

CONVERGENCE RATES FOR DIFFERENCE SCHEMES FOR POLYHEDRAL NONLINEAR PARABOLIC EQUATIONS*

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Abstract

We build finite difference schemes for a class of fully nonlinear parabolic equations. The schemes are polyhedral and grid aligned. While this is a restrictive class of schemes, a wide class of equations are well approximated by equations from this class. For regular ($C^{2,\alpha}$) solutions of uniformly parabolic equations, we also establish of convergence rate of $\mathcal{O}(\alpha)$. A case study along with supporting numerical results is included.

Mathematics subject classification: 65M06, 65M12.

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1. Introduction

Although the theory of viscosity solutions has been well established for a broad class of nonlinear elliptic and parabolic equations, there are no general methods available for building convergent difference schemes to solve these equations. Schemes need to be custom built for each equation, or for classes of equations.

For degenerate, quasilinear equations such as motion of level sets by mean curvature, and the infinity Laplace equations, specialized convergent schemes have been build [13,14]. Convergent schemes have been built for the class of equations which are functions of the eigenvalues of the Hessian [16]. In general, these schemes requires successively wider stencils in order to converge. This means that the approximation error depends on an additional parameter, $d\theta$, the directional resolution. In practice, schemes of width one or two are sufficient, since the $d\theta$ error is small compared to the spatial resolution error.

In this article, we focus on the particular subclass of *polyhedral grid aligned* equations. The subclass is artificial: it is designed for the purpose of building convergent schemes. However, many of the previously mentioned equations can be approximated by this class. We build convergent schemes, and establish error estimates, which depend on the regularity of the solutions.

Related results

Convergence rates for second order elliptic and parabolic equations, without any regularity assumptions, are obtained in example Krylov [8,11], Kuo and Trudinger [12], Barles and Jacobsen [1], and Caffarelli and Souganidis [5] and the references therein. The methods used come from regularity theory for nonlinear elliptic PDEs and are substantially more technical than the methods herein.

Here we obtain convergence rates using available regularity results. This approach simplifies the argument considerably, since it avoids a reiteration of the regularity theory.

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Contents

The remainder of this section recalls the setting for our nonlinear parabolic equations and the necessary regularity results.

Section 2 is a case study with a simple example equation. Error estimates are obtained directly in this simpler setting, and supporting numerical results are presented.

The first part of section 3 recalls general results on nonlinear elliptic schemes. The second part presents new material on error estimates in terms of the residual for perturbed equations, the methods of lines, and finally for fully discrete difference schemes.

The main results are in the section 4. Here the class of schemes is established. The schemes are shown to be elliptic, and consistent, which is enough to prove convergence. Then the error estimates of the previous section are used to obtain a convergence rate.

1.1. Nonlinear parabolic equations

Our results concern the fully nonlinear parabolic Partial Differential Equation (PDE)

$$u_t(x, t) + F[u](x, t) = 0, \quad \text{for } (x, t) \text{ in } \Omega \times [0, T] \tag{PDE}$$

where Ω is a domain in \mathbb{R}^n , along with initial and boundary conditions

$$\begin{cases} u(x, t) = g(x, t), & \text{for } (x, t) \text{ on } \Omega \times \{0\} \\ u(x, t) = h(x, t), & \text{for } (x, t) \text{ on } \partial\Omega \times (0, T). \end{cases} \tag{BC}$$

The fully nonlinear elliptic partial differential operator $F[u]$ is given by

$$F[u](x) \equiv F(D^2u(x), Du(x), u(x), x). \tag{1.1}$$

Here Du and D^2u denote the gradient and Hessian of u , respectively. The function $F(X, p, r, x)$ is defined on $\mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega$, and \mathbb{S}^n is the space of symmetric $n \times n$ matrices. The natural setting for equations of this type is viscosity solutions [7].

Definition 1.1. *The differential operator (1.1) is nonlinear or degenerate elliptic if*

$$F(N, p, r, x) \leq F(M, p, s, x) \text{ whenever } r \leq s \text{ and } M \leq N. \tag{1.2}$$

Here $M \leq N$ means that $M - N$ is a nonnegative definite symmetric matrix. The corresponding parabolic operator (PDE) is called nonlinear or degenerate parabolic.

1.2. Regularity

When the equation F is convex and uniformly parabolic, solutions of (PDE) are $C^{2,\alpha}$, [17, 18]. These results build upon the elliptic regularity [4, 9]. In two dimensions and for special nonconvex equations, more regularity is available [3].

Here we use the convention of [10], where $C^{2,\alpha}$ means $C^{2,\alpha}$ in x and $C^{1,\alpha/2}$ in t .

Remark 1.1. It is often the case (for both theory and numerics) that the time variable scales quadratically with the space variable, as in [6] below.

2. A Case Study with Numerics

Before stating the main results in generality, we begin with a case study of a fully nonlinear equation. We build an elliptic (monotone) scheme, and show directly how the global error is controlled by the consistency error. Then we present computational results which are consistent with a convergence rate of $1 + \alpha$, better than the analytic result which gives a rate of α .

2.1. The one dimensional equation

Let $\Omega = [-1, 1] \subset \mathbb{R}$ and consider the operator

$$F[u] = \max \{u_{xx}, 2u_{xx}\} + f.$$

Let $u(x, t)$ be the viscosity solution of the parabolic equation

$$u_t = \max \{u_{xx}, 2u_{xx}\} + f. \quad (2.1)$$

in Ω , with initial values $u(x, 0) = h(x)$ and zero Dirichlet boundary values. The solution is known to be $C^{2,\alpha}$ but will fail to be C^3 at any point where u_{xx} changes sign.

While exact solutions of the parabolic equation were not available, two viscosity solutions of the stationary equation are

$$-\frac{1}{24} (2x^2 \min(x, 2x) - 3x + 1)$$

which is $C^{2,1}$, corresponding to $f(x) = x$, and

$$-\frac{2}{15} (2|x|^{3/2} \min(x, 2x) - 3x + 1)$$

which is $C^{2,.5}$, corresponding to $f(x) = |x|^{.5}$.

2.2. A monotone finite difference method

Divide the real line into intervals of length h and the time line into intervals of length ρ . Use the notation $u_j^n = u(jh, n\rho)$. Begin with finite differences

$$(\mathcal{D}_{xx}^h u)_j^n = \frac{1}{h^2} (u_{j-1}^n - 2u_j^n + u_{j+1}^n) \quad (2.2)$$

$$(\mathcal{D}_t^\rho u)_j^n = \frac{1}{\rho} (u_j^{n+1} - u_j^n). \quad (2.3)$$

Define the spatial and temporal residuals

$$\begin{aligned} \delta^h[u]_j^n &= (\mathcal{D}_{xx}^h u)_j^n - (u_{xx})_j^n, \\ \delta^\rho[u]_j^n &= (\mathcal{D}_t^\rho u)_j^n - (u_t)_j^n. \end{aligned}$$

The residuals are $\mathcal{O}(h^2)$ and $\mathcal{O}(\rho)$ when $u \in C^4$.

To discretize (2.1), simply insert (2.2) into the operator, to obtain

$$F^h[u] = \max \{\mathcal{D}_{xx}^h u, 2\mathcal{D}_{xx}^h u\} + f. \quad (2.4)$$

Use (2.3) for the time derivative. The the total approximation error is given by

$$\delta^{h,\rho} = \delta^\rho + F^h[u] - F[u].$$

Substituting (2.2, 2.3, 2.4) into (2.1) and using the definition of the approximation errors gives

$$\frac{1}{\rho} (u_j^{n+1} - u_j^n) = \frac{1}{h^2} \max \{u_{j-1}^n - 2u_j^n + u_{j+1}^n, 2(u_{j-1}^n - 2u_j^n + u_{j+1}^n)\} + \delta^{h,\rho}[u]_j^n.$$

Solving for u_j^{n+1} we obtain

$$u_j^{n+1} = (1 - \alpha)u_j^n + \alpha \max \{(u_{j-1}^n + 2u_j^n + u_{j+1}^n)/4, (u_{j-1}^n + u_{j+1}^n)/2\} + \rho\delta^{h,\rho}[u]_j^n,$$

where we have defined

$$\alpha \equiv 4\rho/h^2.$$

For emphasis, define the nonlinear average which appears in the last equation

$$\mathcal{A}(x_0, x_+, x_-; \alpha) = (1 - \alpha)x_0 + \alpha \max \{(x_- + 2x_0 + x_+)/4, (x_- + x_+)/2\}$$

and write

$$u_j^{n+1} = \mathcal{A}(u_j^n, u_{j+1}^n, u_{j-1}^n; \alpha) + \rho\delta^{h,\rho}[u]_j^n. \tag{2.5}$$

For monotonicity [15] we require that the coefficients of the schemes be nonnegative,

$$\alpha \leq 1, \text{ equivalently } \rho \leq \frac{h^2}{4}. \tag{2.6}$$

2.3. Error estimates

Let $U = U^{h,\rho}$ be the solution of the finite difference scheme

$$U_j^{n+1} = \mathcal{A}(U_j^n, U_{j+1}^n, U_{j-1}^n; \alpha) \tag{2.7}$$

along with consistent initial and boundary conditions. Define

$$Z_j^n = U_j^n - u_j^n$$

to be the error at each grid point. Then subtracting (2.5) from (2.7) we obtain

$$Z_j^{n+1} = \mathcal{A}(U_j^n, U_{j+1}^n, U_{j-1}^n; \alpha) - \mathcal{A}(u_j^n, u_{j+1}^n, u_{j-1}^n; \alpha) - \rho\delta^{h,\rho}[u]_j^n. \tag{2.8}$$

along with zero initial and boundary conditions.

Now provided that (2.6) holds, \mathcal{A} is a nonlinear average, which means

$$\mathcal{A}(x_0, x_+, x_-; \alpha) \leq \max\{x_0, x_+, x_-\}$$

and, using the fact that

$$\left| \max_i \{X_i\} - \max_i \{Y_i\} \right| \leq \max_i |X_i - Y_i|$$

it is easy to verify that

$$\left| \mathcal{A}(U_j^n, U_{j+1}^n, U_{j-1}^n; \alpha) - \mathcal{A}(u_j^n, u_{j+1}^n, u_{j-1}^n; \alpha) \right| \leq \mathcal{A}(|Z_j^n|, |Z_{j+1}^n|, |Z_{j-1}^n|; \alpha).$$

Together, these facts imply that

$$|Z_j^{n+1}| \leq \max\{|Z_j^n|, |Z_{j+1}^n|, |Z_{j-1}^n|\} + \rho |\delta^{h,\rho}[u]_j^n|.$$

Since the initial and boundary values of Z are zero, we can induct and conclude

$$\max_j |Z_j^n| \leq \rho \sum_{k=1}^n |\delta^{h,\rho}[u]_k^n|.$$

2.4. Numerical Results

For both cases, the solutions were numerically computed on $t \in [0, .5]$, with initial data given by the exact solution. The error in the maximum norm (from the exact solution) at $t = .5$ is shown as a function of the number of grid points in Figure 2.1. The convergence rate observed is better than $O(\alpha)$, indeed it is consistent with $O(1 + \alpha)$.

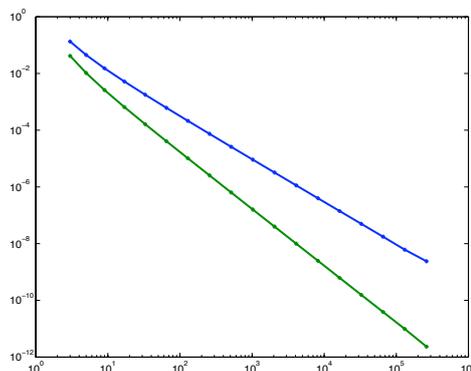


Fig. 2.1. Error in maximum norm versus number of grid points. The bottom and top curves correspond to the solutions, $u \in C^{2,1}$, $v \in C^{2,.5}$. The slopes of the line of best fit are -2.04 and -1.54, respectively.

3. General Elliptic Finite Difference Schemes

Before restricting to polyhedral schemes, we state some results which apply to elliptic schemes in general.

In the first subsection, we recall the framework established in [15] for building monotone finite difference schemes. In the second subsection, we present contraction properties of time-dependent schemes. In the last sections, we present three results on error estimates for increasing discrete schemes, going from perturbed equations, to the method of lines, to fully discrete finite difference schemes. In the last case the nonlinear (CFL) condition appears.

3.1. Characterization of elliptic difference schemes

Let G^h be a suitable finite difference grid on the domain Ω . Let the grid points be indexed by x_i , $i = 1, \dots, N$. For a function $u(x, t)$ defined on $\Omega \times [0, T]$, write $u_i^n = u(x_i, n\rho)$. For a given grid point i , let $i' = i_1, \dots, i_k$ be the list of neighboring grid points. A grid function is a vector $U = (U_1, \dots, U_N)$ of values at the grid points, and a finite difference scheme is a nonlinear function which maps grid functions to grid functions. (A solution of the scheme is a grid function which satisfies $F^h[U] = 0$, the zero grid function). Write the scheme F^h at the grid point i as

$$F^i[U] \equiv F^i(U_i, U_i - U_{i_1}, \dots, U_i - U_{i_k}) \equiv F^i(U_i, U_i - U_{i'}),$$

where $U_i - U_{i'}$ is shorthand for the same expression repeated for each of the neighbors. (When the context is clear, we will drop the superscript h from the scheme and the grid functions).

Definition 3.1. *The nonlinear scheme F^h is elliptic if each component F^i is nondecreasing in each variable, i.e.*

$$X \leq Y \text{ implies } F^i(X) \leq F^i(Y) \tag{3.1}$$

The nonlinear elliptic structure condition on schemes is the discrete version of the same conditions for PDEs. In both cases, the local structure condition implies that the solution operator is a contraction in the maximum norm, see [15] for proofs.

Lemma 3.1. *The function $F(X, p, r, x)$ is degenerate elliptic if and only if*

$$F[u](x) \geq F[v](x), \tag{3.2}$$

whenever x is a nonnegative local maximum of $u - v$, for twice differentiable functions u, v .

Lemma 3.2. *The scheme F is elliptic if and only if*

$$F^i[U] \geq F^i[V], \tag{3.3}$$

whenever i is a nonnegative local maximum of $U - V$, for grid functions U, V .

3.2. Contraction properties is continuous or discrete time

When we discretize (PDE) in space, using a finite difference scheme, the result is the following system of ODEs.

$$\frac{d}{dt}U(t) + F^h[U(t)] = 0, \quad t > 0, U \in \mathbb{R}^N \tag{ODE}$$

where $F^h[U]$ incorporates the boundary conditions as well.

When we also discretize time by using the forward Euler method in (ODE), we get the following explicit method

$$U^{n+1} = U^n - \rho F^h[U^n]. \tag{3.4}$$

The method consists of iteratively applying the explicit map, which we record below.

Definition 3.2 (The explicit Euler map) *For $\rho > 0$, define the map S_ρ which takes grid functions to grid functions, by*

$$S_\rho(U) = U - \rho F[U]. \tag{3.5}$$

Definition 3.3 (Nonlinear CFL condition) *Let $F^h[U]$ be an elliptic scheme, and suppose that it is Lipschitz continuous with constant K^h . The nonlinear Courant-Friedrichs-Lewy condition [6] for the Euler map S_ρ is*

$$\rho \leq \frac{1}{K^h}. \tag{CFL}$$

Remark 3.1. We can also consider locally Lipschitz continuous schemes, in which case $K(h) = K(h, U)$. In some cases we still get global existence of solutions and $K(h, U)$ does not decrease in time. For example, this is the case for the scheme for the equation $u_t = u_x^2$ and for the time-dependent Monge-Ampère equation, see [16].

The contraction properties in time arise as a result from the local structure condition that the operators or schemes be elliptic.

Lemma 3.3 (*L[∞] Stability*) *Let $u(x, t), v(x, t)$ be solutions of (PDE), subject to different initial conditions. Then*

$$\|u(\cdot, t) - v(\cdot, t)\|_\infty \leq \|u(\cdot, s) - v(\cdot, s)\|_\infty, \quad \text{for } s \leq t.$$

Proof. Write

$$N(t) = \max_{x \in \Omega} |u(x, t) - v(x, t)|$$

and choose

$$x^+(t) \in \arg \max_{x \in \Omega} u(x, t) - v(x, t).$$

We will establish that $N(t)$ is a decreasing function of time. Without loss of generality, we can assume that $N(t)$ is achieved at $x^+(t)$. Compute

$$\begin{aligned} & \frac{d}{dt} u(x^+(t), t) - v(x^+(t), t) \\ &= u_t(x^+(t), t) - v_t(x^+(t), t) + \left(Du(x^+(t), t) - Dv(x^+(t), t) \right) \frac{d}{dt} x^+(t) \\ &= -F(D^2u, Du, u, x) + F(D^2v, Dv, v, x) |_{x=x^+(t)} \leq 0, \end{aligned}$$

where we have used the fact that $Du = Dv$ at a local max of $u - v$, in the first step, and (3.2) in the second step. (The argument can be made valid for viscosity solutions, by replacing u, v in the calculation by smooth test functions touching above or below, as necessary). □

Lemma 3.4. *If $F^h[U]$ is an elliptic scheme, then the solution operator of (ODE) is a contraction in the maximum norm. In other words,*

$$N(t) = \max_j |U_j(t) - V_j(t)| \text{ is a decreasing function of time,}$$

whenever $U(t), V(t)$ are solutions of (ODE).

Proof. Let $k \in \arg \max_j U_j(t) - V_j(t)$. Without loss of generality, we can assume that $N(t)$ is achieved at k . Then since the scheme is elliptic, (3.3) holds, which gives

$$\frac{d}{dt} (U_k(t) - V_k(t)) = F^k(U) - F^k(V) \leq 0.$$

This completes the proof of the lemma. □

Lemma 3.5. *Let F be a Lipschitz continuous, degenerate elliptic scheme. Then the Euler map (3.5) is a contraction in \mathbb{R}^N equipped with the maximum norm, provided (CFL) holds.*

Proof. Refer to [15]. □

3.3. Error estimates for perturbed equations

We begin by recording the error estimates in the continuous setting.

Lemma 3.6 (Error estimate) *Let $u(x, t) \in C^{2,\alpha}$ be the viscosity solution of (PDE), and let u^ϵ be the solution of a family of approximate equations, $u_t^\epsilon + F^\epsilon[u^\epsilon] = 0$, where for $\epsilon > 0$, F^ϵ is an elliptic operator, subject to consistent initial and boundary conditions. Then*

$$\|u(\cdot, t) - u^\epsilon(\cdot, t)\|_{L^\infty} \leq \int_0^t \max_x |\delta^\epsilon[u]| \, ds.$$

Proof. Compute

$$\begin{aligned} \frac{d}{dt}(u - u^\epsilon) &= F^\epsilon[u^\epsilon] - F[u] \\ &= F^\epsilon[u^\epsilon] + F^\epsilon[u] - F^\epsilon[u] - F[u] \\ &= F^\epsilon[u^\epsilon] - F^\epsilon[u] + \delta^\epsilon, \end{aligned}$$

where we have used the equation satisfied by u^ϵ , (PDE), and the definition of the truncation error (4.8). As in Lemma 3.3, the first two terms have a favorable sign at a local maximum, or minimum, of $u^\epsilon - u$. As a result,

$$\frac{d}{dx} \|u(\cdot, t) - u^\epsilon(\cdot, t)\|_{L^\infty} \leq \|\delta^\epsilon\|_{L^\infty(\cdot, t)}$$

and the result follows. □

3.4. Error estimates for the method of lines

Lemma 3.7 (Error estimates for the method of lines.) *Let $u(x, t) \in C^{2,\alpha}$ be the viscosity solution of (PDE). Let F^ϵ be an elliptic scheme for (1.1) and let $U^\epsilon(t)$ be the solution of the method of lines (ODE). Then the scheme converges and*

$$\max_j |U_j^\epsilon(t) - u_j(t)| \leq \int_0^t \max_j |\delta^\epsilon[u(s)]_j| \, ds.$$

Proof. Compute

$$\begin{aligned} \frac{d}{dt}(u_j(t) - U_j^\epsilon(t)) &= F^\epsilon[U^\epsilon]_j - F[u]_j \\ &= F^\epsilon[U^\epsilon]_j + F^\epsilon[u]_j - F^\epsilon[u]_j - F[u]_j \\ &= F^\epsilon[U^\epsilon]_j - F^\epsilon[u]_j + \delta_j^\epsilon, \end{aligned}$$

where we have used (ODE), (PDE), and the definition of the truncation error (4.8).

Having set up the equation above, we reiterate the proof that the solution mapping is a contraction, carrying the inhomogeneous term to arrive at the conclusion. Let $N(t) = \max_j |U_j(t) - u_j(t)|$. Assume that $N(t) = \max_j U_j(t) - u_j(t)$. (A similar computation will work if the sign is reversed). Compute for any $k(t) \in \arg \max_j U_j(t) - u_j(t)$,

$$\frac{d}{dt} N(t) = F^\epsilon[U]_k - F^\epsilon[u]_k + \delta_k^\epsilon(t) \leq \delta_k^\epsilon(t)$$

where we have used the fact that F^ϵ is elliptic (3.3). Since $N(0) = 0$, the result follows. □

3.5. Error Estimates for the forward Euler method

Lemma 3.8. (*Error estimates for Forward Euler*) Let $u(x, t) \in C^{2,\alpha}$ be the viscosity solution of (PDE). Let F^h be an elliptic scheme for (1.1) and let $U^h = (U^h)_j^n$ be the solution of the forward Euler method (3.4). Suppose (CFL) holds. Then the scheme converges and,

$$\max_j |(U^h)_j^n - u_j^n| \leq \rho \sum_{m=1}^n \max_j |\delta^{h,\rho}[u]_j^m|. \tag{3.6}$$

Proof. We first show that

$$u^{n+1} = S_\rho(u^n) + \rho \delta^{h,\rho}[u]^n.$$

This is simply a matter of collecting the definitions of the truncation errors and plugging them into the equations,

$$\begin{aligned} u^{n+1} &= u^n + \rho (\delta^\rho[u]^n + (u_t)^n) && \text{from (4.9)} \\ &= u^n + \rho (\delta^\rho[u]^n - F[u]^n) && \text{from (PDE)} \\ &= u^n + \rho (\delta^\rho[u]^n - F^h[u]^n + \delta^h[u]^n) && \text{by (4.8)} \\ &= S_\rho(u^n) + \rho (\delta^\rho[u]^n + \delta^h[u]^n) && \text{by (3.5)} \\ &= S_\rho(u^n) + \rho \delta^{h,\rho}[u]^n && \text{by (4.9)} \end{aligned}$$

From (3.4), dropping the h superscript,

$$U^{n+1} = S_\rho(U^n).$$

So subtracting

$$U^{n+1} - u^{n+1} = S_\rho(U^n) - S_\rho(u^n) - \rho \delta^{h,\rho}[u]^n.$$

By Lemma 3.5, the Euler map is a contraction, thus

$$\|U^{n+1} - u^{n+1}\|_\infty \leq \|U^n - u^n\|_\infty + \rho \|\delta^{h,\rho}[u]^n\|_\infty.$$

Since $U^0 = u^0$, the result follows by induction. □

4. Elliptic Schemes for Polyhedral Equations

4.1. Polyhedral grid aligned equations

We now turn our attention to a restricted class of nonlinear elliptic operators $F[u]$. Suppose that F can be written as a (possibly nonconvex) polyhedral operator

$$F^{d\theta}[u] = \min_{i=1,\dots,K} \max_{j=1,\dots,K} \{L_{ij}[u]\}, \tag{4.1}$$

where each

$$L_{ij}[u] \equiv a_{ij}(x) : D^2u + b_{ij}(x) \cdot Du + c_{ij}(x)u + d_{ij}(x)$$

is a linear, possibly degenerate elliptic equation with bounded coefficients.

Here, in addition to the standard assumption that each coefficient matrix a_{ij} is elliptic (although possibly degenerate), we also assume that at each x each coefficient matrix a_{ij} is grid aligned on a stencil of width W , for each i, j .

Definition 4.1. *The symmetric non-negative definite $n \times n$ matrix a is grid aligned on a stencil of width W if it can be written as*

$$a = \sum_{k=1}^n \lambda_k v_k \otimes v_k, \tag{4.2}$$

for orthogonal vectors v_k of the form

$$v_k = (m_1, \dots, m_n), \quad m_i \in \mathbb{Z}, \quad |m_i| \leq W, \quad i = 1, \dots, n. \tag{4.3}$$

See Figure 4.1.

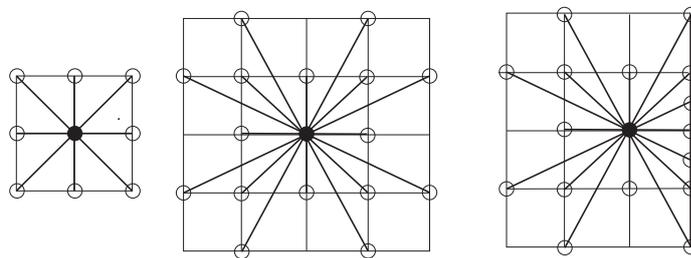


Fig. 4.1. (a) Stencils of width 1 and 2 (with redundant points removed) (b) Modified stencil near the boundary.

These operators arise as the value function of a stochastic differential game problem. Here the polyhedral approximation corresponds to approximating the controlled diffusions by a restricted set of controlled diffusions. In the simpler case, with just a minimum (or a maximum) in (4.1) we obtain a stochastic control problem.

4.2. Elliptic schemes for polyhedral equations

In this section we build the elliptic schemes and demonstrate that the schemes are elliptic using the framework from section 3.

The main tool for building elliptic finite difference schemes is the second finite difference operator in the grid direction v . Generalize (2.2) to define

$$D_{vv}^h u(x) \equiv \frac{u(x + hv) - 2u(x) + u(x - hv)}{h^2 |v|^2}. \tag{4.4}$$

Next, define the finite difference scheme.

Definition 4.2. *Let the nonlinear elliptic operator F be of the polyhedral form (4.1), where each linear operator L_{ij} is grid aligned. For each L_{ij} define the corresponding discretized operator L_{ij}^h by inserting the directional second derivatives (4.4) into the expression (4.2), to obtain*

$$(a : D^2 u)^h \equiv \sum_{k=1}^n \lambda_k D_{v_k}^h u.$$

The first order terms in L_{ij} are discretized by upwinding [15], and the remaining terms are simply evaluated at the grid points. The grid aligned finite difference scheme for F is then given by the pointwise maximum (or minimum) of the L_{ij}^h

$$F^h[u] = \min_{i=1,\dots,K} \max_{j=1,\dots,K} \{L_{ij}^h[u]\}. \quad (4.5)$$

Lemma 4.1. *The polyhedral finite difference scheme defined by (4.5) is elliptic.*

Proof. We need to verify that the scheme satisfies (3.1). Notice first, that by definition, for each grid direction v , the scheme (4.4) is elliptic. This follows since (4.4) is a positive multiple of $(u(x+hv) - u(x)) + (u(x-hv) - u(x))$, so it is non-decreasing in the differences. Next, since each λ_k is non-negative, the expression (4.2) is also elliptic. Finally, the expression (4.5) is a minimum of a maximum of elliptic terms, and both of these operations preserved ellipticity. \square

Remark 4.1. In some cases, where F is convex, but we may nevertheless choose a non-convex approximation, as was done in [16] for the Monge-Ampère equation.

4.3. Approximately polyhedral equations

While the class of polyhedral grid aligned nonlinear elliptic equations is restrictive, many equations can be *approximated* by this class. For example, the schemes in [13, 14, 16] were approximated by equations from (or similar to) this class.

For approximately polyhedral equations, the consistency of the approximation depends on the additional parameter, $d\theta$, the directional resolution of the stencil. (In two dimensions, it is the largest angle of a wedge that avoids grid points in the stencil.) Then we also want to estimate $|u^{d\theta} - u|$, the difference between the solution of the polyhedral equation $F^{d\theta}[u^{d\theta}]$ and the original equation $F[u]$, in terms of the consistency error.

Definition 4.3. *We say the operator (1.1) is approximately polyhedral if, given $d\theta > 0$ there exists a grid with directional resolution $d\theta$ and a polyhedral grid aligned $F^{d\theta}$ such that for all smooth functions ϕ*

$$F^{d\theta}[\phi] - F[\phi] \rightarrow 0 \text{ as } d\theta \rightarrow 0. \quad (4.6)$$

Example 4.1. *To take an example from [16] the directional approximation error on smooth functions, ϕ , is*

$$\delta^{d\theta}[\phi] = (\lambda_{\max}[\phi] - \lambda_{\min}[\phi]) d\theta^2,$$

where the first term on the right hand side is the difference between the largest and smallest eigenvalue of the Hessian of ϕ at x .

4.4. Consistency of the schemes.

In this section consistency is defined in terms of the approximation errors. Consistency will be used to prove convergence, and the approximation errors (when defined) will control the convergence rate.

Without additional hypotheses on the elliptic operator, F , which ensure regularity of solutions, the approximation error may be unbounded on viscosity solutions, making the error estimates meaningless. However the consistency of the approximation need only be verified

on smooth test functions, which allows for convergence results (without a rate) without any regularity assumptions.

We assume that the schemes are either exact in $d\theta$, (i.e., the equation is polyhedral and grid aligned) or that the equation is nearly grid aligned, which is equivalent to consistency. The previously cited references give examples of approximations by polyhedral grid aligned functions on wide stencil grids.

Definition 4.4. Let F^ϵ be an approximation scheme for the equation (1.1), where $\epsilon = (d\theta, h, \rho)$, to indicate that the scheme depends on the directional, spatial, and temporal resolution. The total approximation error $\delta^\epsilon[u]$ for the function u is the function

$$\delta^\epsilon = \delta^h + \delta^\rho + \delta^{d\theta}, \tag{4.7}$$

which is the sum of the spatial, temporal and directional approximation errors,

$$\delta^h[u] \equiv F^h[u] - F[u], \tag{4.8}$$

$$\delta^\rho[u] \equiv u_t - \mathcal{D}_t^\rho u, \tag{4.9}$$

$$\delta^{d\theta}[\phi] \equiv F^{d\theta}[\phi] - F[\phi]. \tag{4.10}$$

The scheme F is consistent if

$$\delta^\epsilon[\phi](x, t) \rightarrow 0 \text{ uniformly on compact subset of } \Omega \times (0, T)$$

for all smooth $\phi(x, t)$.

Lemma 4.2. Consider functions $u(x, t) \in C_1^{2,\alpha}(\Omega \times [0, t])$, and $\phi(x, t)$ smooth. Let F^ϵ be the finite difference scheme for (PDE) comprised of the finite difference method (4.5) for the spatial discretization and the forward Euler method in time (2.3). Suppose also that the CFL condition (CFL) holds. Then

$$\delta^h[\phi] + \delta^\rho[\phi] = \mathcal{O}(h^2), \tag{4.11}$$

$$\delta^h[u] + \delta^\rho[u] = \mathcal{O}(h^\alpha). \tag{4.12}$$

Proof. 1. It is a standard result from finite differences that for smooth functions $\phi(x, t)$ the accuracy of the centered second finite difference scheme (4.4) is of $\mathcal{O}(h^2)$ and the forward Euler method (2.3) is $\mathcal{O}(\rho)$

$$\begin{aligned} \mathcal{D}_{vv}^h \phi - \partial_{vv} \phi &= \mathcal{O}(h^2) \\ \delta^\rho[\phi] &= \mathcal{O}(\rho). \end{aligned}$$

2. For functions $u(x) \in C^{2,\alpha}$, we establish

$$\mathcal{D}_{vv}^h u(x) - \frac{\partial^2}{\partial v^2} u(x) = \mathcal{O}(h^\alpha).$$

The application of Taylor series to $u(x) \in C^{2,\alpha}(\Omega)$ in the direction $\pm hv$ gives

$$\begin{aligned} u(x + hv) &= u(x) + h|v|u_v(x) + \frac{h^2}{2}|v|^2u_{vv}(x) + \mathcal{O}(h^{2+\alpha}) \\ u(x - hv) &= u(x) - h|v|u_v(x) + \frac{h^2}{2}|v|^2u_{vv}(x) + \mathcal{O}(h^{2+\alpha}). \end{aligned}$$

Adding these equations and dividing by h^2 gives the identity

$$\frac{u(x + hv) - 2u(x) + u(x - hv)}{h^2|v|^2} - u_{vv} = \mathcal{O}(h^\alpha), \quad u \in C^{2,\alpha}.$$

3. Next, we estimate the spatial and temporal approximation errors. Notice that by virtue of (4.1) and the boundedness of the coefficients in the L_{ij} , F is Lipschitz continuous as a function of the derivatives, with constant K . Correspondingly, F^ϵ is Lipschitz continuous as a function of the finite difference terms, with constant K^h . Thus

$$\delta^h[u] = \mathcal{O}(h^\alpha) \quad \delta^h[\phi] = \mathcal{O}(h^2)$$

Finally, the Lipschitz constant of K^h as a function of h is $O(h^2)$, provided not all second order terms are zero. Thus from (CFL), $\rho = \mathcal{O}(h^2)$, so (4.11) and (4.12) hold. \square

4.5. Convergence

In order to prove convergence, we use the theorem of [2], which says that consistent, monotone, stable schemes converge. The results of [15], make this verification simple: it is enough to check that the scheme is elliptic, and that (CFL) holds. We summarize the results of the previous sections in the following theorem.

Theorem 4.1. *Let F be an approximately polyhedral elliptic operator, and let $u(x, t)$ be the unique solution of (PDE), (BC). Let F^ϵ be the finite difference equation for (PDE) which is comprised of (4.5) for the spatial discretization (including the approximation in $d\theta$ by a polyhedral operator) and the forward Euler method in time (2.3). Write U^ϵ for the solution of F^ϵ . Then*

$$U^\epsilon \rightarrow u, \quad \text{uniformly on compact subsets of } \Omega \times [0, t].$$

Proof. Consistency in h , ρ has been established in lemma 4.2: only the fact that the left hand terms of (4.11) go to zero as h , ρ is needed. Consistency in $d\theta$ is assumed. Since we assume F is approximately polyhedral, (4.6) holds. Stability follows from (CFL). The fact that the scheme is elliptic has been established in lemma (4.1). Together these facts ensure convergence. \square

4.6. A convergence rate

Given the regularity of solutions, we obtain a corresponding convergence rate for the numerical schemes, up to an additional directional approximation error term.

Theorem 4.2. *Let F be a convex uniformly elliptic operator, which is also approximately polyhedral. Let $u(x, t)$ be the unique solution of (PDE), (BC). Then $u \in C_1^{2,\alpha}(\Omega \times ([0, t]))$. Let F^ϵ be the finite difference equation for (PDE) which is comprised of (4.5) for the spatial discretization (including the approximation in $d\theta$ by a polyhedral operator) and the forward Euler method in time (2.3). Write U^ϵ for the solution of F^ϵ . Then*

$$\max_j |U^\epsilon - u|_j^n \leq \rho \sum_{m=1}^n \max_j |\delta^\epsilon[u]_j^m| \leq tO(h^\alpha) + t\delta^{d\theta}[u], \tag{4.13}$$

where $t = n\rho$.

Proof. The estimate in terms of the residual is provided by Lemma (3.8). The first inequality of (4.13) is a restatement of (3.6). The entire approximation error, δ^ϵ , is defined in (4.7) as

$$\delta^\epsilon = \delta^h + \delta^\rho + \delta^{d\theta}.$$

We will carry the third term, the directional approximation error, and estimate the first two.

The cited references [17, 18] give the $C^{2,\alpha}$ regularity for uniformly elliptic equations. The consistency results of our scheme, applied to functions u with the regularity above for the spatial and temporal terms, are given in Lemma 4.2. Then (4.12) gives

$$\delta^h[u] + \delta^\rho[u] = \mathcal{O}(h^\alpha).$$

The result follows from inserting the last equation into (3.6), after observing that the summation is a quadrature approximation to the time integral, and the error terms for the quadrature and higher order and therefore negligible. \square

Remark 4.2. When the equation is polyhedral, we obtain a convergence rate of α as an immediate corollary of the previous theorem.

References

- [1] G. Barles and E. R. Jakobsen, Error bounds for monotone approximation schemes for parabolic Hamilton-Jacobi-Bellman equations, *Math. Comput.*, **76**:260 (2007), 1861-1893 (electronic).
- [2] G. Barles and P. E. Souganidis, Convergence of approximation schemes for fully nonlinear second order equations, *Asymptotic Anal.*, **4**:3 (1991), 271-283.
- [3] X. Cabré and L. A. Caffarelli, Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations, *J. Math. Pure. Appl. (9)*, **82**:5 (2003), 573-612.
- [4] L. A. Caffarelli and X. Cabré, *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, 1995.
- [5] L. A. Caffarelli and P. E. Souganidis, A rate of convergence for monotone finite difference approximations to fully nonlinear, uniformly elliptic PDEs, *Commun. Pur. Appl. Math.*, **61**:1 (2008), 1-17.
- [6] R. Courant, K. O. Friedrichs, and H. Lewy, On the partial difference equations of mathematical physics, *IBM J. Res. Dev.*, **11** (1967), 215-234.
- [7] M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)*, **27**:1 (1992), 1-67.
- [8] H. Dong and N. V. Krylov, The rate of convergence of finite-difference approximations for parabolic Bellman equations with Lipschitz coefficients in cylindrical domains, *Appl. Math. Opt.*, **56**:1 (2007), 37-66.
- [9] L. C. Evans, Classical solutions of fully nonlinear, convex, second-order elliptic equations, *Commun. Pur. Appl. Math.*, **35**:3 (1982), 333-363.
- [10] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [11] N. V. Krylov, On the rate of convergence of finite-difference approximations for Bellman's equations, *Algebra i Analiz*, **9**:3 (1997), 245-256.
- [12] H.-J. Kuo and N. S. Trudinger, On the discrete maximum principle for parabolic difference operators, *RAIRO Modél. Math. Anal. Numér.*, **27**:6 (1993), 719-737.
- [13] A. M. Oberman, A convergent monotone difference scheme for motion of level sets by mean curvature, *Numer. Math.*, **99**:2 (2004), 365-379.

- [14] A. M. Oberman, A convergent difference scheme for the infinity Laplacian: construction of absolutely minimizing Lipschitz extensions, *Math. Comput.*, **74**:251 (2005), 1217-1230 (electronic).
- [15] A. M. Oberman, Convergent difference schemes for degenerate elliptic and parabolic equations: Hamilton-Jacobi equations and free boundary problems, *SIAM J. Numer. Anal.*, **44**:2 (2006), 879-895 (electronic).
- [16] A. M. Oberman, Wide stencil finite difference schemes for the elliptic Monge-Ampère equation and functions of the eigenvalues of the Hessian, *Discrete Cont. Dyn-B*, **10**:1 (2008), 221-238.
- [17] L.-H. Wang, On the regularity theory of fully nonlinear parabolic equations, *Bull. Amer. Math. Soc. (N.S.)*, **22**:1 (1990), 107-114.
- [18] L.-H. Wang, On the regularity theory of fully nonlinear parabolic equations. I, *Commun. Pur. Appl. Math.*, **45**:1 (1992), 27-76.