

## CONSTRUCTION OF GEOMETRIC PARTIAL DIFFERENTIAL EQUATIONS FOR LEVEL SETS \*

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### Abstract

Geometric partial differential equations of level-set form are usually constructed by a variational method using either Dirac delta function or co-area formula in the energy functional to be minimized. However, the equations derived by these two approaches are not consistent. In this paper, we present a third approach for constructing the level-set form equations. By representing various differential geometry quantities and differential geometry operators in terms of the implicit surface, we are able to reformulate three classes of parametric geometric partial differential equations (second-order, fourth-order and sixth-order) into the level-set forms. The reformulation of the equations is generic and simple, and the resulting equations are consistent with their parametric form counterparts. We further prove that the equations derived using co-area formula are also consistent with the parametric forms. However, these equations are of much complicated forms than these given by the equations we derived.

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*Key words:* Geometric partial differential equations, Level set, Differential geometry operators.

### 1. Introduction

In many scientific research and application areas of surfaces, such as geometry design (see, e.g., [1–4]), shape deformation (see, e.g., [5–9]), surface reconstruction (see, e.g., [10, 11]), surface restoration (see, e.g., [12, 13]) and image processing (see, e.g., [14, 15]), geometric partial differential equations (PDEs), which govern the motion of surfaces, have played a very important role. Using geometric PDEs, a number of efficient and effective numerical methods have been obtained, usually called geometric PDE method. The basic theory and numerical methods concerned geometric PDEs can be found in many references. We suggest the interested readers to refer [3, 14, 16, 17].

Depending on the nature of the problems to be solved, geometric PDEs are represented as either parametric form or level-set form. By virtue of surface variation techniques, a vast geometric PDEs have been successfully derived by minimizing certain energy functionals defined

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on parametric surfaces (see [3] and literatures therein). However, less efforts have been made for implicit surfaces. Currently, two variational methods have been developed. The first one utilizes the Dirac delta function in the energy functional to convert the energy functional defined on a surface to an energy functional defined on a 3D volume. Then the first order variational calculus is conducted over the 3D domain and an Euler-Lagrange equation is finally derived. The second technique employs the co-area formula to convert the surface energy to the volume energy (see [13, 16]) and from there the Euler-Lagrange equation is obtained. By the second approach, the second-order, the fourth-order and sixth-order geometric PDEs in the general form have been constructed (see [16]). It has been behind that the two approaches result in the same equations and they are all equivalent to the parametric form ones for the same energy functional. However, these claims are not theoretically proved. Our recent study shows that the equations constructed via these two approaches are very different.

We insist that the right equation of the level-set form should coincide with the parametric form if the starting energy to be minimized is the same. This motivates strongly the current research. The aim of this paper is to construct geometric PDEs in the level-set form such that the constructed equations are equivalent to their parametric form counterpart for the same energy functional. Our strategy is that converting directly these geometric PDEs from parametric form to implicit form, so that the constructed geometric PDEs in level-set forms are equal to those in the parametric forms. In order to accomplish this task, we need to convert all the required geometric quantities and various differential geometry operators, that are used to describe geometric PDEs and previously defined for the parametric surface, from the parametric formulations to the level-set ones.

In the classical differential geometry, various geometric quantities on parametric surface have been introduced and widely used. For instance, various curvatures and geometric operators (see Section 2) are well understood (see [18–20]). For implicit surface, some of these geometric entries, such as Gaussian curvature, mean curvature and principal curvatures, have been given in the literatures (see [20–24]). But some of the others, such as the principal directions, the second and third tangent operators, and Giaquinta-Hildebrandt operator, have not been defined for the implicit surfaces, to the best of our knowledge. These operators are necessary for representing geometric PDEs in the implicit form.

Our contributions in this paper include: (i) Generalize the parametric form differential operators to level-set surface; (ii) Convert geometric PDEs in parametric forms directly into level-set forms; and (iii) Prove the equivalent relationship between the equations derived using co-area formula and the equations obtained via our approach.

The rest of the paper is organized as follows. In Section 2, we first review some results on parametric form differential geometry, and then in Section 3 we represent some useful differential quantities and differential operators in the level-set form. In Section 4 we reformulate the parametric form geometric PDEs in the level-set form. The equivalency of our PDEs and the ones using co-area approach is discussed in Section 5. Section 6 concludes the paper.

## 2. Mathematical Preliminaries

This section introduces the notations and fundamental mathematics used in the following context. We first give in Subsection 2.1 some of the main geometric notions and various results of differential operators defined on parametric surfaces. Subsection 2.2 presents some geometric partial differential equations in parametric form which are frequently used in computational

geometry (see [3, 20]).

### 2.1. Several Differential Operators for Parametric Surfaces

Suppose that  $\mathcal{S} := \{\mathbf{x}(u^1, u^2) \in \mathbb{R}^3 : (u^1, u^2) \in \mathcal{D} \subset \mathbb{R}^2\}$  is a sufficiently smooth, regular and orientable parametric surface. Let  $g_{\alpha\beta} = \langle \mathbf{x}_{u^\alpha}, \mathbf{x}_{u^\beta} \rangle$  and  $b_{\alpha\beta} = \langle \mathbf{n}, \mathbf{x}_{u^\alpha u^\beta} \rangle$  be the coefficients of the first and second fundamental forms of surface  $\mathcal{S}$  with

$$\begin{aligned} \mathbf{x}_{u^\alpha} &= \frac{\partial \mathbf{x}}{\partial u^\alpha}, \quad \mathbf{x}_{u^\alpha u^\beta} = \frac{\partial^2 \mathbf{x}}{\partial u^\alpha \partial u^\beta}, \quad \alpha, \beta = 1, 2, \\ \mathbf{n} &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}, \quad (u, v) := (u^1, u^2), \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$ ,  $\|\cdot\|$  and  $\cdot \times \cdot$  denote the usual inner product, norm and outer product in the Euclidean space  $\mathbb{R}^3$ , respectively.

**Curvatures.** Let us set  $[g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}$ ,  $[b^{\alpha\beta}] = [b_{\alpha\beta}]^{-1}$ ,  $g = \det[g_{\alpha\beta}]$ ,  $b = \det[b_{\alpha\beta}]$ . Then we can define the mean curvature  $H$  and the Gaussian curvature  $K$  of surface  $\mathcal{S}$  as follows

$$H = \frac{1}{2}[g^{\alpha\beta}]:[b_{\alpha\beta}] \quad \text{and} \quad K = \frac{b}{g}, \quad (2.1)$$

where  $A:B$  stands for the trace of  $A^T B$ . Let  $\mathbf{H} = H\mathbf{n}$  and  $\mathbf{K} = K\mathbf{n}$ , which are called mean curvature normal and Gaussian curvature normal, respectively.

**Tangential gradient operator.** Suppose  $f \in C^1(\mathcal{S})$ . Then the tangential gradient operator  $\nabla_s$  acting on  $f$  is defined as

$$\nabla_s f = [\mathbf{x}_u, \mathbf{x}_v][g^{\alpha\beta}][f_u, f_v]^T, \quad (2.2)$$

where  $\nabla_s f \in \mathbb{R}^3$ . For a vector-valued function  $\mathbf{v} = [v_1, v_2, v_3]^T \in C^1(\mathcal{S})^3$ , we define  $\nabla_s \mathbf{v} = [\nabla_s v_1, \nabla_s v_2, \nabla_s v_3]$ .

**Second tangent operator.** Assume that  $f \in C^1(\mathcal{S})$ . Then the second tangent operator  $\diamond$  applying to  $f$  is given by

$$\diamond f = [\mathbf{x}_u, \mathbf{x}_v][h_{\alpha\beta}][f_u, f_v]^T \in \mathbb{R}^3, \quad (2.3)$$

where

$$[h_{\alpha\beta}] = \frac{1}{g} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}.$$

**Third tangent operator.** Assume  $f \in C^1(\mathcal{S})$ . Then the third tangent operator  $\circ$  acting on  $f$  is defined by

$$\circ f = [\mathbf{x}_u, \mathbf{x}_v][g^{\alpha\beta}]S[f_u, f_v]^T \in \mathbb{R}^3, \quad (2.4)$$

where  $S = [b_{\alpha\beta}][g^{\alpha\beta}]$  is the coefficient matrix of Weingarten transform.

**Tangential divergence operator.** Suppose that  $\mathbf{v}$  is a smooth vector field defined on  $\mathcal{S}$ . Then the tangential divergence operator  $\text{div}_s$  applying to  $\mathbf{v}$  is defined as

$$\text{div}_s(\mathbf{v}) = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] \left[ \sqrt{g} [g^{\alpha\beta}] [\mathbf{x}_u, \mathbf{x}_v]^T \mathbf{v} \right]. \quad (2.5)$$

For a matrix-valued function  $\mathbf{M} = [M_1, M_2, M_3] \in C^1(\mathcal{S})^{3 \times 3}$ , we have

$$\text{div}_s \mathbf{M} = [\text{div}_s M_1, \text{div}_s M_2, \text{div}_s M_3]^T.$$

Using these first-order differential operators introduced above, we can obtain several second-order ones, which are important in computational geometry (see [3, 16]).

**Laplace-Beltrami operator (LBO).** Suppose that  $f \in C^2(\mathcal{S})$ . The Laplace-Beltrami operator  $\Delta_s$  acting on  $f$  is defined as  $\Delta_s f = \text{div}_s(\nabla_s f)$ . According to (2.2) and (2.5), we can deduce that

$$\Delta_s f = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] [\sqrt{g} [g^{\alpha\beta}] (f_u, f_v)^T]. \quad (2.6)$$

**Giaquinta-Hildebrandt operator (GHO).** Assume  $f \in C^2(\mathcal{S})$ . The Giaquinta-Hildebrandt operator  $\square$  applying to  $f$  is given by  $\square f = \text{div}_s(\diamond f)$ . It follows from (2.3) and (2.5) that

$$\square f = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] [\sqrt{g} [h_{\alpha\beta}] (f_u, f_v)^T]. \quad (2.7)$$

**$\boxtimes$  operator.** Let  $f \in C^2(\mathcal{S})$ . Then the  $\boxtimes$  operator acting on  $f$  is defined as  $\boxtimes f = \text{div}_s(\oslash f)$ . Using (2.4), (2.5), we have

$$\boxtimes f = \frac{1}{\sqrt{g}} \left[ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] [\sqrt{g} [g^{\alpha\beta}] S (f_u, f_v)^T]. \quad (2.8)$$

**Remark 2.1.** All of the differential operators presented above are geometric intrinsic. That is, although they are defined using the local parametrization of surfaces, they do not depend on the concrete choice of the parametrization. We call this property geometric intrinsic. For more details, we refer the reader to [3]. The intrinsic property of the differential operators ensures that the generalization of differential operators from parametric surface to level-set surface is feasible and valid.

Applying the differential operators to  $\mathbf{x}$  and  $\mathbf{n}$ , where  $\mathbf{x}$  is a point of the surface,  $\mathbf{n}$  is the corresponding unit normal vector on surface, we obtain the following equalities:

$$\nabla_s \mathbf{x} = [\mathbf{x}_u, \mathbf{x}_v] [g^{\alpha\beta}] [\mathbf{x}_u, \mathbf{x}_v]^T, \quad (2.9a)$$

$$\nabla_s \mathbf{n} = -[\mathbf{x}_u, \mathbf{x}_v] [g^{\alpha\beta}] S [\mathbf{x}_u, \mathbf{x}_v]^T, \quad (2.9b)$$

$$\diamond \mathbf{x} = [\mathbf{x}_u, \mathbf{x}_v] [h_{\alpha\beta}] [\mathbf{x}_u, \mathbf{x}_v]^T, \quad (2.9c)$$

$$\diamond \mathbf{n} = -[\mathbf{x}_u, \mathbf{x}_v] [h_{\alpha\beta}] S [\mathbf{x}_u, \mathbf{x}_v]^T, \quad (2.9d)$$

$$\oslash \mathbf{x} = [\mathbf{x}_u, \mathbf{x}_v] [g^{\alpha\beta}] S [\mathbf{x}_u, \mathbf{x}_v]^T, \quad (2.9e)$$

$$\oslash \mathbf{n} = -[\mathbf{x}_u, \mathbf{x}_v] [g^{\alpha\beta}] S^2 [\mathbf{x}_u, \mathbf{x}_v]^T. \quad (2.9f)$$

From the definitions of  $[g^{\alpha\beta}]$ ,  $[h_{\alpha\beta}]$  and  $S$ , we know that they are all 3 by 3 symmetric matrices, which have been studied in [3] and the following results are established.

**Lemma 2.1.** *Suppose  $\mathcal{S}$  is a sufficiently smooth, regular and orientable parametric surface. We have*

$$\nabla_s \mathbf{n} + \oslash \mathbf{x} = \mathbf{0}, \quad (2.10a)$$

$$\diamond \mathbf{n} + K \nabla_s \mathbf{x} = \mathbf{0}, \quad (2.10b)$$

$$2H \nabla_s \mathbf{x} + \nabla_s \mathbf{n} - \diamond \mathbf{x} = \mathbf{0}, \quad (2.10c)$$

$$2H\nabla_s \mathbf{n} - \odot \mathbf{n} - \diamond \mathbf{n} = \mathbf{0}, \quad (2.10d)$$

$$\nabla_s \mathbf{x} + \mathbf{n}\mathbf{n}^T = \mathbf{I}, \quad (2.10e)$$

where  $\mathbf{I}$  is a unit operator in space  $\mathbb{R}^3$ .

**Lemma 2.2.** *If  $\mathbf{f}, \mathbf{h} \in C^2(\mathcal{S})^3$ , we have (see [3])*

$$\operatorname{div}_s(\nabla_s \mathbf{f} \mathbf{h}) = \langle \Delta_s \mathbf{f}, \mathbf{h} \rangle + \nabla_s \mathbf{f} : \nabla_s \mathbf{h}. \quad (2.11)$$

**Lemma 2.3.** *If  $\mathcal{S}$  is a sufficiently smooth, regular and orientable parametric surface, we have (see [3])*

$$\Delta_s \mathbf{x} = 2H\mathbf{n} = 2\mathbf{H}, \quad (2.12)$$

$$\square \mathbf{x} = 2K\mathbf{n} = 2\mathbf{K}. \quad (2.13)$$

## 2.2. Geometric Partial Differential Equations in Parametric Forms

Suppose  $\mathcal{S} := \{\mathbf{x}(u^1, u^2) \in \mathbb{R}^3 : (u^1, u^2) \in \mathcal{D} \subset \mathbb{R}^2\}$  is a sufficiently smooth, regular and orientable parametric surface. For a given Lagrange function  $F$ , we define the general energy functional on the surface as

$$\mathcal{E}(\mathcal{S}) = \int_{\mathcal{S}} F dA. \quad (2.14)$$

To minimize (2.14), a  $L^2$ -gradient flow is derived (see [3]) by a variational technique. Choosing different  $F$ , various geometric flows have been constructed. In the following, we present three classes of them.

**Second order geometric PDE.** Assume that  $F = h(\mathbf{x}, \mathbf{n})$  is a properly smooth function defined in  $\mathbb{R}^3 \times \mathbb{R}^3$ . The geometric PDE (second order geometric flow) in the sense of  $L^2$  can be written as

$$\frac{\partial \mathbf{x}}{\partial t} = - \left( \mathbf{n}^T \nabla_{\mathbf{x}} h + \operatorname{div}_s(\nabla_{\mathbf{n}} h) - 2hH \right) \mathbf{n}, \quad (2.15)$$

where  $\mathbf{x} = [x_1, x_2, x_3]^T$ ,  $\mathbf{n} = [n_1, n_2, n_3]^T$ ,  $\nabla_{\mathbf{x}} h = [h_{x_1}, h_{x_2}, h_{x_3}]^T$  and  $\nabla_{\mathbf{n}} h = [h_{n_1}, h_{n_2}, h_{n_3}]^T$ .

**Fourth order geometric PDE.** Suppose that  $F = f(H, K)$  is a smooth function defined on  $\mathbb{R}^2$ . The geometric PDE (fourth order geometric flow) in the  $L^2$  sense is given by

$$\frac{\partial \mathbf{x}}{\partial t} = - \left( \square f_K + \frac{1}{2} \Delta_s f_H + 2HK f_K + (2H^2 - K) f_H - 2Hf \right) \mathbf{n}. \quad (2.16)$$

**Sixth order geometric PDE.** Suppose that  $F = \|\nabla_s f(H, K)\|^2$  is a smooth function defined on  $\mathbb{R}^2$ . The geometric PDE (sixth order geometric flow) in the  $L^2$  sense is given by

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial t} = & - \left( \Delta_s (f_H \Delta_s f) + 2\square (f_K \Delta_s f) + 4KH f_K \Delta_s f + 4H^2 f_H \Delta_s f - 2K f_H \Delta_s f \right. \\ & \left. - 2H \|\nabla_s f\|^2 + 2\langle \nabla_s f, \diamond f \rangle \right) \mathbf{n}. \end{aligned} \quad (2.17)$$

### 3. Several Differential Operators for Level-set Surfaces

This section is concerned with several differential operators for level-set surfaces. The aim is to generalize various differential operators to the implicit surface. Assume that

$$\Gamma_c := \{\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3 : \phi(\mathbf{x}) = c\},$$

where  $\phi(\mathbf{x})$  is a properly smooth function defined on  $\mathbb{R}^3$ ,  $c$  is an arbitrarily given constant. Suppose that  $\|\nabla\phi\| \neq 0$  on  $\Gamma_c$ , thus, according to the implicit function theorem, the level-set surface could be locally parameterized. For each point on the surface, we can obtain the unit normal vector (see [3, 22])

$$\mathbf{n} = \frac{\nabla\phi}{\|\nabla\phi\|}.$$

The mean curvature  $H$  and the Gaussian curvature  $K$  for the level-set surface can be deduced as (see [3, 22])

$$H = -\frac{1}{2}\operatorname{div}\left(\frac{\nabla\phi}{\|\nabla\phi\|}\right), \quad K = -\|\nabla\phi\|^{-4} \det \begin{bmatrix} \nabla^2\phi & \nabla\phi \\ \nabla\phi^T & 0 \end{bmatrix},$$

where  $\nabla$  and  $\operatorname{div}$  denote the classical gradient and divergence operators, respectively,  $\nabla^2\phi$  stands for the gradient of  $\nabla\phi$ .

Note that the principal curvatures  $k_1, k_2$ . The mean curvature  $H$  and the Gaussian curvature  $K$  have connections as follows

$$k_1, k_2 = H \pm \sqrt{H^2 - K}.$$

Therefore, the principal curvatures have been generalized.

Next, we consider principal directions corresponding to the principal curvatures defined on level-set surfaces. The following lemma plays a key role in obtaining the principal directions.

**Lemma 3.1.** *Let  $\mathcal{S} := \{\mathbf{x}(u^1, u^2) \in \mathbb{R}^3 : (u^1, u^2) \in \mathcal{D} \subset \mathbb{R}^2\}$  be a sufficiently smooth, regular and orientable parametric surface. Then we have (see [3])*

$$\begin{aligned} \lambda(\nabla_s \mathbf{x}) &= \{1, 1, 0\}, & \lambda(\nabla_s \mathbf{n}) &= \{-k_1, -k_2, 0\}, \\ \lambda(\diamond \mathbf{x}) &= \{k_2, k_1, 0\}, & \lambda(\diamond \mathbf{n}) &= \{-K, -K, 0\}, \\ \lambda(\circlearrowleft \mathbf{x}) &= \{k_1, k_2, 0\}, & \lambda(\circlearrowleft \mathbf{n}) &= \{-k_1^2, k_2^2, 0\}, \end{aligned}$$

the corresponding eigenvectors are  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$ , where  $\lambda(\cdot)$  stands for the spectrum of a matrix,  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are principal directions with respect to principal curvatures  $k_1$  and  $k_2$ , respectively.

According to Lemma 3.1, it is easy to observe that  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are eigenvectors with respect to eigenvalues  $k_1$  and  $k_2$  of  $\circlearrowleft \mathbf{x}$ , respectively. Hence, if  $\circlearrowleft \mathbf{x}$  can be converted to the implicit surface, then  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are well defined.

In the following, we will present several geometric differential operators in level-set forms. As a matter of fact, all of the differential operators presented above are geometric intrinsic, that is, they do not depend on the concrete choice of the parametrization. Hence the conversion of differential operators from parametric surface to level-set surface is meaningful. In fact, some differential operators in level-set forms have been presented in several references, such as [3]. However, the operators in level-set forms do not equal to those in parametric forms.

**Definition 3.1.** Given a level-set function  $\phi$ , suppose  $\|\nabla\phi(\mathbf{x})\| \neq 0$  in a neighborhood  $\Omega$  of the level-set surface  $\Gamma_c = \{\mathbf{x} \in \mathbb{R}^3 : \phi(\mathbf{x}) = c\}$ . For any  $\mathbf{x} \in \Gamma_c$ , let  $\mathbf{x}(u, v)$  be a local parametrization of  $\Gamma_c$  around  $\mathbf{x}$ . Then for the level-set surface  $\Gamma_c$ , we define the tangential gradient operator  $\nabla_\phi$ , the second tangent operator  $\diamond_\phi$ , the third tangent operator  $\circlearrowleft_\phi$ , the tangential divergence operator  $\text{div}_\phi$ , Laplace-Beltrami operator  $\Delta_\phi$ , Giaquinta-Hildebrandt operator  $\square_\phi$  and  $\boxtimes$  operator  $\boxtimes_\phi$  around  $\mathbf{x}$  as  $\nabla_s$ ,  $\diamond_s$ ,  $\circlearrowleft_s$ ,  $\text{div}_s$ ,  $\Delta_s$ ,  $\square_s$  and  $\boxtimes$  for the parametric surface  $\mathbf{x}(u, v)$ , respectively.

In the following, we give explicit representations for each of the differential operators of the level-set surface. These are represented as theorems. In this section, we always assume  $\|\nabla\phi(\mathbf{x})\| \neq 0$  in a neighborhood  $\Omega$  of the level-set surface  $\Gamma_c$ .

**Theorem 3.1.** Suppose  $f(\mathbf{x})$  is a differentiable function in  $\Omega$ . Then the tangential gradient of  $f(\mathbf{x})$  on the level-set surface  $\Gamma_c \subset \Omega$  can be written as

$$\nabla_\phi f(\mathbf{x}) = \mathbf{P}(\mathbf{x})\nabla f(\mathbf{x}),$$

where  $\mathbf{x} = [x_1, x_2, x_3]^T \in \Gamma_c$ ,  $\nabla f = [f_{x_1}, f_{x_2}, f_{x_3}]^T$ ,  $\mathbf{P} = \mathbf{I} - \mathbf{nn}^T$  is a projection operator onto the tangent space,  $\mathbf{n} = [n_1, n_2, n_3]^T$  is the unit normal vector on the surface and  $\mathbf{I}$  is a unit operator in  $\mathbb{R}^3$ .

*Proof.* Since  $\|\nabla\phi\| \neq 0$ , according to the implicit function theorem, the level-set surface can be locally parameterized. Since the tangential gradient operator on the parametric surface is geometric intrinsic, the tangential gradient operator in level-set form satisfies

$$\nabla_\phi f(\mathbf{x}) = \nabla_s f(\mathbf{x}).$$

By definition,

$$\nabla_s f = [\mathbf{x}_u, \mathbf{x}_v][g^{\alpha\beta}][f_u, f_v]^T.$$

Thanks to the chain rule, we have

$$[f_u, f_v]^T = [\mathbf{x}_u, \mathbf{x}_v]^T \nabla f.$$

Noticing (2.9a), we get

$$\nabla_s f = \nabla_s \mathbf{x} \nabla f.$$

From (2.10e), we obtain

$$\nabla_s f = (\mathbf{I} - \mathbf{nn}^T) \nabla f,$$

and finally,

$$\nabla_\phi f = \nabla_s f = (\mathbf{I} - \mathbf{nn}^T) \nabla f = \mathbf{P} \nabla f. \quad (3.1)$$

This completes the proof of the theorem.  $\square$

**Remark 3.1.** In Theorem 3.1, the tangential gradient of  $f(\mathbf{x})$  depends only on the function values on  $\Gamma_c$ . To prove this conclusion, we introduce a perturbation term  $\psi$  to  $f$ , where  $\psi$  is a differentiable function defined on  $\Omega$  satisfying  $\psi(\mathbf{x}) = 0$  on  $\Gamma_c$ . Since  $\Gamma_1 = \{\mathbf{x} \in \Gamma_c : \phi(\mathbf{x}) = c\}$  and  $\Gamma_2 = \{\mathbf{x} \in \Gamma_c : \psi(\mathbf{x}) = 0\}$  are the same level sets, they have the same unit normal vector on each corresponding point. Hence

$$\mathbf{P} \nabla \psi = \mathbf{0}, \nabla_\phi(f + \psi) = \mathbf{P} \nabla f + \mathbf{P} \nabla \psi = \mathbf{P} \nabla f.$$

Therefore, the tangential gradient of  $f$  is independent of its values outer of  $\Gamma_c$ . In conclusion, the tangential gradient of  $f$  is well defined.

**Theorem 3.2.** *Suppose  $f(\mathbf{x})$  is a differentiable function in  $\Omega$ . Then the second tangent operator applying to  $f(\mathbf{x})$  on the level-set surface  $\Gamma_c \subset \Omega$  can be written as*

$$\diamond_{\phi} f(\mathbf{x}) = \mathbf{P}(2H\mathbf{I} + \nabla \mathbf{n})\nabla f(\mathbf{x}), \quad (3.2)$$

where  $\mathbf{x} \in \Gamma_c$ ,  $\nabla \mathbf{n} = [\nabla n_1, \nabla n_2, \nabla n_3]$ .

*Proof.* Since  $\|\nabla \phi\| \neq 0$ , according to the implicit function theorem, the level-set surface can be locally parameterized. We know  $\diamond$  is geometric intrinsic for the parametric surface, so the second tangent operator in level-set form satisfies

$$\diamond_{\phi} f = \diamond f.$$

By definition,

$$\begin{aligned} \diamond f &= [\mathbf{x}_u, \mathbf{x}_v][h_{\alpha\beta}][f_u, f_v]^T \\ &= [\mathbf{x}_u, \mathbf{x}_v][h_{\alpha\beta}][\mathbf{x}_u, \mathbf{x}_v]^T \nabla f. \end{aligned}$$

Noticing (2.9c), we have

$$\diamond f = \diamond \mathbf{x} \nabla f.$$

Applying Theorem 3.1, we have  $\nabla_s \mathbf{n} = \mathbf{P} \nabla \mathbf{n}$ . Using (2.10c) and (2.10e), we obtain

$$\begin{aligned} \diamond \mathbf{x} &= 2H \nabla_s \mathbf{x} + \nabla_s \mathbf{n} \\ &= 2H\mathbf{P} + \mathbf{P} \nabla \mathbf{n} \\ &= \mathbf{P}(2H\mathbf{I} + \nabla \mathbf{n}). \end{aligned}$$

Therefore

$$\diamond_{\phi} f = \diamond f = \mathbf{P}(2H\mathbf{I} + \nabla \mathbf{n})\nabla f.$$

This completes the proof of the theorem.  $\square$

**Remark 3.2.** As Remark 3.1, we can prove in Theorem 3.2 that, the second tangent operator applying to  $f(\mathbf{x})$  depends only on the function values on  $\Gamma_c$ . Let  $\psi$  be a differentiable function defined on  $\Omega$  satisfying  $\psi(\mathbf{x}) = 0$  on  $\Gamma_c$ . Then

$$\begin{aligned} \diamond_{\phi} \psi(\mathbf{x}) &= \mathbf{P}(2H\mathbf{I} + \nabla \mathbf{n})\nabla \psi(\mathbf{x}) \\ &= \nabla_s \mathbf{n} \nabla \psi(\mathbf{x}) = \mathbf{0}. \end{aligned}$$

Hence  $\diamond_{\phi}(f + \psi) = \diamond_{\phi} f + \diamond_{\phi} \psi = \diamond_{\phi} f$ . Therefore, the second tangent operator applying to  $f$  is independent of its values outer of  $\Gamma_c$ . Hence, the second tangent operator is well defined.

**Theorem 3.3.** *Suppose  $f(\mathbf{x})$  is a differentiable function in  $\Omega$ . Then the third tangent operator applying to  $f(\mathbf{x})$  on the level-set surface  $\Gamma_c \subset \Omega$  can be written as*

$$\circlearrowleft_{\phi} f(\mathbf{x}) = -\mathbf{P} \nabla \mathbf{n} \nabla f(\mathbf{x}), \quad \mathbf{x} \in \Gamma_c.$$

*Proof.* Since  $\|\nabla \phi\| \neq 0$ , by the implicit function theorem, the level-set surface can be locally parameterized. Again since  $\circlearrowleft$  is geometric intrinsic on parametric surface, the level-set formulations of second tangent operator satisfies

$$\circlearrowleft_{\phi} f = \circlearrowleft f.$$

By definition

$$\circlearrowleft f = [\mathbf{x}_u, \mathbf{x}_v][g^{\alpha\beta}] S [f_u, f_v]^T.$$

From the chain rule, the above formula becomes

$$\circlearrowleft f = [\mathbf{x}_u, \mathbf{x}_v][g^{\alpha\beta}] S [\mathbf{x}_u, \mathbf{x}_v]^T \nabla f.$$

Applying (2.9e), we have

$$\circlearrowleft f = \circlearrowleft \mathbf{x} \nabla f.$$

From (2.10a), we obtain  $\circlearrowleft \mathbf{x} = -\nabla_s \mathbf{n}$ . Using (3.1), we get  $\nabla_s \mathbf{n} = \mathbf{P} \nabla \mathbf{n}$ . Again applying above formula yields

$$\circlearrowleft_\phi f = -\mathbf{P} \nabla \mathbf{n} \nabla f. \quad (3.3)$$

The proof is thereby complete.  $\square$

**Remark 3.3.** It is easy to prove in Theorem 3.3 that, the third tangent operator applying to  $f(\mathbf{x})$  depends only on the function values on  $\Gamma_c$ . The proof is similar to that of Remark 3.2, and we omit the details.

**Remark 3.4.** Because  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are the eigenvectors of  $\circlearrowleft \mathbf{x}$  corresponding to the eigenvalues  $k_1$  and  $k_2$ , respectively, by Theorem 3.3,  $\circlearrowleft \mathbf{x} = -\mathbf{P} \nabla \mathbf{n}$ . We obtain  $-\mathbf{P} \nabla \mathbf{n} \mathbf{e}_i = k_i \mathbf{e}_i$ ,  $i = 1, 2$ . Therefore the principal directions of the level-set surface can be obtained via solving the linear systems.

Let us now consider the tangential divergence operator.

**Theorem 3.4.** *Assume that  $\mathbf{v}$  is a smoothing vector field defined in  $\Omega$ . Then the tangential divergence operator applying to  $\mathbf{v}$  on  $\Gamma_c \subset \Omega$  is*

$$\operatorname{div}_\phi(\mathbf{v}) = \langle 2H\mathbf{n}, \mathbf{v} \rangle + \operatorname{div}(\mathbf{v}) - \mathbf{n}^T(\nabla \mathbf{v})\mathbf{n}.$$

*In particular, when  $\mathbf{v}$  is a tangent vector field defined on level-set surface  $\Gamma_c$ , we have*

$$\operatorname{div}_\phi(\mathbf{v}) = \operatorname{div}(\mathbf{v}) - \mathbf{n}^T(\nabla \mathbf{v})\mathbf{n}. \quad (3.4)$$

*Furthermore, if  $\mathbf{v}$  is a normal vector field defined on level-set surface  $\Gamma_c$ , then  $\operatorname{div}_\phi(\mathbf{v}) = 0$ , where  $\nabla \mathbf{v} = [\nabla v_1, \nabla v_2, \nabla v_3]$ .*

*Proof.* Because  $\|\nabla \phi\| \neq 0$ , according to the implicit function theorem, the level-set surface can be locally parameterized. Since  $\operatorname{div}_s$  is geometric intrinsic on surface, the tangential divergence operator in level-set form satisfies

$$\operatorname{div}_\phi(\mathbf{v}) = \operatorname{div}_s(\mathbf{v}).$$

From  $[\mathbf{x}_u, \mathbf{x}_v]^T = [\mathbf{x}_u, \mathbf{x}_v]^T \nabla_s \mathbf{x}$  and (2.5), we have

$$\operatorname{div}_s(\mathbf{v}) = \operatorname{div}_s(\nabla_s \mathbf{x} \mathbf{v}).$$

Applying Lemma 2.2 with  $\mathbf{f} = \mathbf{x}$ ,  $\mathbf{h} = \mathbf{v}$ , we have

$$\operatorname{div}_s(\nabla_s \mathbf{x} \mathbf{v}) = \langle \Delta_s \mathbf{x}, \mathbf{v} \rangle + \nabla_s \mathbf{x} : \nabla_s \mathbf{v}.$$

Using (2.9a), (2.10e) and Lemma 2.3, we finally have

$$\begin{aligned}
\operatorname{div}_s(\mathbf{v}) &= \operatorname{div}_s(\nabla_s \mathbf{x} \mathbf{v}) \\
&= \langle 2H\mathbf{n}, \mathbf{v} \rangle + \operatorname{tr}(\nabla_s \mathbf{x}^T \nabla_s \mathbf{v}) \\
&= \langle 2H\mathbf{n}, \mathbf{v} \rangle + \operatorname{tr}([\mathbf{x}_u, \mathbf{x}_v][g^{\alpha\beta}][\mathbf{v}_u, \mathbf{v}_v]^T) \\
&= \langle 2H\mathbf{n}, \mathbf{v} \rangle + \operatorname{tr}([\mathbf{x}_u, \mathbf{x}_v][g^{\alpha\beta}][\mathbf{x}_u, \mathbf{x}_v]^T \nabla \mathbf{v}) \\
&= \langle 2H\mathbf{n}, \mathbf{v} \rangle + \operatorname{tr}((\mathbf{I} - \mathbf{n}\mathbf{n}^T)\nabla \mathbf{v}) \\
&= \langle 2H\mathbf{n}, \mathbf{v} \rangle + \operatorname{div}(\mathbf{v}) - \mathbf{n}^T(\nabla \mathbf{v})\mathbf{n}.
\end{aligned} \tag{3.5}$$

Hence

$$\operatorname{div}_\phi(\mathbf{v}) = \langle 2H\mathbf{n}, \mathbf{v} \rangle + \operatorname{div}(\mathbf{v}) - \mathbf{n}^T(\nabla \mathbf{v})\mathbf{n}.$$

In particular, if  $\mathbf{v}$  is a tangent vector field defined on level-set surface  $\Gamma_c$ , we obtain  $\langle 2H\mathbf{n}, \mathbf{v} \rangle = 0$ . Consequently, (3.4) holds. On the other hand, if  $\mathbf{v}$  is a normal vector field defined on level-set surface  $\Gamma_c$ , we obviously have from (2.5) that  $\operatorname{div}_\phi(\mathbf{v}) = 0$ .  $\square$

**Remark 3.5.** Using (3.5), we can easily prove in Theorem 3.4 that, the tangential divergence operator applying to  $f(\mathbf{x})$  depends only on the vector field values on  $\Gamma_c$ . That is, if  $\mathbf{v}$  is a vector field satisfying  $\mathbf{v}(\mathbf{x}) = \mathbf{0}$  for all  $\mathbf{x} \in \Gamma_c$ , then  $\operatorname{div}_\phi(\mathbf{v}) = 0$ . Hence,  $\operatorname{div}_\phi$  is well defined.

**Remark 3.6.** Note that an arbitrary vector on level-set surface can be decomposed into a tangent component and a normal component, that is,  $\mathbf{v} = \mathbf{v}_t + \mathbf{v}_n$ . Applying Theorem 3.4, we have  $\operatorname{div}_\phi(\mathbf{v}) = \operatorname{div}(\mathbf{v}_t) - \mathbf{n}^T(\nabla \mathbf{v}_t)\mathbf{n}$ .

To verify the correctness of above generalization of tangent operators, we check a relationship among these first-order differential operators. This relation is valid for the parametric surface (see [3]).

**Theorem 3.5.** *For any differentiable function  $f$  defined on  $\Omega$ , we have*

$$2H\nabla_\phi f - \diamond_\phi f - \circlearrowleft_\phi f = \mathbf{0}. \tag{3.6}$$

*Proof.* From Theorems 3.1-3.3, we have

$$\begin{aligned}
\nabla_\phi f &= \mathbf{P}\nabla f, \\
\diamond_\phi f &= \mathbf{P}(2H\mathbf{I} + \nabla \mathbf{n})\nabla f, \\
\circlearrowleft_\phi f &= -\mathbf{P}\nabla \mathbf{n}\nabla f.
\end{aligned}$$

Therefore (3.6) can be verified.  $\square$

**Theorem 3.6.** *Suppose  $f$  is a second order differentiable function on  $\Omega$ . Then the level-set form Laplace-Beltrami operator  $\Delta_\phi$  applying to  $f$  is given by*

$$\Delta_\phi f = \operatorname{div}_\phi(\nabla_\phi f). \tag{3.7}$$

*Proof.* According to Theorems 3.1 and 3.4, we can obtain (3.7).  $\square$

**Theorem 3.7.** *Assume  $f$  is a second order differentiable function on  $\Omega$ . Then the level-set form Giaquinta-Hildebrandt operator  $\square_\phi$  acting on  $f$  is*

$$\square_\phi f = \operatorname{div}_\phi(\diamond_\phi f). \quad (3.8)$$

*Proof.* According to Theorems 3.2 and 3.3, the conclusion is obvious.  $\square$

**Theorem 3.8.** *Suppose  $f$  is a second order differentiable function on  $\Omega$ . Then the level-set form  $\boxtimes$  operator  $\boxtimes_\phi$  applying to  $f$  is written as*

$$\boxtimes_\phi f = \operatorname{div}_\phi(\circlearrowleft_\phi f). \quad (3.9)$$

*Proof.* The desired result follows from Theorems 3.2 and 3.4.  $\square$

## 4. Construction of Geometric PDEs for Level-set Surfaces

In this section we consider the general energy functional defined on the level-set surface

$$\mathcal{E}(\Gamma_c) = \int_{\Gamma_c} F(\mathbf{x}) dA. \quad (4.1)$$

To minimize (4.1), we conduct the first-order total variation  $\delta(\mathcal{E}_c(\Gamma_c), \Phi)$  or normal variation  $\delta(\mathcal{E}(\Gamma_c), \phi)$ , and obtain Euler-Lagrange equation. Then we construct the following weak form geometric flow in  $L^2$  sense

$$\int_{\Gamma_c} \left\langle \frac{\partial \mathbf{x}}{\partial t}, \Phi \right\rangle dA = - \int_{\Gamma_c} \langle \mathcal{E}_c(\Gamma_c), \Phi \rangle dA,$$

and the strong form geometric flow

$$\frac{\partial \mathbf{x}}{\partial t} = -\mathcal{E}_c(\Gamma_c).$$

Without loss of generality, we just consider the situation of total variation.

Assume  $\phi = \phi(\mathbf{x}, t)$ . Then we have the following level-set equation

$$\phi(\mathbf{x}(t), t) = c.$$

Taking a temporal derivative of the entire equation, we have

$$\frac{\partial \phi}{\partial t} + (\nabla \phi)^T \frac{\partial \mathbf{x}}{\partial t} = 0. \quad (4.2)$$

In the previous section we have generalized several differential operators to the implicit surface. Now we can get the geometric PDEs in the level-set form, which are equivalent to those for the parametric surface.

### 4.1. Second-order geometric PDE in level-set form

By Theorem 3.4, we can obtain an equivalent form of (2.15) in level-set form. Suppose  $F = h(\mathbf{x}, \mathbf{n})$  is a smooth function defined on  $\mathbb{R}^3 \times \mathbb{R}^3$ . Then in  $L^2$  sense the second-order geometric PDE in level-set form is of the form

$$\frac{\partial \phi}{\partial t} = \left( \mathbf{n}^T \nabla_{\mathbf{x}} h + 2H \nabla_{\mathbf{n}} h^T \mathbf{n} + \operatorname{div}(\nabla_{\mathbf{n}} h) - \mathbf{n}^T (\nabla(\nabla_{\mathbf{n}} h)) \mathbf{n} - 2hH \right) \|\nabla \phi\|. \quad (4.3)$$

**Example 4.1. (Mean curvature flow in the level-set form).** Let  $F = h(\mathbf{x}, \mathbf{n})=1$ . Then (4.3) can be written as

$$\frac{\partial \phi}{\partial t} = -2H \|\nabla \phi\|. \quad (4.4)$$

**Example 4.2. (Weighted mean curvature flow in the level-set form).** Suppose  $h(\mathbf{x}, \mathbf{n})$  is a positive  $t$ -order homogeneous function with respect to the second variable  $t$ , that is,

$$h(\mathbf{x}, \lambda \mathbf{n}) = \lambda^t h(\mathbf{x}, \mathbf{n}), \quad \forall \mathbf{x} \in \mathbb{R}^3, \mathbf{n} \in \mathbb{R}^3 \setminus \{0\}, \lambda > 0.$$

Then (4.3) becomes

$$\frac{\partial \phi}{\partial t} = \left( \mathbf{n}^T \nabla_{\mathbf{x}} h + 2hH(t-1) + \mathbf{P} : (\nabla_{\mathbf{x}\mathbf{n}}^2 h) + \mathbf{P} \nabla \mathbf{n} : (\nabla_{\mathbf{n}\mathbf{n}}^2 h) \right) \|\nabla \phi\|, \quad (4.5)$$

where  $\nabla_{\mathbf{x}\mathbf{n}}^2 h = \nabla_{\mathbf{x}} \nabla_{\mathbf{n}} h \in \mathbb{R}^{3 \times 3}$ ,  $\nabla_{\mathbf{n}\mathbf{n}}^2 h = \nabla_{\mathbf{n}} \nabla_{\mathbf{n}} h \in \mathbb{R}^{3 \times 3}$ .

#### 4.2. Fourth-order geometric PDE in the level-set form

Let  $F = f(H, K)$  be a smooth function defined on  $\mathbb{R}^2$ . Then by (2.16), Theorems 3.4, 3.6, and 3.7, we have the fourth-order geometric PDE in level-set form

$$\begin{aligned} \frac{\partial \phi}{\partial t} = & \left( \operatorname{div}(\mathbf{P}(2H\mathbf{I} + \nabla \mathbf{n}) \nabla f_K) - \mathbf{n}^T \nabla(\mathbf{P}(2H\mathbf{I} + \nabla \mathbf{n}) \nabla f_K) \mathbf{n} + \frac{1}{2} \operatorname{div}(\mathbf{P} \nabla f_H) \right. \\ & \left. - \frac{1}{2} \mathbf{n}^T \nabla(\mathbf{P} \nabla f_H) \mathbf{n} + 2HK f_K + (2H^2 - K) f_H - 2Hf \right) \|\nabla \phi\|. \end{aligned} \quad (4.6)$$

**Example 4.3 (Willmore flow in the level-set form).** Suppose  $f(H, K) = H^2$ . Then (4.6) becomes

$$\frac{\partial \phi}{\partial t} = \left( \operatorname{div}(\mathbf{P} \nabla H) - \mathbf{n}^T \nabla(\mathbf{P} \nabla H) \mathbf{n} + 2H(H^2 - K) \right) \|\nabla \phi\|. \quad (4.7)$$

**Example 4.4 (Gaussian curvature flow in the level-set form).** Assume  $f(H, K) = H$ . Then (4.6) is

$$\frac{\partial \phi}{\partial t} = -K \|\nabla \phi\|. \quad (4.8)$$

#### 4.3. Sixth-order geometric PDE in the level-set form

Let  $F = \|\nabla_s f(H, K)\|^2$  be a smooth function defined on  $\mathbb{R}^2$ . Then by (2.17), Theorems 3.1, 3.2, 3.4, 3.6 and 3.7, we have the sixth-order geometric PDE in level-set form

$$\begin{aligned} \frac{\partial \phi}{\partial t} = & \left( \Delta_{\phi}(f_H \Delta_{\phi} f) + 2\Box_{\phi}(f_K \Delta_{\phi} f) + 4KH f_K \Delta_{\phi} f + 4H^2 f_H \Delta_{\phi} f - 2K f_H \Delta_{\phi} f \right. \\ & \left. - 2H \|\nabla_{\phi} f\|^2 + 2\langle \nabla_{\phi} f, \diamond_{\phi} f \rangle \right) \|\nabla \phi\|. \end{aligned} \quad (4.9)$$

**Example 4.5 (Minimal mean curvature variation flow).** Suppose  $f = H$ . Then (4.9) can be written as

$$\frac{\partial \phi}{\partial t} = \left( \Delta_{\phi}^2 H + 2(2H^2 - K) \Delta_{\phi} H + 2\langle \nabla_{\phi} H, \diamond_{\phi} H \rangle - 2H \|\nabla_{\phi} H\|^2 \right) \|\nabla \phi\|. \quad (4.10)$$

**Remark 4.1.** Apart from the three classes of geometric PDEs described above, any parametric form geometric PDE

$$\frac{\partial \mathbf{x}}{\partial t} = V(\mathbf{x}, \mathbf{n}, H, K)\mathbf{n}$$

can be converted to the level-set form

$$\frac{\partial \phi}{\partial t} + V(\mathbf{x}, \mathbf{n}, H, K)\|\nabla \phi\| = 0,$$

provided that the velocity function  $V$  can be represented by the differential operators introduced in Section 2. Therefore, the  $H^{-1}$  gradient flows for the first, second and third order energy functionals can also be converted to the level-set forms. For instance, the surface diffusion flow

$$\frac{\partial \mathbf{x}}{\partial t} = -2\Delta_s H\mathbf{n},$$

which is the  $H^{-1}$  gradient flow of the area functional, is converted to the following level-set form

$$\frac{\partial \phi}{\partial t} + \Delta_\phi \left[ \operatorname{div} \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) \right] \|\nabla \phi\| = 0.$$

## 5. Equivalencies of Geometric PDEs in the Level-set Forms

As a matter of fact, the construction methods of geometric PDEs in level-set form have been introduced in some literatures. In [3], two energy functionals using Dirac delta function and co-area formula have been presented. To explain them briefly, we first recall the *Heaviside* function in one dimensional space as

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases} \quad (5.1)$$

Then we define the Dirac delta function  $\delta$  (to make a difference from the variation notation  $\delta$ ) as the generalized derivative of the *Heaviside* function, i.e.,

$$\delta(x) = H'(x). \quad (5.2)$$

Therefore, one type energy functional using function  $\delta$  is expressed as

$$\mathcal{E}_1(\Gamma_c) = \int_{\Gamma_c} f(\mathbf{x})dA = \int_{\mathbb{R}^3} f(\mathbf{x})\delta(\phi(\mathbf{x}) - c)\|\nabla \phi(\mathbf{x})\|d\mathbf{x}, \quad (5.3)$$

where  $d\mathbf{x}$  is volumetric element. From (5.3),  $\mathcal{E}_1(\Gamma_c)$  can be denoted by  $\mathcal{E}_1(\phi)$ .

To introduce the second type energy functional, we consider a family of level-set surfaces

$$\Gamma_a = \{\mathbf{x} : \phi(\mathbf{x}) = a\}, \quad a \in [c - \omega, c + \omega].$$

Applying the co-area formula (see [25, 26]), we have

$$\int_{-\omega}^{\omega} \left[ \int_{\Gamma_a} f(\mathbf{x})dA \right] da = \int_{\Omega} f(\mathbf{x})\|\nabla \phi(\mathbf{x})\|d\mathbf{x},$$

where  $\Omega = \bigcup_{a \in [c - \omega, c + \omega]} \{\mathbf{x} : \phi(\mathbf{x}) = a\}$ . Suppose  $\|\nabla \phi(\mathbf{x})\| > 0, \forall \mathbf{x} \in \Omega$ . Then the energy functional becomes

$$\mathcal{E}_2(\phi) = \int_{\Omega} f(\mathbf{x})\|\nabla \phi(\mathbf{x})\|d\mathbf{x}, \quad \forall \phi \in C^1(\Omega). \quad (5.4)$$

In (5.3) and (5.4), assuming  $f(\mathbf{x}) = h(\mathbf{x}, \mathbf{n})$  and  $h(\mathbf{x}, \mathbf{n})$  is a properly smooth function, we have

$$\begin{aligned}\mathcal{E}_1(\phi) &= \int_{\mathbb{R}^3} h(\mathbf{x}, \mathbf{n}) \delta(\phi(\mathbf{x}) - c) \|\nabla\phi(\mathbf{x})\| d\mathbf{x}, \\ \mathcal{E}_2(\phi) &= \int_{\Omega} h(\mathbf{x}, \mathbf{n}) \|\nabla\phi(\mathbf{x})\| d\mathbf{x}.\end{aligned}$$

To minimize energy functional  $\mathcal{E}_1(\phi)$ , we compute its first-order variation and we obtain

$$\begin{aligned}\delta(\mathcal{E}_1(\phi, \psi)) &= \frac{d}{d\epsilon} \mathcal{E}_1(\phi + \epsilon\psi) \Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \int_{\mathbb{R}^3} h\left(\mathbf{x}, \frac{\nabla(\phi + \epsilon\psi)}{\|\nabla(\phi + \epsilon\psi)\|}\right) \delta(\phi + \epsilon\psi) \|\nabla(\phi + \epsilon\psi)\| d\mathbf{x} \Big|_{\epsilon=0} \\ &= \int_{\mathbb{R}^3} \delta(\phi) \left[ (\nabla_{\mathbf{n}} h)^T \mathbf{P} \nabla \psi - \frac{(\nabla h)^T \nabla \phi}{\|\nabla \phi\|} \psi - \operatorname{div} \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) h \psi \right] d\mathbf{x},\end{aligned}$$

where  $\psi(\mathbf{x})$  is an arbitrary properly smooth function. Then we obtain the following Euler-Lagrange equation

$$\mathcal{E}_1(\phi) = -\delta(\phi) \operatorname{div}(\mathbf{P} \nabla_{\mathbf{n}} h) - \delta(\phi) \frac{(\nabla h)^T \nabla \phi}{\|\nabla \phi\|} - \delta(\phi) \operatorname{div} \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) h = 0. \quad (5.5)$$

Computing first-order variation of  $\mathcal{E}_2(\phi)$  (see [3]), we have the following Euler-Lagrange equation

$$\mathcal{E}_2(\phi) = -\operatorname{div} \left( \frac{h \nabla \phi}{\|\nabla \phi\|} + \mathbf{P} \nabla_{\mathbf{n}} h \right) = 0.$$

By some calculations we have

$$\mathcal{E}_2(\phi) = -\operatorname{div}(\mathbf{P} \nabla_{\mathbf{n}} h) - \frac{(\nabla h)^T \nabla \phi}{\|\nabla \phi\|} - \operatorname{div} \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right) h = 0. \quad (5.6)$$

**Remark 5.1.** Note that function  $\delta$  is used in (5.5). Since  $\delta(\phi) = 0$  almost everywhere, except on the lower-dimensional interface, which has measure zero,  $\delta(\phi)$  is a generalized function that leads to troublesome in the real computation. Therefore,  $\delta(\phi)$  is often approximated by a continuous function. For instance,  $\delta(\phi)$  is chosen as  $\|\nabla\phi(\mathbf{x})\|$  in [11], while in literatures [23, 27],  $\delta(\phi)$  is chosen as

$$\delta(\phi) = \begin{cases} 0, & \phi < -\epsilon, \\ \frac{1}{2\epsilon} + \frac{1}{2\epsilon} \cos\left(\frac{\pi\epsilon}{\phi}\right), & -\epsilon \leq \phi \leq \epsilon, \\ 0, & \epsilon < \phi, \end{cases}$$

where  $\epsilon$  is a parameter that determines the size of the bandwidth of numerical smearing. Obviously, taking  $\delta(\phi)$  in such a way, (5.5) and (5.6) are not equal, i.e., the variations of  $\mathcal{E}_1(\phi)$  and  $\mathcal{E}_2(\phi)$  are not equivalent.

Suppose  $\phi(\mathbf{x})$  is a signed distance function (see [27]), that is,  $\|\nabla\phi(\mathbf{x})\| = 1$ . Taking  $\delta(\phi) = \|\nabla\phi(\mathbf{x})\|$  in (5.5), then (5.5) and (5.6) become

$$\mathcal{E}_i(\phi) = -(\nabla_{\mathbf{x}} h)^T \mathbf{n} - 2H(\nabla_{\mathbf{n}} h)^T \mathbf{n} - \operatorname{div}(\nabla_{\mathbf{n}} h) + \mathbf{n}^T \nabla(\nabla_{\mathbf{n}} h) \mathbf{n} + 2Hh = 0, \quad i = 1, 2. \quad (5.7)$$

Then we can construct the following geometric PDE

$$\frac{\partial \phi}{\partial t} = \left( \mathbf{n}^T \nabla_{\mathbf{x}} h + 2H \nabla_{\mathbf{n}} h^T \mathbf{n} + \operatorname{div}(\nabla_{\mathbf{n}} h) - \mathbf{n}^T (\nabla(\nabla_{\mathbf{n}} h)) \mathbf{n} - 2hH \right) \|\nabla \phi\|. \quad (5.8)$$

The concrete construction process can be found in [3].

Then we can find (4.3) and (5.8) are the same, that is, the second-order geometric PDEs are equivalent via two different methods, respectively.

To verify that the fourth-order and the sixth-order geometric PDEs are also consistent, we prove the following result.

**Theorem 5.1.** *Assume that  $\Gamma_c := \{\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3 : \phi(\mathbf{x}) = c\}$ , where  $\phi(\mathbf{x})$  is a properly smooth function defined on  $\mathbb{R}^3$  and  $\|\nabla \phi\| \neq 0$  on  $\Gamma_c$ , where  $c$  is an arbitrary constant. Then we have*

$$K = 2H^2 - \nabla H^T \mathbf{n} - \frac{1}{2} \operatorname{div}(\nabla \mathbf{n}^T \mathbf{n}). \quad (5.9)$$

*Proof.* From Section 3 we have

$$\mathbf{n} = \frac{\nabla \phi}{\|\nabla \phi\|}, \quad H = -\frac{1}{2} \operatorname{div} \left( \frac{\nabla \phi}{\|\nabla \phi\|} \right), \quad K = -\|\nabla \phi\|^{-4} \det \begin{bmatrix} \nabla^2 \phi & \nabla \phi \\ \nabla \phi^T & 0 \end{bmatrix}.$$

Expanding each of the three terms of the right-hand side of (5.9), we have

$$\begin{aligned} 2H^2 &= \frac{1}{2} \left( \frac{\Delta \phi}{\|\nabla \phi\|} \right)^2 - \Delta \phi \frac{\nabla \phi^T \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^4} + \frac{1}{2} \frac{(\nabla \phi^T \nabla^2 \phi \nabla \phi)^2}{\|\nabla \phi\|^6}, \\ \nabla H^T \mathbf{n} &= -\frac{1}{2} \frac{(\nabla(\Delta \phi))^T \nabla \phi}{\|\nabla \phi\|^2} + \frac{1}{2} \frac{\nabla \phi^T \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^4} \Delta \phi + \frac{1}{2} \nabla \left( \frac{\nabla \phi^T \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^3} \right)^T \mathbf{n}, \\ \frac{1}{2} \operatorname{div}(\nabla \mathbf{n}^T \mathbf{n}) &= \frac{1}{2} \operatorname{div} \left( \frac{\mathbf{P} \nabla^2 \phi \mathbf{n}}{\|\nabla \phi\|} \right) \\ &= \frac{1}{2} \frac{(\nabla(\Delta \phi))^T \nabla \phi}{\|\nabla \phi\|^2} + \frac{1}{2} \frac{\nabla^2 \phi : \nabla^2 \phi}{\|\nabla \phi\|^2} - \frac{\nabla \phi^T \nabla^2 \phi \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^4} \\ &\quad - \frac{1}{2} \nabla \left( \frac{\nabla \phi^T \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^3} \right)^T \mathbf{n} - \frac{1}{2} \frac{\nabla \phi^T \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^4} \Delta \phi + \frac{1}{2} \frac{(\nabla \phi^T \nabla^2 \phi \nabla \phi)^2}{\|\nabla \phi\|^6}. \end{aligned}$$

Substituting these into the right-hand side of (5.9) gives

$$\begin{aligned} &2H^2 - \nabla H^T \mathbf{n} - \frac{1}{2} \operatorname{div}(\nabla \mathbf{n}^T \mathbf{n}) \\ &= \frac{1}{2} \left( \frac{\Delta \phi}{\|\nabla \phi\|} \right)^2 - \Delta \phi \frac{\nabla \phi^T \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^4} - \frac{1}{2} \frac{\nabla^2 \phi : \nabla^2 \phi}{\|\nabla \phi\|^2} + \frac{\nabla \phi^T \nabla^2 \phi \nabla^2 \phi \nabla \phi}{\|\nabla \phi\|^4}. \end{aligned} \quad (5.10)$$

Then expanding the left-hand side of (5.9), we have

$$K = \|\nabla \phi\|^{-4} \nabla \phi^T (\nabla^2 \phi)^* \nabla \phi,$$

where  $(\nabla^2 \phi)^*$  stands for the adjoint matrix of  $\nabla^2 \phi$ . The further calculations show that the right-hand side of the above equality coincide with the right-hand side of (5.10). Hence, (5.9) holds.  $\square$

Using Theorem 5.1, we can prove the following theorem.

**Theorem 5.2.** *For any given first, second and third order Lagrange functions  $F = h(\mathbf{x}, \mathbf{n})$ ,  $F = f(H, K)$  and  $F = \|\nabla_s f(H, K)\|^2$ , the corresponding level set form geometric partial differential equations derived from converting the parametric forms and the equations derived from varying (5.4) are equivalent.*

*Proof.* For the first order Lagrange functions  $F = h(\mathbf{x}, \mathbf{n})$ , the equivalence of the two sets equations has been given above. Hence, we only need to show the correctness of the theorem for  $F = f(H, K)$  and  $F = \|\nabla_s f(H, K)\|^2$ . Using Eq. (5.9), we know that both  $H$  and  $K$  can be regarded as functions of  $\mathbf{n}$ . Therefore,  $f(H, K)$  and  $\|\nabla_s f(H, K)\|^2$  can be written as  $g(\mathbf{n})$ , where  $g$  is a function involving differentiations. Using the same technique of the equivalency proof of the second order equations, we can complete the proof.  $\square$

## 6. Conclusions

We have shown that the level-set form geometric PDEs constructed using Delta function and co-area formula are not the same, while the equations constructed using co-area formula are consistent with the parametric form equations. By representing several differential quantities and differential operators in the implicit form, we have reformulated successfully the parametric form geometric PDEs, from the second-order to sixth-order, into level-set forms. The derived equations are consistent with the parametric ones and they are simpler in form than the equations derived using co-area formula. Therefore, one does not need to derive the level-set form equations if the parametric form ones are available. It should be pointed out that our conversion approach from parametric to implicit is effective for any parametric geometric PDEs as long as the PDEs are represented in terms of the differential operators as we discussed.

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## References

- [1] R. Schneider and L. Kobbelt, Generating fair meshes with  $G^1$  boundary conditions, Geometric Modeling and Processing, pages 251-261, Hong Kong, China, 2000.
- [2] R. Schneider and L. Kobbelt, Geometric fairing of irregular meshes for free-form surface design, *Comput. Aided Geom. D.*, **18**:4 (2001), 359-379.
- [3] G. Xu, Geometric Partial Differential Equation Methods in Computational Geometry, Science Press, Beijing, China, 2008.
- [4] G. Xu and Q. Zhang, A general framework for surface modeling using geometric partial differential equations, *Comput. Aided Geom. D.*, **25**:3 (2008), 181-202.
- [5] R. Whitaker and D. Breen, Level set models for the deformation of solid objects, Proceedings of the 3rd International Workshop on Implicit Surfaces, Eurographics Association, pages 19-35, June 1998.
- [6] D.L. Chopp and J.A. Sethian, Motion by intrinsic Laplacian of curvature, *Interface. Free Bound.*, **1** (1999), 1-18.
- [7] K. Museth, D. Breen, R. Whitaker and A. Barr, Level set surface editing operators, SIGGRAPH, pages 330-338, 2002.
- [8] S.J. Osher and R.P. Fedkiw, Level set methods, CAM Report 00-07, UCLA, Mathematics Department, 2000.

- [9] Q. Pan and G. Xu, Metamorphosis based on the level-set methods, Proceedings of National 15th CAD/CG Conference, pages 339-343, Dalian, China, May 24-27 2008.
- [10] H.K. Zhao, S. Osher and R. Fedkiw, Fast surface reconstruction using the level set method, CAM Report 01-01, UCLA, Mathematics Department, 2001.
- [11] H.K. Zhao, S. Osher, B. Merriman and M. Kang, Implicit and non-parametric shape reconstruction from unorganized points using variational level set method, *Comput. Vis. Image Und.*, **80**:3 (2000), 295-319.
- [12] U. Clarenz, U. Diewald, G. Dziuk, M. Rumpf and R. Rusu, A finite element method for surface restoration with boundary conditions, *Comput. Aided Geom. D.*, **21**:5 (2004), 427-445.
- [13] M. Droske and M. Rumpf, A level set formulation for Willmore flow, *Interface. Free Bound.*, **6**:3 (2004), 361-378.
- [14] G. Sapiro, Geometric Partial Differential Equations and Image Analysis, Cambridge University Press, Cambridge, 2001.
- [15] C. Bajaj and G. Xu, Anisotropic diffusion of subdivision surfaces and functions on surfaces, *ACM T. Graphic.*, **22**:1 (2003), 4-32.
- [16] G. Xu and Q. Zhang, Construction of geometric partial differential equations in computational geometry, *Mathematica Numerica Sinica*, **28**:4 (2006), 337-356.
- [17] Q. Zhang and G. Xu, Geometric partial differential equations for minimal curvature variation surfaces, Research Report No. ICM-06-03, Institute of Computational Mathematics, Chinese Academy of Sciences, 2006.
- [18] G. Farin, Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide, Academic Press, San Diego, CA, fifth edition, 2002.
- [19] J. Hoschek and D. Lasser, Fundamentals of Computer Aided Geometric Design, A. K. Peters, 1993.
- [20] M. Spivak, A Comprehensive Introduction to Differential Geometry, volume 3, Publish or Perish, Boston, third edition, 1999.
- [21] W. Fulton, Algebraic Curves: An Introduction to Algebraic Geometry, Mathematical Lecture Notes Series, 1974.
- [22] R. Goldman, Curvature formulas for implicit curves and surfaces, *Comput. Aided Geom. D.*, **22**:7 (2005), 632-658.
- [23] S. Osher and R. Fedkiw, Level Set Methods and Dynamic Implicit Surfaces, volume 153 of *Applied Mathematical Science*, Springer-Verlag, New York, 2003.
- [24] T.J. Willmore, An Introduction to Differential Geometry, Clarendon Press, Oxford, 1959.
- [25] G. Aubert and P. Kornprobst, Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations, Springer: Applied Mathematical Sciences, 2002.
- [26] L.C. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.
- [27] D. Peng, B. Merriman, S. Osher, H. Zhao and M. Kang, A PDE-based fast local level set method, *J. Comput. Phys.*, **155**:2 (1999), 410-438.