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EQUIVALENCE OF SEMI-LAGRANGIAN AND LAGRANGE–GALERKIN SCHEMES UNDER CONSTANT ADVECTION SPEED*

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Abstract

We compare in this paper two major implementations of large time-step schemes for advection equations, i.e., Semi-Lagrangian and Lagrange–Galerkin techniques. We show that SL schemes are equivalent to exact LG schemes via a suitable definition of the basis functions. In this paper, this equivalence will be proved assuming some simplifying hypoteses, mainly constant advection speed, uniform space grid, symmetry and translation invariance of the cardinal basis functions for interpolation. As a byproduct of this equivalence, we obtain a simpler proof of stability for SL schemes in the constant-coefficient case.

Mathematics subject classification: 65N12, 65M10, 49L25.
Key words: Semi-Lagrangian schemes, Lagrange–Galerkin schemes, Stability.

1. Introduction

High-order, large time-step schemes for first order equation have gained an increasing popularity in the last two decades. Schemes based on characteristics for hyperbolic PDEs have been proposed by Courant–Isaacson–Rees in [3] and have prompted the development of a number of specific techniques, including Semi-Lagrangian (SL) methods, which have first appeared in the framework of Numerical Weather Prediction problems (see the pioneering paper [18] and the review [15]). SL methods have another well estabilished field of application in plasma physics, where their use has been suggested in [2] and widely studied since (see, e.g., [1,8,16]), and their popularity is also growing in other fields such as general Computational Fluid Dynamics (CFD) problems, Hamilton–Jacobi equations and level-set methods [7,17], conservation laws [13]. On the other hand, Lagrange–Galerkin (LG) methods have been proposed independently in [5,12], and currently their main field of application is CFD (including Numerical Weather Prediction) in a finite element setting.

The main ideas of this work will be sketched focusing on the model problem of the constant-coefficient, evolutive advection equation

$$\begin{cases} v_t(x,t) + a\cdot\nabla v(x,t) = 0, & \text{in } \mathbb{R}^N\times\mathbb{R} \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

in which the transport velocity is given by the constant vector $a = (a_1 \cdots a_N)^T \in \mathbb{R}^N$.

The construction of large time-step schemes for (1.1) stems from the application of the method of characteristics (see, e.g., [5,6,12]), which uses the property of the solution of (1.1)

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to be constant along the characteristics lines $(x - a\tau, t - \tau)$ of the $x - t$ space. This means that the following time–discrete representation formula

$$v(x,t) = v(x - a\Delta t, t - \Delta t) \tag{1.2}$$

holds for the solution v.

In large time-step schemes the discretization is performed on the representation formula (1.2) rather than on the Eq. (1.1). The discretization of (1.2) is obtained by introducing a numerical reconstruction to approximate the value $v(x - a\Delta t, t - \Delta t)$, since in general the foot of the characteristic $x_j - a\Delta t$ does not coincide with any grid point (we note that in the more general setting of variable coefficient equations, characteristics are no longer straight lines, and the position of the foot of characteristics needs itself to be approximated).

As we will recall in the next section, the space reconstruction is precisely what discriminates between SL and LG schemes, and is also the crucial point in proving stability of the scheme, which is not a trivial result when using high-order techniques. In this respect, LG schemes allow for a cleaner and more general analysis, whereas high-order SL schemes have only be proved to be stable in the Von Neumann sense. Therefore, the equivalence of the two formulations (whenever provable) is also a way of obtaining a more general stability result for SL schemes.

In this paper, we will prove such an equivalence for the simplified case of (1.1), although in fact L^2 stability of SL schemes is a known fact for such a model – so, while expecting that this technique may also work in greater generality, this paper only presents a simpler stability proof, along with a deeper insight in the relationship between the two classes of schemes.

The paper is organized as follows. Section 2 reviews the construction and basic convergence theory of SL and LG schemes for (1.1), and section 3 sets up a framework in which the equivalence of SL and LG schemes can be proved. In section 4, we give some practical examples of reconstruction bases falling in this theory.

2. A Comparison of Semi-Lagrangian and Lagrange–Galerkin Schemes

This section presents the main ideas in the construction of SL and LG schemes. We point out that this is presentation is focused on the *basic* form of the schemes. Recent years have witnessed a number of developments, especially aimed at improving the performances of the schemes in presence of discontinuous solutions. Among such advances, we mention the use of nonlinear (non-oscillatory or monotone) reconstructions, the introduction of Discontinuous Galerkin type techniques, the study of *a posteriori* error estimates and adaptive grids. It is worth to emphasize that, at least at a technical level, the ideas of this paper do not allow for a straightforward adaptation to such complex situations.

In the SL scheme, (1.2) is discretized as

$$v_i^{n+1} = I[V^n](x_i - a\Delta t), \tag{2.1}$$

where v_i^{n+1} is an approximation of $v(x_i, t^{n+1})$ and the interpolation $I[V^n](x)$ is computed as

$$I[V^n](x) = \sum_j v_j^n \psi_j(x) \tag{2.2}$$

with $\psi_j(x_i) = \delta_{ij}$. In particular, this form holds for Lagrange interpolation, as we will show in the examples. Using (2.2) in (2.1), we obtain at last

$$v_i^{n+1} = \sum_j v_j^n \psi_j(x_i - a\Delta t), \tag{2.3}$$

or in matrix form,

$$V^{n+1} = \Psi V^n, \tag{2.4}$$

where the matrix Ψ has elements ψ_{ij} defined by

$$\psi_{ij} = \psi_j(x_i - a\Delta t). \tag{2.5}$$

Typically, in a uniform structured space grid the basis functions are affine transformations of the same reference function, and multiple space dimensions are treated by separation of variables. Such a structure of the basis functions will be discussed in detail in the next section.

However, despite the straightforwardness of the construction (and the numerical evidence of unconditional stability), analysis of high-order SL schemes is quite technical and stability has actually been proved only in the Von Neumann setting (see [1,6]). In particular, no theoretical result exists about stability of SL schemes under variable advection speed.

In the LG scheme, once written the approximate solution at time t^k as $\sum_j v_j^k \phi_j(x)$, (1.2) is discretized instead by integrating the product of both sides of (1.2) with a basis of test functions (in particular, the basis $\{\phi_i\}$) so that the equality

$$\int_{\mathbb{R}^N} \sum_j v_j^{n+1} \phi_j(\xi)\phi_i(\xi)d\xi = \int_{\mathbb{R}^N} \sum_j v_j^n \phi_j(\xi - a\Delta t)\phi_i(\xi)d\xi \tag{2.6}$$

must hold for all i. More explicitly, condition (2.6) is actually enforced as

$$\sum_j v_j^{n+1} \int_{\mathbb{R}^N} \phi_j(\xi)\phi_i(\xi)d\xi = \sum_j v_j^n \int_{\mathbb{R}^N} \phi_j(\xi - a\Delta t)\phi_i(\xi)d\xi, \tag{2.7}$$

which can be written in matrix form as

$$MV^{n+1} = \Phi V^n, \tag{2.8}$$

where M is the mass matrix and the matrix Φ has elements ϕ_{ij} defined by

$$\phi_{ij} = \int_{\mathbb{R}^N} \phi_j(\xi - a\Delta t)\phi_i(\xi)d\xi. \tag{2.9}$$

Stability analysis is considerably easier in the case of LG schemes, since reconstruction is actually performed by means of an L^2 projection. However, it should be noted that in the variable coefficient case, the evaluation of the integrals (2.9) cannot in general be performed exactly, and a rigorous analysis becomes more complex (see [10]).

3. Interpreting SL as LG Schemes

In this section we look for conditions under which the SL scheme is equivalent to the LG scheme for the model problem (1.1). We restrict therefore to the easier case of structured grids, so that the basis functions for both the SL and the LG scheme can be obtained by affine transformations of a reference function, corresponding to the case $\Delta x = 1$ and $x_j = 0$, multiple space dimensions being treated by tensor product of a one-dimensional basis. We will use $j = (j_1, \ldots, j_N)^T$ as a multiindex and set $x_j = (j_1\Delta x_1, \ldots, j_N \Delta x_N)^T$. Moreover, in the sequel x and ξ will be used as variables in the physical space, y and η as variables in the reference space.

3.1. Preliminary material and basic assumptions

- The space grid is supposed to be infinite, orthogonal and uniform, that is, for $j \in \mathbb{Z}^N$:

$$x_j = (j_1\Delta x_1, \ldots, j_N \Delta x_N)^T, \tag{3.1}$$

- The basis functions ψ_j for the SL case are defined by:

$$\psi_j(\xi) = \psi\left(\frac{\xi_1}{\Delta x_1} - j_1\right) \cdots \psi\left(\frac{\xi_N}{\Delta x_N} - j_N\right) = \prod_{k=1}^{N} \psi\left(\frac{\xi_k}{\Delta x_k} - j_k\right), \tag{3.2}$$

where we have denoted by ψ the (one-dimensional) reference function and by Δx_k the space step along the k–th direction. The LG basis will be assumed to have (unless for a normalizing factor whose role will be clear in the sequel) a structure which parallels (3.2), that is

$$\begin{aligned} \phi_j(\xi) &= \frac{1}{\sqrt{\Delta x_1 \cdots \Delta x_N}}\ \phi\left(\frac{\xi_1}{\Delta x_1} - j_1\right) \cdots \phi\left(\frac{\xi_N}{\Delta x_N} - j_N\right) \\ &= \prod_{k=1}^{N} \frac{1}{\sqrt{\Delta x_k}}\ \phi\left(\frac{\xi_k}{\Delta x_k} - j_k\right). \end{aligned} \tag{3.3}$$

- The function ψ is assumed to satisfy the conditions:

$$\psi \in W^{1,\infty}(\mathbb{R}) \cap L^1(\mathbb{R}), \tag{3.4}$$

$$\psi(y) = \psi(-y), \tag{3.5}$$

$$\psi(i) = \begin{cases} 1 & \text{if} \quad i = 0, \\ 0 & \text{if} \quad i \in \mathbb{Z},\ i \neq 0. \end{cases} \tag{3.6}$$

Let us also give a definition which will be useful to characterize the solution.

Definition 3.1. *A complex-valued function $g : \mathbb{R}^d \to \mathbb{R}$ is said to be positive semi-definite if*

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k g(x_k - x_j)\bar{\alpha}_j \geq 0 \tag{3.7}$$

for any $x_k \in \mathbb{R}^d$, $\alpha_k \in \mathbb{C}$ $(k = 1, \ldots, n)$ and for all $n \in \mathbb{N}$.

As we said before, the arguments used for the proof (and the basic assumptions (3.1)–(3.6)) rule out some situation of practical interest. In particular, this theory cannot handle the case of unstructured grids, and more in general, reconstructions performed by finite elements, which violate the assumptions of symmetry and translation invariance. Also, the theory cannot be applied to nonlinear reconstructions, since linearity of the numerical scheme is widely exploited in the proof. On the other hand, the technique of proof seems to be also suitable for the case of variable coefficient equations (this latter extension will be the object of a forthcoming study).

3.2. The main result

Our purpose is to prove that it is possible to find a basis for the LG scheme such that (2.4) and (2.8) coincide, that is, left multiplying both sides of (2.4) by M and comparing with (2.8), such that

$$\Phi = M\Psi. \tag{3.8}$$

Although this formulation could lend itself to a more general theory, in the present setting it will suffice to satisfy the more restrictive conditions

$$\begin{cases} M = I, \\ \Phi = \Psi. \end{cases} \tag{3.9}$$

Since we have assumed the structure (3.3), the problem essentially comes down to define a suitable reference function ϕ. This is contained in the following theorem.

Theorem 3.1. *Let the basic assumptions* (3.1)–(3.6) *hold. Then, there exists a real function* ϕ *for which* (2.7) *is equivalent to* (2.3) *(in the sense of* (3.9)*) if and only if either of the conditions*
i) The function $\psi(y)$ *has a real nonnegative Fourier transform:* $\hat{\psi}(\omega) \geq 0$*,*
ii) The function $\psi(y)$ *is positive semi-definite is satisfied.*

Proof. First, we note that conditions *i)* and *ii)* are equivalent due to a theorem of Bochner (see [14]). Comparing (2.3) and (2.7), we obtain the set of conditions to be satisfied:

$$\int_{\mathbb{R}^N} \phi_j(\xi)\phi_i(\xi)d\xi = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{3.10}$$

$$\int_{\mathbb{R}^N} \phi_j(\xi - a\Delta t)\phi_i(\xi)d\xi = \psi_j(x_i - a\Delta t). \tag{3.11}$$

The proof is split in two steps. In the first the problem is reduced to a single space dimension, whereas in the second we prove the claim.

Step 1. Note that, by assumption (3.6), condition (3.10) is in fact included in (3.11), as it can be seen setting $\Delta t = 0$. Using the definitions (3.2), (3.3) for ψ_j and ϕ_j, (3.11) can be rewritten as

$$\begin{aligned} &\int_{\mathbb{R}^N} \prod_{k=1}^{N} \frac{1}{\sqrt{\Delta x_k}} \; \phi\left(\frac{\xi_k - a_k\Delta t}{\Delta x_k} - j_k\right) \prod_{k=1}^{N} \frac{1}{\sqrt{\Delta x_k}} \; \phi\left(\frac{\xi_k}{\Delta x_k} - i_k\right) d\xi \\ &= \prod_{k=1}^{N} \psi\left(\frac{x_{i,k} - a_k\Delta t}{\Delta x_k} - j_k\right), \end{aligned} \tag{3.12}$$

and working by separation of variables,

$$\prod_{k=1}^{N} \frac{1}{\Delta x_k} \int_{\mathbb{R}} \phi\left(\frac{\xi_k - a_k\Delta t}{\Delta x_k} - j_k\right) \phi\left(\frac{\xi_k}{\Delta x_k} - i_k\right) d\xi_k = \prod_{k=1}^{N} \psi\left(\frac{x_{i,k} - a_k\Delta t}{\Delta x_k} - j_k\right). \tag{3.13}$$

Therefore, matching corresponding terms of the products above, we obtain that (3.11) is satisfied if, for $k = 1, \ldots, N$:

$$\frac{1}{\Delta x_k} \int_{\mathbb{R}} \phi\left(\frac{\xi_k - a_k\Delta t}{\Delta x_k} - j_k\right) \phi\left(\frac{\xi_k}{\Delta x_k} - i_k\right) d\xi_k = \psi\left(\frac{x_{i,k} - a_k\Delta t}{\Delta x_k} - j_k\right) \tag{3.14}$$

that is, after setting $\eta = \xi_k/\Delta x_k - i_k$:

$$\int_{\mathbb{R}} \phi\left(\eta - \frac{a_k\Delta t}{\Delta x_k} + i_k - j_k\right) \phi(\eta) d\eta = \psi\left(-\frac{a_k\Delta t}{\Delta x_k} + i_k - j_k\right). \tag{3.15}$$

This ultimately amounts to find a function ϕ such that

$$\int_{\mathbb{R}} \phi(\eta + y)\phi(\eta) d\eta = \psi(y) \tag{3.16}$$

Step 2. The left-hand side of (3.16) is the autocorrelation integral (see [11]) of the unknown function ϕ. Working in the Fourier domain and transforming both sides of (3.16) we have:

$$|\hat{\phi}(\omega)|^2 = \hat{\psi}(\omega). \tag{3.17}$$

Now, since ψ is a real and even function of y, its Fourier transform $\hat{\psi}$ is also a real and even function of ω (see [11]). Moreover, the assumption $\psi \in L^1(\mathbb{R})$ implies that $\hat{\psi}(\omega)$ is bounded, whereas $\psi \in W^{1,\infty}(\mathbb{R})$ implies that it decays like $\mathcal{O}(\omega^{-2})$ for $\omega \to \pm\infty$. Therefore, $\hat{\psi}$ being also nonnegative by assumption *i*), its square root $\hat{\psi}(\omega)^{1/2}$ is real, even and nonnegative. In addition, $\hat{\psi}^{1/2}$ is also bounded and decays like $\mathcal{O}(\omega^{-1})$, and this implies that $\hat{\psi}^{1/2} \in L^2(\mathbb{R})$.

Finally, looking at the inverse Fourier transform $\mathcal{F}^{-1}$ as an operator mapping $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$, we obtain that the solution ϕ defined by

$$\phi(y) = \mathcal{F}^{-1}\left\{\hat{\psi}(\omega)^{1/2}\right\} \tag{3.18}$$

is a well-defined even real function of $L^2(\mathbb{R})$ solving (3.16). □

Remark. We explicitly note that both symmetry condition (3.5) and positive definiteness (which actually implies (3.5) for real functions) are required to ensure that ψ may be regarded as an autocorrelation. In fact, any autocorrelation integral of a real function must be even and have a positive Fourier transform. On the other hand, once these conditions are satisfied, (3.18) needs not have a unique solution. Equating $\hat{\psi}$ and $|\hat{\phi}|^2$, we neglect any information about the phase diagram of ϕ, and this in turn makes it possible to have multiple solutions, as it will be shown in the examples.

4. Some Practical Examples

We apply in this section the general theory to some situation of practical interest, in particular Lagrange interpolations of odd order, interpolatory wavelets and (although with some caution) spline interpolation. Note that the reference LG basis function obtained in the various cases is not even supposed to be (and in fact, unless for splines, is not) continuous. This is not a major problem here, since the construction of a LG scheme for (1.1) just requires a basis of functions in $L^2(\mathbb{R})$.

4.1. $\mathbb{P}_1$ interpolation

We first show the simple case of $\mathbb{P}_1$ interpolation, for which explicit computation can be performed. This will allow to illustrate some remarkable point, in particular the lack of uniqueness for the solution. We recall that in this case all the base functions for interpolation in the SL scheme are affine transformations of the function

$$\psi^{[1]}(y) = \begin{cases} 1+y & \text{if} \quad -1 \le y \le 0, \\ 1-y & \text{if} \quad 0 \le y \le 1, \\ 0 & \text{elsewhere}, \end{cases} \tag{4.1}$$

(here and in the sequel, we will use the notation $\psi^{[k]}$ or $\phi^{[k]}$ to distinguish among different interpolation orders). The Fourier transform of (4.1) is in turn

$$\hat{\psi}^{[1]}(\omega) = \frac{2-2\cos\omega}{\omega^2} = \frac{\sin\left(\frac{\omega}{2}\right)^2}{\left(\frac{\omega}{2}\right)^2}. \tag{4.2}$$

Now, taking the square root of this transform we get

$$\hat{\phi}^{[1]}(\omega) = \left|\sin\frac{\omega}{2}\right| / \left|\frac{\omega}{2}\right| \tag{4.3}$$

and accordingly,

$$\phi^{[1]}(y) = \mathcal{F}^{-1}\left\{\left|\sin\frac{\omega}{2}\right| / \left|\frac{\omega}{2}\right|\right\}. \tag{4.4}$$

However, a different (and possibly more natural) solution can be picked up by noting that $\hat{\psi}^{[1]}(\omega)$ is also the squared magnitude of

$$\hat{\phi}^{[1]}(\omega) = \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} \tag{4.5}$$

whose inverse Fourier transform is explicitly computable as

$$\phi^{[1]}(y) = \begin{cases} 1 & \text{if} \quad -\frac{1}{2} \le y \le \frac{1}{2}, \\ 0 & \text{elsewhere}, \end{cases} \tag{4.6}$$

that is, the base function for piecewise constant ($\mathbb{P}_0$) reconstruction. This example shows more precisely the effect of losing uniqueness by neglecting the phase information – in fact, we can obtain an infinity of solutions to (3.16) which differ in the phase term. This is not a major problem, however, since we only need existence of a solution in this setting. Conventionally, we will refer in the sequel to the particular solution of (3.16) for which the phase is identically zero. This corresponds to the solution obtained by (3.18).

4.2. Lagrange interpolation of higher (odd) order

This kind of interpolation is performed using an equal number of nodes on both sides of the point x, this stencil of nodes depending on the interval where x is placed. Although at a first glance it could seem that this interpolation does not use the same basis on each interval, yet being a linear operator with respect to the values to be interpolated, such a basis can still be suitably defined. In fact, the interpolation operator $I[\cdot]$ is a linear map from the space l^∞ of bounded sequences into the space of continuous functions on $\mathbb{R}$:

$$I[\cdot] : l^\infty \to C^0(\mathbb{R}) \tag{4.7}$$

so that, by elementary linear algebra arguments, any basis function ψ_j is nothing but the image of the base element e_j (that is, the interpolation of a sequence such that $v_j = 1$, $v_i = 0$ for $i \neq j$).

For example, cubic reconstruction performs an interpolation using the four nearest nodes (two on the left and two on the right of the interval containing x). The value at a node x_j affects the reconstruction only in the interval (x_{j-2}, x_{j+2}). More precisely, using the notation $L_j^{k,k+3}$ to denote the Lagrange basis function associated to the node x_j and built on the nodes ranging from x_k to x_{k+3}, we have

$$\psi_j^{[3]}(x) = \begin{cases} L_j^{j-1,j+2}(x) & \text{if} \quad x_j \le x \le x_{j+1}, \\ L_j^{j,j+3}(x) & \text{if} \quad x_{j+1} \le x \le x_{j+2}, \\ L_j^{j-2,j+1}(x) & \text{if} \quad x_{j-1} \le x \le x_j, \\ L_j^{j-3,j}(x) & \text{if} \quad x_{j-2} \le x \le x_{j-1}, \\ 0 & \text{elsewhere}, \end{cases} \tag{4.8}$$

(note that all the local definitions are weighting functions associated to x_j, but that they are built on different sets of nodes depending on the interval considered). Referring again to the case in which $x_j = 0$ and $\Delta x = 1$, the explicit form of the reference base function $\psi^{[3]}$ is

$$\psi^{[3]}(y) = \begin{cases} \frac{1}{2}(y+1)(y-1)(y-2) & \text{if} \quad 0 \le y \le 1, \\ -\frac{1}{6}(y-1)(y-2)(y-3) & \text{if} \quad 1 \le y \le 2, \\ 0 & \text{if} \quad y > 2, \end{cases} \tag{4.9}$$

and extended by symmetry for $y < 0$. The Fourier transform of this function has the form

$$\hat{\psi}^{[3]}(\omega) = \frac{8(6+\omega^2)\sin\left(\frac{\omega}{2}\right)^4}{3\omega^4}. \tag{4.10}$$

The transform (4.10) (along with all the transforms for the higher order cases) has been obtained by symbolic integration with Mathematica. The positivity of $\hat{\psi}^{[3]}(\omega)$ immediately follows from its structure.

More in general, we can give the form of the basis functions for an arbitrary order of interpolation. For n odd, the form of $\psi^{[n]}$ is

$$\psi^{[n]}(y) = \begin{cases} \displaystyle\prod_{k\neq 0, k=-[n/2]}^{[n/2]+1} \frac{y-k}{-k} & \text{if } 0 \leq y \leq 1 \\ \vdots & \\ \displaystyle\prod_{k=1}^{n} \frac{y-k}{-k} & \text{if } [n/2] \leq y \leq [n/2]+1 \\ 0 & \text{if } y > [n/2]+1 \end{cases} \tag{4.11}$$

and extended by symmetry for $y < 0$. The study of the Fourier transforms gives, for $n = 5, 7, 9$:

$$\hat{\psi}^{[5]}(\omega) = \frac{16(60 + 15\omega^2 + 2\omega^4)\sin\left(\frac{\omega}{2}\right)^6}{15\omega^6}, \tag{4.12}$$

$$\hat{\psi}^{[7]}(\omega) = \frac{32(840 + 280\omega^2 + 49\omega^4 + 6\omega^6)\sin\left(\frac{\omega}{2}\right)^8}{105\omega^8}, \tag{4.13}$$

$$\hat{\psi}^{[9]}(\omega) = \frac{64(15120 + 6300\omega^2 + 1365\omega^4 + 205\omega^6 + 24\omega^8)\sin\left(\frac{\omega}{2}\right)^{10}}{945\omega^{10}}. \tag{4.14}$$

Although we have no general result stating the positivity of the transforms $\hat{\psi}^{[n]}(\omega)$, for all odd numbers $n \leq 13$ they have the structure

$$\hat{\psi}^{[n]}(\omega) = p(\omega^2)\,\frac{\sin\left(\frac{\omega}{2}\right)^{n+1}}{\left(\frac{\omega}{2}\right)^{n+1}} \tag{4.15}$$

with $p(\omega^2)$ a polynomial of degree $[n/2]$ with positive coefficients. A natural conjecture would be that this structure holds for any odd value of n, however rather than proving such a property, we simply note here that it holds for any order of practical interest.

Lastly, we show in Figure 4.2 the reference basis functions for interpolation, $\psi^{[n]}(y)$, and their counterparts for LG scheme, the solutions $\phi^{[n]}(y) = \mathcal{F}^{-1}\{\hat{\psi}^{[n]}(\omega)^{1/2}\}$ for $n = 1, 3, 5$. The computations have been carried out by FFT.

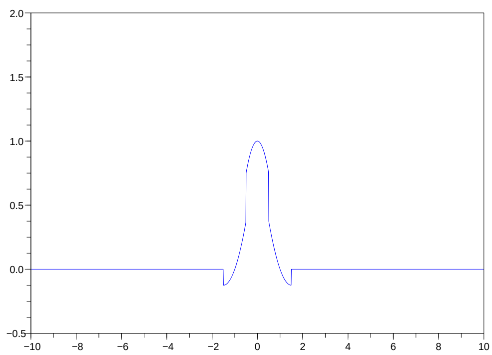

Fig. 4.1. The basis function for symmetric quadratic interpolation

Remark. Note that a similar construction can also be performed for an even degree of interpolation, but this requires to use an odd number of nodes for the reconstruction and results in disregarding some of the basic assumptions. In fact, a first choice would be to use an asymmetric stencil of points, and this would violate (3.5). A second choice (see [1]) preserves symmetry by defining around a generic node x_j a "cell" $[x_j - \Delta x/2, x_j - \Delta x/2)$, in which the reconstruction is performed using the values on a symmetric set of nodes around x_j (for example, a quadratic

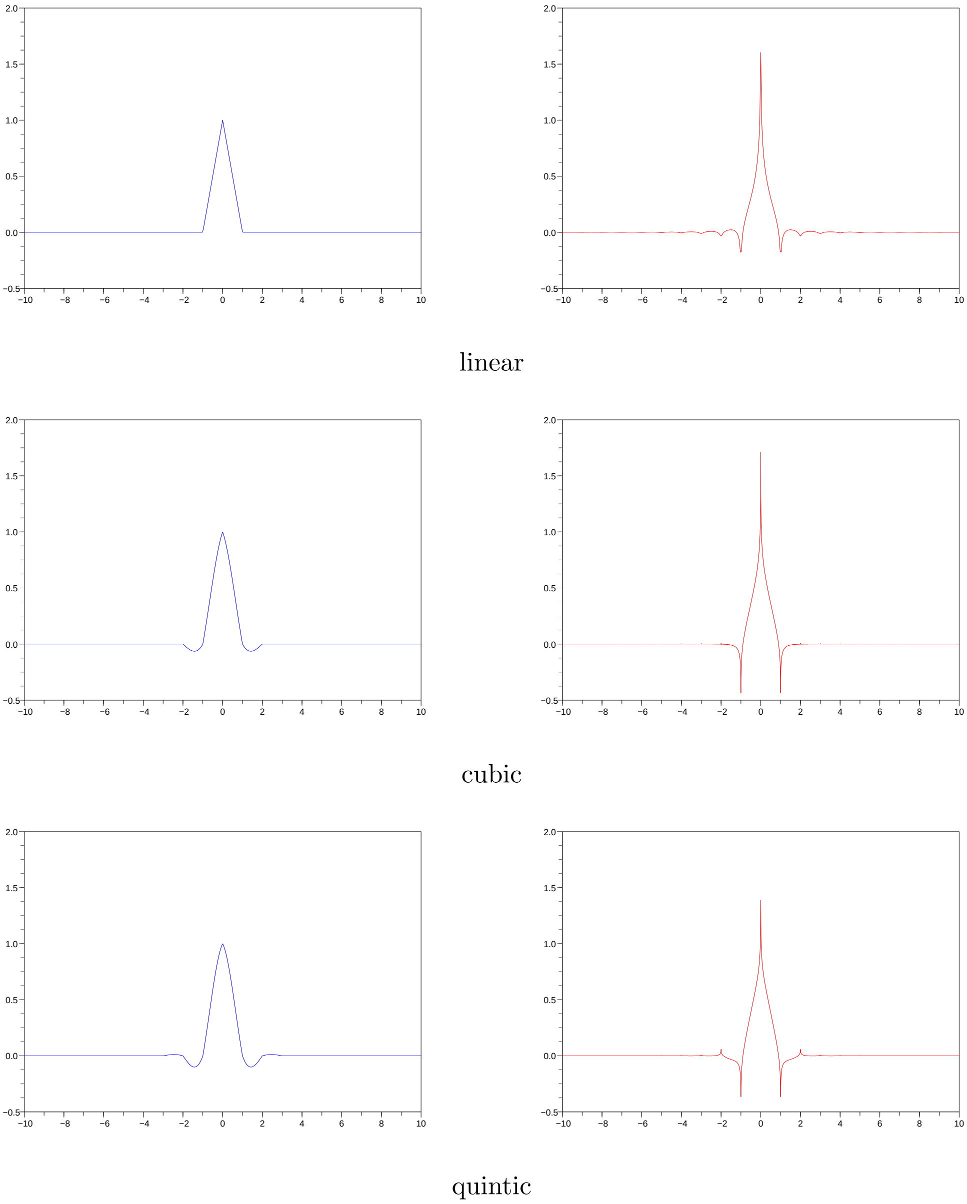

Fig. 4.2. The basis functions for linear, cubic, quintic interpolation (left) and the corresponding zero-phase LG basis (right)

reconstruction would use the nodes x_{j-1}, x_j and x_{j+1}). But, when applying this reconstruction to the sequence e_j in order to construct ψ_j, it turns out that this basis function is discontinuous at the interface between two cells (see the quadratic case in Figure 4.1), and by a theorem of Riesz (see [14]) such a function cannot be positive definite.

4.3. Cardinal splines

A second framework in which this theory seems to be applicable is spline interpolation. Restricting to Hermite cubic splines with continuous second derivative, the cardinal basis may be obtained as before by interpolating the sequence e_0 with space step $\Delta x = 1$ (in order to mimic the behaviour of such cardinal functions on the whole real axis, periodic conditions have been imposed at the boundary of a sufficiently large interval). The author is unaware of results stating that this basis function is positive definite; nevertheless, the numerical solution by Fast Fourier Transform apparently shows that this is the case, and a corresponding LG basis could be suitably defined. The first row of figure 4.3 shows both the reference basis function for cubic spline interpolation, and the corresponding LG basis. Note that the reference LG basis function is smoother then in the Lagrange case. This is implied by the increased smoothness of the SL basis function, which causes a faster decay of its Fourier transform.

4.4. Interpolatory wavelets

Interpolatory wavelets are another situation which can be treated within this theory, since they are usually defined to be positive definite functions. The simplest case is the Shannon (or sinc) wavelet which is defined, in the reference case, by

$$\psi(y) = \frac{\sin(\pi y)}{\pi y} \tag{4.16}$$

whose Fourier transform is given by

$$\hat{\psi}(\omega) = \begin{cases} 1 & \text{if} \quad -\pi \le \omega \le \pi \\ 0 & \text{elsewhere.} \end{cases} \tag{4.17}$$

Since $\hat{\psi}(\omega)^{1/2} = \hat{\psi}(\omega)$, we also have $\phi(y) = \psi(y)$ which implies that the basis for interpolation in the SL scheme coincides with the equivalent basis of LG scheme (note that, strictly speaking, $\psi \notin L^1(\mathbb{R})$, but in this special case $\hat{\psi}$ is bounded and therefore the main theorem applies).

Another example of positive definite interpolatory wavelet has been proposed by Deslauriers and Dubuc in [4], to which the reader is referred for its detailed definition. Second and third row of figure 4.3 show the basis functions for interpolation, along with the corresponding LG basis functions, for both the sinc and the (cubic) Deslauriers–Dubuc wavelets. Note that in the latter case, the numerical computation shows some instability due to the occurrence of small negative values in the Fast Fourier Transform.

We finally note that, as for the case of symmetric Lagrange interpolation of §4.2, L^2 stability of the SL scheme with wavelet interpolation has already been proved with Von Neumann analysis arguments, with similar conclusions. The result is contained in an unpublished work (see [9]).

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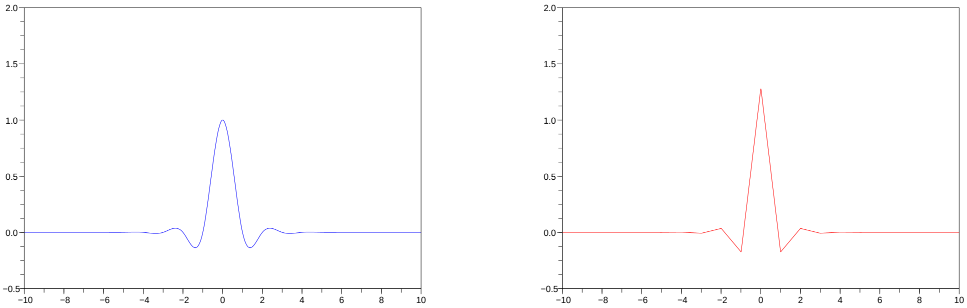
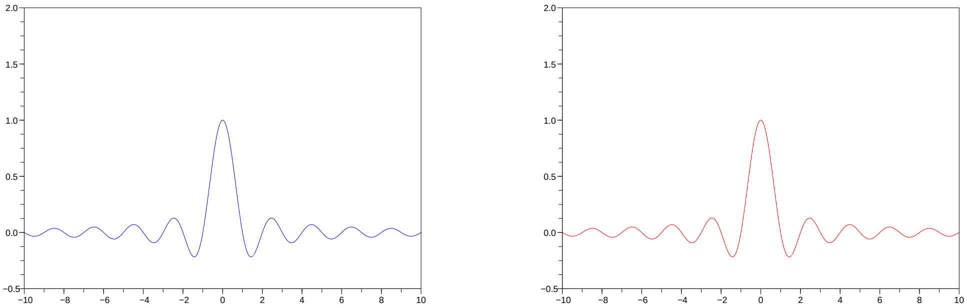
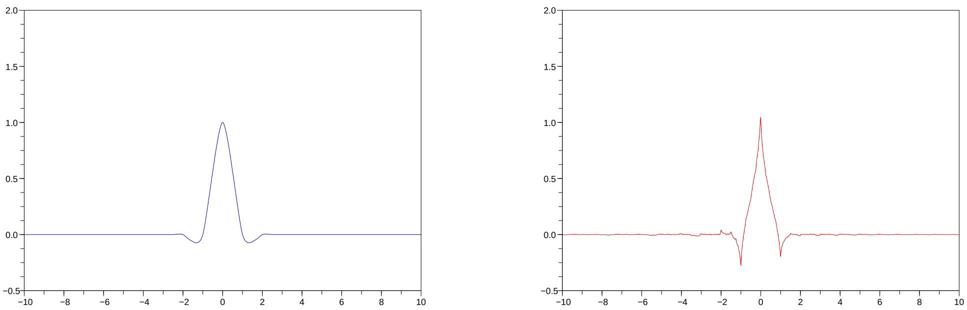


Cubic cardinal spline



Shannon wavelet



Cubic Deslauriers–Dubuc wavelet

Fig. 4.3. Cubic cardinal splines, interpolatory wavelets (left) and the corresponding zero-phase LG basis (right)

References

[1] N. Besse, M. Mehrenberger, Convergence of classes of high-order semi-Lagrangian schemes for the Vlasov–Poisson system, *Math. Comput.*, **77** (2008), 93-123.

[2] C.Z. Cheng and G. Knorr, The integration of the Vlasov equation in configuration space, *J. Comput. Phys.*, **22** (1976), 330-351.

[3] R. Courant, E. Isaacson, M. Rees, On the solution of nonlinear hyperbolic differential equations

by finite differences, *Commun. Pur. Appl. Math.*, **5** (1952), 243-255.

[4] G. Deslauriers, S. Dubuc, Symmetric iterative interpolation processes, *Constr. Approx.*, **5** (1989), 49-68.

[5] J. Douglas, T.F. Russell, Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures, *SIAM J. Numer. Anal.*, **19** (1982), 871-885.

[6] M. Falcone, R. Ferretti, Convergence analysis for a class of semi–lagrangian advection schemes, *SIAM J. Numer. Anal.,* **35** (1998), 909-940.

[7] M. Falcone, R. Ferretti, Semi–Lagrangian schemes for Hamilton–Jacobi equations, discrete representation formulae and Godunov methods, *J. Comput. Phys.*, **175** (2002), 559-575.

[8] A. Ghizzo, P. Bertrand, M.M. Shoucri, T. W. Johnston, E. Fijalkow and M.R. Feix, A Vlasov code for the numerical simulation of stimulated Raman scattering, *J. Comput. Phys.*, **90** (1990), 431-457.

[9] M. Mehrenberger, High order semi-Lagrangian schemes, "GdR CHANT Workshop on numerical methods for kinetic, hyperbolic and Hamilton-Jacobi equations", 23-25 novembre 2005, Strasbourg (available online at: `http://www-irma.u-strasbg.fr/∼mehrenbe/strasbg05.pdf`)

[10] K.W. Morton, A. Priestley and E. Süli, Stability of the Lagrange–Galerkin method with non-exact integration, M^2AN, **22** (1988), 625-653.

[11] A. Papoulis, Signal Analysis, MacGraw–Hill, New York, 1977.

[12] O. Pironneau, On the transport–diffusion algorithm and its application to the Navier–Stokes equations, *Numer. Math.*, **38** (1982), 309-332.

[13] J.-M. Qiu, C.-W. Shu, Convergence of Godunov-type schemes for scalar conservation laws under large time steps, *SIAM J. Numer. Anal.*, **46** (2008), 2211-2237.

[14] M. Schlather, Introduction to positive definite functions and to unconditional simulation of random fields, Technical Report ST-99-10, Dept. Math. Stats., Lancaster Univ., Lancaster, UK (1999) (available online at: `http://www.stochastik.math.uni-goettingen.de/∼schlather/`).

[15] A.N. Staniforth, J. Côtè, Semi–Lagrangian integration schemes for atmospheric models – A review, *Mon. Weather. Rev.*, **119** (1991), 2206-2223.

[16] E. Sonnendrücker, J. Roche, P. Bertrand and A. Ghizzo, The semi-Lagrangian method for the numerical resolution of the Vlasov equation, *J. Comput. Phys.*, **149** (1999), 201-220.

[17] J. Strain, Semi-Lagrangian methods for level set equations, *J. Comput. Phys.*, **151** (1999), 498–533.

[18] A. Wiin-Nielsen, On the application of trajectory methods in numerical forecasting, *Tellus*, **11** (1959), 180-196.