and uses some preset δ_1, δ_2 with $0 < \delta_2 < \delta_1 < 1$ and performs a three-step adaptive *hp*-refinement as follows (see [8] for details):

(i) if $\theta_j \leq \delta_2\,\theta_{\max}$ do nothing;
(ii) if $\delta_2\,\theta_{\max} \leq \theta_j \leq \delta_1\theta_{\max}$ increase p by one;
(iii) if $\delta_1\,\theta_{\max} \leq \theta_j$ reduce h.

Fig. 3.1 shows the refined mesh and and the polynomial degrees of the boundary element solution of the hypersingular integral equation (1.2) after 5 adaptive *hp*-refinements for the right hand side function

$$g(x) = \Big(\mathrm{dist}\,\big(x, (-0.1, -0.1)\big)\Big)^{-1} \text{ on } \Gamma = (-1/2, 1/2)^2 \times \{0\}.$$

Finally we consider as an example for this adaptive boundary element method the Signorini contact problem for the Lamé operator Δ^* in linear elasticity describing the deformation of an elastic bar when it is pushed down on a fixed foundation by given boundary traction $\mathcal{T}(u)$, see Fig. 3.2. The problem under consideration reads:

$$\begin{aligned}&\Delta^* u = 0 && \text{in } \Omega,\\ &\mathcal{T}(u) = (0, -160) && \text{on } \Gamma_{N,1},\\ &\mathcal{T}(u) = (0, 0) && \text{on } \Gamma_{N,2},\\ &u_n \geq 0,\ \ \mathcal{T}(u)_n \geq 0,\ \ u_n \cdot \mathcal{T}(u)_n = 0 && \text{on } \Gamma_S.\end{aligned}$$

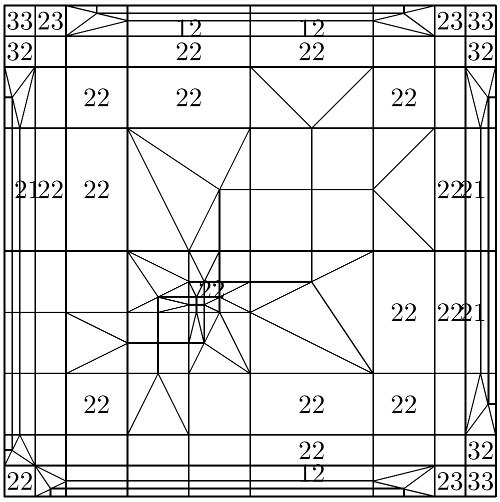

Fig. 3.1. Refined mesh and polynomial degree distribution for the hp-version applied to the hypersingular equation (1.2) after 5 refinements. Here, e.g. 32 means polynomial degree 3 in x_1 and degree 2 in x_2.

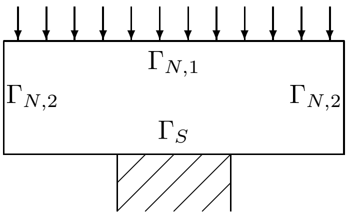

Fig. 3.2. Elastic bar with Signorini contact.

In [10] Maischak and Stephan reduce the above contact problem to the following system with a variational inequality on the boundary Γ with the convex set of admissible functions

$$K_\Gamma := \left\{ v \in \mathbf{H}^{1/2}(\Gamma) \,:\, v|_{\Gamma_D} = g|_{\Gamma_D}, v \cdot \vec{n}|_{\Gamma_S} \leq g_n|_{\Gamma_S} \right\}.$$

This uniquely solvable system reads: find $(u, \varphi) \in K_\Gamma \times \mathbf{H}^{-1/2}(\Gamma)$,

$$\langle u, W(v-u)\rangle + \langle \varphi, (I+K)(v-u)\rangle \geq 2l(v-u), \tag{3.1a}$$

$$\langle \psi, V\varphi\rangle - \langle \psi, (I+K)u\rangle = 0, \quad \forall (v,\psi) \in K_\Gamma \times \mathbf{H}^{-1/2}(\Gamma). \tag{3.1b}$$

Here, the boundary integral operators are the hypersingular operator W, the single layer potential V and the double layer potential K. The hp-version of the boundary element method is performed by solving the system (3.1) using for u Lagrange polynomials enforcing the contact conditions at the Gauss-Lobatto points only and taking for ϕ Legendre polynomials. The numerical experiments in Fig. 3.3 show correct mesh refinement and distribution of polynomial degrees are also obtained for contact problems with our three-step hp-adaptive algorithm described above. In [12].we have used the steering parameters $\delta_2 = 0.8$ and $\delta_1 = 0.9$, i.e., 20% of all elements will be refined and the polynomial degrees of 10% of all elements will be increased, whereas the 10% of all elements with the largest indicator values will be bisected.

The variational inequality is solved by an iterative solver, the Polyak algorithm (a modified CG scheme). As a stopping criterion we use that the last relative change of the solution vector is less than 10^{-8}. Consequently, the numerical solution will not be completely symmetric, which leads to slightly unsymmetric values of the error indicators. For further details, see [12].

As shown in our article [12] *hp*-versions of boundary element methods are powerful tools for solving contact problems in linear elasticity. The error-controlled adaptive algorithm presented here (see, also [12]), which uses local error estimators of hierarchical type, can also be extended to problems with friction and inhomogeneous problems by appropriate use of symmetric FEM-BEM coupling (see, e.g., [3, 11]). These local error estimators are easily computed by enriching the boundary element space with bubble functions or local mesh refinements.

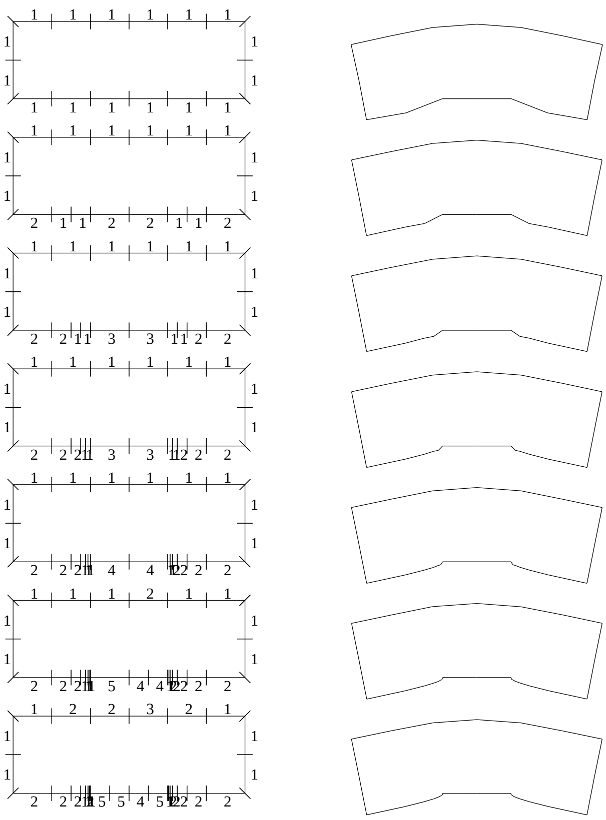

Fig. 3.3. *hp*-adaptive generated meshes for Lamé-BEM (Bubble estimator).

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