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# MULTI-LEVEL ADAPTIVE CORRECTIONS IN FINITE DIMENSIONAL APPROXIMATIONS *

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**Abstract**

Based on the Boolean sum technique, we introduce and analyze in this paper a class of multi-level iterative corrections for finite dimensional approximations. This type of multi-level corrections is adaptive and can produce highly accurate approximations. For illustration, we present some old and new finite element correction schemes for an elliptic boundary value problem.

*Mathematics subject classification:* 65B05, 65D15, 65J05, 65N15, 65N30.
*Key words:* Adaptive, Boolean sum, Correction, Finite dimensional, Multi-level

## 1. Introduction

Our multi-level corrections are based on the Boolean sum technique. The idea of applying the Boolean sum technique to construct highly accurate finite dimensional approximations may be dated back to [22,23], in which some local two-level and three-level finite element correction schemes were derived. In this paper, we shall propose a type of multi-level iterative corrections for finite dimensional approximations. This type of schemes is adaptive and is proposed to produce highly accurate approximations based on some simple postprocesses.

Let us give a little more detailed description of the main idea. Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space and $A$ and $B$ be two operators on $\mathcal{H}$. It is known that the so-called Boolean sum of $A$ and $B$ is defined by $A \oplus B = A + B - AB$. It is easy to see that

$$I - (A \oplus B) = (I - A)(I - B).$$

Hence as an operator from a subspace of $\mathcal{H}$ to another subspace, there may hold that

$$\|I - (A \oplus B)\| < \|I - B\|$$

for some proper operator $A$, which is the key that motivates our multi-level corrections. More precisely, let $u \in \mathcal{H}$ and $Bu$ be an approximation to $u$. Then $(A \oplus B)\,u$ may be a better approximation than $Bu$ for some simple operator $A$, where both $Au$ and $ABu$ are computable in application. Note that the construction of $A$ is associated with some subspace of $\mathcal{H}$ and the Boolean sum technique in the multi-level correction in this paper is indeed a successive subspace correction approach (see Section 2 for details).

We should mention that the Boolean sum technique has been applied to design efficient numerical schemes in approximation theory, numerical integration, numerical partial differential

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and numerical integral equations, etc., see, e.g., [2, 4, 5, 8–11, 13, 15–18, 20, 25–27, 29, 35–37, 39, 40, 43] and references therein. We refer to [28, 42] for other interesting connections.

Throughout this paper, we shall use the letter $C$ (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences. For convenience, the symbol $\lesssim$ will be used in this paper. The notation that $x_1 \lesssim y_1$ means that $x_1 \le C y_1$ for some positive constant $C$ that is independent of mesh parameters.

## 2. Multi-Level Correction

We shall discuss the multi-level corrections in a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ that can be compactly embedded into an inner product space $(\mathcal{H}, <\cdot,\cdot>)$, where associated norms are $\|\cdot\|$ and $|\cdot|$, respectively.

Let $\mathcal{K}$ be an operator on $\mathcal{H}$ defined by

$$(\mathcal{K}w, v) = <w, v> \quad \forall w \quad \forall v \in \mathcal{H}.$$

Then $\mathcal{K}$ is compact on $(\mathcal{H}, \|\cdot\|)$. Let $\mathcal{V} \subset \mathcal{H}$ be a finite dimensional subspace of $\mathcal{H}$ and $P_{\mathcal{V}} : \mathcal{H} \longrightarrow \mathcal{V}$ be a projection operator (namely $P_{\mathcal{V}}^2 = P_{\mathcal{V}}$) satisfying

$$\|u - P_{\mathcal{V}} u\| \lesssim \inf\{\|u - v\| : v \in \mathcal{V}\} \quad \forall\, u \in \mathcal{H}. \tag{2.1}$$

Set

$$\rho_{\mathcal{V}} = \sup_{u \in \mathcal{H}, \|u\|=1} |u - P_{\mathcal{V}} u|. \tag{2.2}$$

Then

$$\begin{aligned} |u - P_{\mathcal{V}} u| &\lesssim \rho_{\mathcal{V}} \|u\|, && \forall\, u \in \mathcal{H}, \\ |u - P_{\mathcal{V}} u| &\lesssim \rho_{\mathcal{V}} \|u - P_{\mathcal{V}} u\|, && \forall\, u \in \mathcal{H}. \end{aligned} \tag{2.3}$$

Consequently,

$$\inf\{|u - v| : v \in \mathcal{V}\} \lesssim \rho_{\mathcal{V}} \inf\{\|u - v\| : v \in \mathcal{V}\}, \quad \forall\, u \in \mathcal{H}. \tag{2.4}$$

**Lemma 2.1.** *There hold*

$$\rho_{\mathcal{V}} \lesssim \left(\|(I - P_{\mathcal{V}})\mathcal{K}\| + \|\mathcal{K}(I - P_{\mathcal{V}})\|\right)^{1/2}, \tag{2.5}$$

$$\lim_{\mathcal{V} \to \mathcal{H}} \rho_{\mathcal{V}} = 0, \tag{2.6}$$

*where* $\mathcal{V} \to \mathcal{H}$ *means that*

$$\inf_{v \in \mathcal{V}} \|u - v\| \to 0 \quad \forall\, u \in \mathcal{H}. \tag{2.7}$$

*Proof.* We divide the proof into four steps. First, note that for any $u \in \mathcal{H}$, there hold

$$\begin{aligned} & |(I - P_{\mathcal{V}})u|^2 \\ =& (\mathcal{K}(I - P_{\mathcal{V}})u, (I - P_{\mathcal{V}})u) \\ =& (\mathcal{K}(I - P_{\mathcal{V}})u - P_{\mathcal{V}}\mathcal{K}(I - P_{\mathcal{V}})u, (I - P_{\mathcal{V}})u) + (P_{\mathcal{V}}\mathcal{K}(I - P_{\mathcal{V}})u, (I - P_{\mathcal{V}})u), \end{aligned}$$

which may be estimated as follows

$$
\begin{aligned}
|(I-P_{\mathcal{V}})u|^2 &\lesssim \|(I-P_{\mathcal{V}})\mathcal{K}\|\|(I-P_{\mathcal{V}})u\|^2 + \|\mathcal{K}(I-P_{\mathcal{V}})u\|\|(I-P_{\mathcal{V}})u\| \\
&\lesssim \left(\|(I-P_{\mathcal{V}})\mathcal{K}\| + \|\mathcal{K}(I-P_{\mathcal{V}})\|\right)\|u\|^2, \ \forall\, u\in\mathcal{H}.
\end{aligned}
$$

This proves (2.5). Next we conclude from (2.1) and (2.7) that

$$\lim_{\mathcal{V}\to\mathcal{H}} \|(I-P_{\mathcal{V}})u\| = 0 \quad \forall\, u\in\mathcal{H}, \tag{2.8}$$

and hence

$$\lim_{\mathcal{V}\to\mathcal{H}} \sup_{u\in\mathcal{H},\|u\|=1} \|(I-P_{\mathcal{V}})\mathcal{K}u\| = 0, \tag{2.9}$$

or equvilently

$$\lim_{\mathcal{V}\to\mathcal{H}} \|(I-P_{\mathcal{V}})\mathcal{K}\| = 0. \tag{2.10}$$

We now apply a contradiction argument to prove

$$\lim_{\mathcal{V}\to\mathcal{H}} \|\mathcal{K}(I-P_{\mathcal{V}})\| = 0. \tag{2.11}$$

If (2.11) is not true, then there exist an $\epsilon_0>0$, a sequence $\{\mathcal{V}_j\}_{j=1}^{\infty}$ which converges to $\mathcal{H}$, and a sequence $\{w_j\}_{j=1}^{\infty}\subset\mathcal{H}$ with $\|w_j\|\le 1$ $(j=1,2,\cdots)$ such that

$$\|\mathcal{K}(I-P_{\mathcal{V}_j})w_j\| \ge \epsilon_0, \qquad j=1,2,\cdots \tag{2.12}$$

Since $\mathcal{H}$ is a Hilbert space, there exist $w\in\mathcal{H}$ and a weakly convergent subsequence of $\{w_j\}_{j=1}^{\infty}$, which we also denote by $\{w_j\}_{j=1}^{\infty}$ for convenience, such that $w_j$ converges weakly to $w$, namely

$$(w-w_j,v)\to 0 \quad as\ j\to\infty\ \forall v\in\mathcal{H}.$$

From (2.8) and the identity

$$(P_{\mathcal{V}_j}(w-w_j),v) = ((P_{\mathcal{V}_j}-I)(w-w_j),(I-P_{\mathcal{V}_j})v) + (w-w_j,v) \quad \forall v\in\mathcal{H},$$

we obtain that $P_{\mathcal{V}_j}(w-w_j)$ converges weakly to zero. Note that

$$(I-P_{\mathcal{V}_j})w_j = w_j - w + (I-P_{\mathcal{V}_j})w + P_{\mathcal{V}_j}(w-w_j),$$

we find that $(I-P_{\mathcal{V}_j})w_j$ converges weakly to zero. Using the compactness of $\mathcal{K}$, we conclude that

$$\lim_{j\to\infty} \|\mathcal{K}(I-P_{\mathcal{V}_j})w_j\| = 0,$$

which is a contradiction to (2.12).

Finally, we get (2.6) by using (2.8) and (2.11). This completes the proof. □

Lemma 2.1 may be viewed as a generalization of that in [33,40] to general finite dimensional approximations. The following interesting conclusion can be derived from (2.4) and Lemma 2.1 directly.

**Corollary 2.1.** *For any $u \in \mathcal{H}$, there holds*

$$\inf\{|u-v| : v \in \mathcal{V}\} \ll \inf\{\|u-v\| : v \in \mathcal{V}\} \tag{2.13}$$

*when $\mathcal{V} \to \mathcal{H}$.*

Let $\mathcal{U}_j$ $(j = 1, 2, \cdots)$ be some finite dimensional subspaces of $\mathcal{H}$ satisfying

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_j \subset \cdots \subset \mathcal{H}$$

and $Q_j (\equiv Q_{\mathcal{U}_j}) : \mathcal{H} \longrightarrow \mathcal{U}_j$ $(j = 1, 2, \cdots)$ be a group of operators satisfying

$$|Q_j u| \lesssim |u| \quad \forall\, u \in \mathcal{H}, j = 1, 2, \cdots \tag{2.14}$$

Define $R_n : \mathcal{H} \longrightarrow \mathcal{V} \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_n (n = 1, 2, \cdots)$ by

$$\begin{aligned} &R_0 = P_{\mathcal{V}}, \\ &R_{2n-1} = Q_n R_{2n-2}, \qquad n = 1, 2, \cdots \\ &R_{2n} = P_{\mathcal{V}} \oplus R_{2n-1}, \qquad n = 1, 2, \cdots \end{aligned}$$

For instance,

$$R_1 = Q_1 P_{\mathcal{V}}, \quad R_2 = P_{\mathcal{V}} + Q_1 P_{\mathcal{V}} - P_{\mathcal{V}} Q_1 P_{\mathcal{V}}.$$

**Theorem 2.1.** *For $n = 1, 2, \cdots$, there hold*

$$|(I - R_{2n})u| \lesssim \rho_{\mathcal{V}} \|(I - R_{2n-1})u\|. \tag{2.15}$$

*Moreover, if*

$$\sup_{1 \le j \le n+1} \|Q_j w\| \lesssim \eta^{-1} |w| \quad \forall\, w \in \mathcal{H}, \tag{2.16}$$

*then*

$$\|(I - R_{2n+1})u\| \lesssim \|(I - Q_{n+1})u\| + \rho_{\mathcal{V}} \eta^{-1} \|(I - R_{2n-1})u\|. \tag{2.17}$$

*Proof.* It is seen from the definition of $R_n$ that

$$I - R_{2n} = (I - P_{\mathcal{V}})(I - R_{2n-1}), \ n = 1, 2, \cdots, \tag{2.18}$$

from which and (2.13) we get (2.15). Note that (2.16) implies

$$\begin{aligned} &\|Q_{n+1}(I - R_{2n})u\| \\ \lesssim &\eta^{-1} |Q_{n+1}(I - R_{2n})u| \lesssim \eta^{-1} |(I - R_{2n})u|, \end{aligned}$$

which together with (2.15) and the identity

$$I - Q_{n+1} R_{2n} = (I - Q_{n+1}) + Q_{n+1}(I - R_{2n}) \tag{2.19}$$

leads to (2.17). This completes the proof. □

It is seen from Theorem 2.1 that $R_n u$ $(n = 1, 2, \cdots)$ would be used as highly accurate approximations to $u$, as compared to $P_{\mathcal{V}} u$, and (2.17) implies that the iteration $\|(I - R_{2n+1})u\|$ $(n = 1, 2, \cdots)$ is quasi-contractive if $\rho_{\mathcal{V}} \eta^{-1} < 1/C$ for some constant $C$. In applications, obviously, we may use $\|(I - Q_{n+1}) R_{2n} u\|$ as a posteriori error estimate for $\|(I - R_{2n})u\|$, from which the multi-level adaptive correction schemes for highly accurate approximations $R_n u (n = 1, 2, \cdots)$ are then followed.

## 3. Applications to Finite Element Approximation

Let $\Omega$ be a bounded domain in $\mathbb{R}^d (d \geq 1)$. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms, see, e.g., [1,7]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v \mid_{\partial\Omega} = 0\}$, where $v \mid_{\partial\Omega} = 0$ is in the sense of trace, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$ and $\|\cdot\|_\Omega = \|\cdot\|_{0,2,\Omega}$.

Assume that $T^h(\Omega) = \{\tau\}$ is a mesh of $\Omega$ with mesh-size function $h(x)$ whose value is the diameter $h_\tau$ of the element $\tau$ containing $x$. Denote $h = \max\limits_{x\in\Omega} h(x)$ the (largest) mesh size of $T^h(\Omega)$.

Let $T^h(\Omega)$ consist of shape-regular simplices and define $S^h(\Omega)(\equiv S^{h,r}(\Omega))$ to be a space of continuous functions on $\Omega$ such that for $v \in S^h(\Omega), v$ restricted to each $\tau$ is a polynomial of total degree $\leq r$, namely

$$S^h(\Omega) = \{v \in C(\bar{\Omega}) : v \mid_\tau \in P_\tau^r \quad \forall\, \tau \in T^h(\Omega)\}, \tag{3.1}$$

where $P_\tau^r$ is the space of polynomials of degree not greater than a positive integer $r$. Set $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$. These are Lagrange finite element spaces and we refer to [40] (see also [31, 32]) for their basic properties that will be used in our analysis.

Let $(\cdot,\cdot)$ be the standard inner-product of $L^2(\Omega)$. If $Q_h(\equiv Q_h^{(r)}) : L^2(\Omega) \mapsto S^h(\Omega)$ is the $L^2-$projection operator defined by

$$(w - Q_h w, v) = 0 \quad \forall\, v \in S^h(\Omega), \tag{3.2}$$

then we have the following standard error estimations

$$\|w - Q_h w\|_{0,\Omega} + h\|w - Q_h w\|_{1,\Omega} \lesssim h^{1+r}|w|_{1+r,\Omega} \ \text{ if } \ w \in H^{1+r}(\Omega). \tag{3.3}$$

### 3.1. A linear elliptic boundary value problem

In this subsection, we shall present some basic properties of a second order elliptic boundary value problem and its finite element approximations, which will be used in this paper.

We consider the homogeneous boundary value problem

$$\begin{cases} Lu = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{3.4}$$

In (3.4), $L$ is a general linear second order elliptic operator:

$$Lu = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu,$$

where $a_{ij}, b_i \in W^{1,\infty}(\Omega), c \in L^\infty(\Omega)$, and $(a_{ij})$ is uniformly positive definite on $\Omega$.

The weak form of (3.4) is as follows: Find $u \equiv L^{-1}f \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall\, v \in H_0^1(\Omega), \tag{3.5}$$

where $a(u,v) = a_0(u,v) + N(u,v)$ with

$$a_0(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} \quad \text{and} \quad N(u,v) = \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} v + cuv.$$

Note that

$$\|w\|_{1,\Omega}^2 \lesssim a_0(w,w) \quad \forall w \in H_0^1(\Omega)$$

and

$$\begin{aligned} a_0(u,v) &\lesssim \|u\|_{1,\Omega} \lesssim \|u\|_{1,\Omega}\|v\|_{1,\Omega}, \\ N(u,v) &\lesssim \|u\|_{0,\Omega}\|v\|_{1,\Omega} \quad \forall u,v \in H_0^1(\Omega). \end{aligned}$$

Our basic assumption is that (3.5) is well-posed, namely (3.5) is uniquely solvable for any $f \in H^{-1}(\Omega)$. (A simple sufficient condition for this assumption to be satisfied is that $c \geq 0$.) An application of the open-mapping theorem yields

$$\|w\|_{1,\Omega} \lesssim \|Lw\|_{-1,\Omega} \quad \forall\, w \in H_0^1(\Omega). \tag{3.6}$$

It is seen that if $L$ satisfies the above assumption and the above estimates, so does its formal adjoint

$$L^* u = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \sum_{i=1}^{d} \frac{\partial (b_i u)}{\partial x_i} + cu.$$

A sufficient and necessary condition for the well-posedness of (3.5) is that

$$\|w\|_{1,\Omega} \lesssim \sup_{\phi \in H_0^1(\Omega)} \frac{a(w,\phi)}{\|\phi\|_{1,\Omega}} \quad \forall\, w \in H_0^1(\Omega) \tag{3.7}$$

and

$$\|w\|_{1,\Omega} \lesssim \sup_{\phi \in H_0^1(\Omega)} \frac{a(\phi,w)}{\|\phi\|_{1,\Omega}} \quad \forall\, w \in H_0^1(\Omega). \tag{3.8}$$

We have (c.f. [12]) the following estimate for the regularity of the solution of (3.4) or (3.5)

$$\|u\|_{1+\alpha,\Omega} \lesssim \|f\|_{-1+\alpha,\Omega} \tag{3.9}$$

for some $\alpha \in (0,1]$ depending on $\Omega$ and the coefficients of $L$.

It is well-known (c.f. [39,40]) that if $h \ll 1$, then

$$\|w_h\|_{1,\Omega} \lesssim \sup_{\phi \in S_0^h(\Omega)} \frac{a(w_h,\phi)}{\|\phi\|_{1,\Omega}} \quad \forall\, w_h \in S_0^h(\Omega)$$

and

$$\|w_h\|_{1,\Omega} \lesssim \sup_{\phi \in S_0^h(\Omega)} \frac{a(\phi,w_h)}{\|\phi\|_{1,\Omega}} \quad \forall\, w_h \in S_0^h(\Omega).$$

Throughout this paper, we will assume that $h \ll 1$ holds so that the above two estimates hold. From the above two estimates, we can then define Galerkin-projections $P_h : H_0^1(\Omega) \mapsto S_0^h(\Omega)$ and $P_h^* : H_0^1(\Omega) \mapsto S_0^h(\Omega)$ by

$$a(u - P_h u, v) = 0 \text{ and } a(v, u - P_h^* u) = 0 \quad \forall\, v \in S_0^h(\Omega) \tag{3.10}$$

and apparently

$$\|P_h u\|_{1,\Omega} \lesssim \|u\|_{1,\Omega} \text{ and } \|P_h^* u\|_{1,\Omega} \lesssim \|u\|_{1,\Omega} \quad \forall\, u \in H_0^1(\Omega). \tag{3.11}$$

From (3.11), various a global priori error estimates can be obtained from the approximate properties of the finite element subspaces $S^h(\Omega)$ (c.f. [6,7,40]). Particularly, if $u \in H_0^1(\Omega)$, then

$$\|u - P_h u\|_{1,\Omega} = o(1) \text{ and } \|u - P_h^* u\|_{1,\Omega} = o(1) \quad \text{as } h \to 0. \tag{3.12}$$

Now we introduce the following quantity:

$$\rho(h) = \max(\rho_{_L}(h), \rho_{_{L^*}}(h)),$$

where

$$\rho_{_L}(h) = \sup_{\substack{f \in L^2(\Omega) \\ \|f\|_{0,\Omega}=1}} \inf_{v \in S_0^h(\Omega)} \|L^{-1} f - v\|_{1,\Omega},$$

$$\rho_{_{L^*}}(h) = \sup_{\substack{f \in L^2(\Omega) \\ \|f\|_{0,\Omega}=1}} \inf_{v \in S_0^h(\Omega)} \|(L^*)^{-1} f - v\|_{1,\Omega}.$$

It is derived from the Aubin-Nitsche duality argument that $\rho_{\mathcal{V}}$ defined by (2.2) satisfies

$$\rho_{\mathcal{V}} \lesssim \rho(h) \text{ for } \mathcal{V} = S_0^{h,1}(\Omega)$$

(c.f. [40]). The following results can be found in [40] (c.f. also [3,33,41]).

**Proposition 3.1.** *There hold*

$$\rho(h) \lesssim h^{\alpha}, \tag{3.13}$$

$$\|(I - P_h)L^{-1} f\|_{1,\Omega} \lesssim \rho(h)\|f\|_{0,\Omega} \quad \forall\ f \in L^2(\Omega), \tag{3.14}$$

$$\|u - P_h u\|_{0,\Omega} \lesssim \rho(h)\|u - P_h u\|_{1,\Omega} \quad \forall\ u \in H_0^1(\Omega). \tag{3.15}$$

### 3.2. Correction schemes

In this subsection, for illustration, we will provide some multi-level adaptive correction schemes to produce highly accurate finite element approximations by using the general approaches presented in Section 2.

In our discussion, we set $\mathcal{H} = H_0^1(\Omega)$, $\|\cdot\| = \|\cdot\|_{1,\Omega}$, and $|\cdot| = \|\cdot\|_{0,\Omega}$. For simplicity, we assume that $\Omega$ is convex. It is seen that $\alpha$ in (3.9) equals to 1. Let $u_h \in S_0^{h,1}(\Omega)$ be the finite element solution of (3.4) or (3.5), namely $u_h = P_h u$ or satisfying

$$a(u_h, v) = (f, v) \quad \forall\ v \in S_0^{h,1}(\Omega). \tag{3.16}$$

To apply the general subspace correction approach, we choose $\mathcal{V} = S_0^{h,1}(\Omega)$ and hence $\rho_{\mathcal{V}}$ defined by (2.2) satisfies $\rho_{\mathcal{V}} \lesssim h$. We may employ another finite element triangulation $\mathcal{T}^H(\Omega)$ with mesh size $H$ so that

$$\mathcal{T}^H(\Omega) \subset \mathcal{T}^h(\Omega)$$

and $H \gg h$. In following examples, we will choose $H = \mathcal{O}(h^{2/3})$ and $\mathcal{U}_j$ to be some $S_0^{H,r}(\Omega)(r = 2, 3, \cdots)$ and $Q_j$ to be the associated $L^2-$projection operators $(j = 1, 2, \cdots)$. As a result, the quantity $\eta$ in (2.16) satisfies $\eta^{-1} \lesssim H^{-1}$.

Below we make some remarks on some estimates that depend on the solution regularity.

- If $u \in H_0^1(\Omega) \cap H^3(\Omega)$, then we may choose $\mathcal{U}_1 = S_0^{H,2}(\Omega)$ and $Q_1 = Q_H^{(2)}$ to obtain

$$\|u - Q_H^{(2)} u_h\|_{1,\Omega} \lesssim h^{4/3}, \tag{3.17}$$

$$\|u - u_h - Q_H^{(2)} u_h + P_h Q_H^{(2)} u_h\|_{0,\Omega} \lesssim h^{7/3}. \tag{3.18}$$

It should be pointed out that (3.17) was first shown in [38].

- If $u \in H_0^1(\Omega) \cap H^4(\Omega)$, then we may choose $\mathcal{U}_j = S_0^{H,j+1}(\Omega)$ and $Q_j = Q_H^{(j+1)} (j = 1, 2)$ to obtain

$$\|u - Q_H^{(2)} u_h - Q_H^{(3)} u_h + Q_H^{(3)} P_h Q_H^{(2)} u_h\|_{1,\Omega} \lesssim h^{5/3} \tag{3.19}$$

$$\|u - R_{h,2} u\|_{0,\Omega} \lesssim h^{8/3}, \tag{3.20}$$

where

$$\begin{aligned} R_{h,2} = & u_h - P_h Q_H^{(2)} u_h - P_h Q_H^{(3)} u_h + P_h Q_H^{(3)} P_h Q_H^{(2)} u_h \\ & + Q_H^{(2)} u_h + Q_H^{(3)} u_h - Q_H^{(3)} P_h Q_H^{(2)} u_h. \end{aligned}$$

Note that if the triangulation $\mathcal{T}^h(\Omega)$ is uniform, then we may choose $H = 2h$. For illustration, we consider the case of two dimensions. Let $T^h(\Omega)$ be derived from $T^{2h}(\Omega)$ as follows: for each element $e \in T^{2h}(\Omega)$, connect the edge midpoints of $e$ and obtain 4 subelements. Denote $\Pi_h$ to be the standard quadratic interpolation operator associated with $T^{2h}(\Omega)$, namely,

$$\Pi_h u \mid_e \in P_2 \quad \forall\, e \in T^{2h}(\Omega)$$

and

$$\Pi_h u = u \quad \text{on } \partial^2 T^h(\Omega),$$

where $P_2 = \text{span}\{x_1^i x_2^j : \ 0 \le i + j \le 2\}$ and $\partial^2 T^h(\Omega)$ is the set of nodal points of $T^h(\Omega)$.

Although (2.14) and (2.16) are not true for $\mathcal{U}_1 = S_0^{2h}(\Omega)$ and $Q_1 = \Pi_{2h}$, highly accurate finite element approximations can also be constructed by using the Boolean sum technique. (see, e.g., [21, 22, 24, 30])

$$\|u - \Pi_{2h} u_h\|_{1,\Omega} \lesssim h^2, \tag{3.21}$$

$$\|u - u_h - \Pi_{2h} u_h + P_h \Pi_{2h} u_h\|_{0,\Omega} \lesssim h^3, \tag{3.22}$$

provided that $u \in H_0^1(\Omega) \cap H^3(\Omega)$. We refer to [23] for a local three-level correction scheme.

## 4. Concluding Remarks

In this paper we have proposed and analyzed a multi-level adaptive correction approach to finite dimensional approximations. We have successfully applied the approach to solve an elliptic boundary value problem based on finite element discretizations. Although we are able to utilize the two-level correction to get highly accurate approximations for elliptic eigenvalue problems (see, e.g., [14, 19, 25, 27, 34, 41]), it is still open if some multi-level (more than two levels) correction scheme can be designed to produce approximations with higher accuracy for elliptic eigenvalue problems. Note that the Boolean sum can be defined on any Banach algebra, we may expect similar results hold when a Hilbert space $(\mathcal{H}, \|\cdot\|)$ is replaced by some continuous function spaces. Anyway, we believe that the approach presented in this paper is a general and powerful technique that can be used for a variety of equations with different types of discretization methods.

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