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A STOPPING CRITERION FOR HIGHER-ORDER SWEEPING SCHEMES FOR STATIC HAMILTON-JACOBI EQUATIONS*

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Abstract

We propose an effective stopping criterion for higher-order fast sweeping schemes for static Hamilton-Jacobi equations based on ratios of three consecutive iterations. To design the new stopping criterion we analyze the convergence of the first-order Lax-Friedrichs sweeping scheme by using the theory of nonlinear iteration. In addition, we propose a fifth-order Weighted PowerENO sweeping scheme for static Hamilton-Jacobi equations with convex Hamiltonians and present numerical examples that validate the effectiveness of the new stopping criterion.

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1. Introduction

Consider the following static Hamilton-Jacobi (H-J) equation:

$$\begin{cases} H(\nabla\phi(x), x) = f(x), & x \in \Omega \setminus \Gamma, \\ \phi(x) = g(x), & x \in \Gamma \subset \Omega, \end{cases} \tag{1.1}$$

where $g(x)$ is a positive, Lipschitz continuous function, Ω is an open, bounded polygonal domain in R^d, and Γ is a subset of Ω. $H(p, x)$ is Lipschitz continuous in both arguments, and it is convex and homogeneous of degree one in the first argument.

This class of first-order nonlinear PDEs arise in many applications such as optimal control, differential games, computer vision, geometric optics, and geophysical applications. Thus it is essential to develop efficient high-order accurate numerical methods for such equations. Based on [22, 27] we propose a fifth-order sweeping scheme for the equation. To design an effective stopping criterion for the sweeping scheme, we analyze convergence of the first-order Lax-Friedrichs scheme in terms of theory of nonlinear iterative methods.

Fast sweeping methods are a family of efficient methods for solving static Hamilton-Jacobi equations [3, 7–9, 11, 18, 19, 25, 28, 29], and some essential ideas of these methods may trace

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back to [2, 20]. In [28] the fast sweeping method was systematically analyzed for eikonal equations. Since then the fast sweeping methods have undergone intensive development for general static Hamilton-Jacobi equations in [3, 8, 9, 11, 18, 19, 25, 28, 29] and have found many different applications; see [10] for example. On the other hand, the fast marching method and its relatives consist of another family of numerical methods for solving static Hamilton-Jacobi equations [5, 23, 24, 26].

A fast sweeping method consists of the following three essential ingredients: 1) an efficient local solver for a Hamilton-Jacobi equation on a given Cartesian mesh or triangulation, 2) systematic orderings of solution nodes according to some pre-determined information-flowing directions, and 3) Gauss-Seidel type iterations based on a given order of solution nodes. However, among all the above cited works most of the methods are of first-order accuracy, and only [11, 27] consider higher-order sweeping schemes for such equations. In [27] a third-order WENO scheme [6] is incorporated into Godunov and Lax-Friedrichs numerical Hamiltonians. In [11] a second-order discontinuous-Galerkin discretization is used to design a fast sweeping method for eikonal equations. Here we propose a fifth-order accurate sweeping method in terms of the Weighted PowerENO reconstruction procedure for H-J equations [22] in the same way as proposed in [27] for third-order accurate sweeping methods. The fifth-order fast sweeping method is able to approximate up to high accuracy the solution of multidimensional H-J equations.

Since Gauss-Seidel iteration requires of a criterion to stop the iterative procedure, an absolute stopping criterion to determine the convergence was proposed in [28] such that the algorithm stops when the L^1-norm of the difference between two consecutive iterates is smaller than a given small number. This stopping criterion behaves consistently for first-order fast sweeping methods based on monotone Godunov and Lax-Friedrichs Hamiltonians [8, 28]. However, high-order versions of these methods do not inherit monotonicity and therefore the convergence may be oscillatory. In this case, an absolute stopping criterion is not robust enough to determine the optimal iterate to which the scheme converges.

To design an effective stopping criterion we analyze the convergence of first-order Lax-Friedrichs sweeping methods based on classical results of nonlinear functional analysis [4, 13]. The convergence of the nonlinear Jacobi or Gauss-Seidel iteration resulting from the Lax-Friedrichs sweeping can be proved by using the Banach fixed-point theorem through the explicit expression of the Jacobian matrix. Based on this convergence analysis, we then propose a consistent relative stopping criterion to determine the converged solution of the iterative procedure. We present a series of numerical experiments to validate the proposed new higher order sweeping scheme.

The paper is organized as follows. In Section 2 we review fast sweeping methods and propose a high-order sweeping scheme based on the fifth-order weighted PowerENO scheme. In Section 3 we analyze the convergence of Lax-Friedrichs fast sweeping methods and propose a stopping criterion consistent with the convergence analysis. In Section 4 we present numerical experiments to demonstrate higher order accuracy of the proposed scheme.

2. Fast Sweeping Methods

2.1. A generic formulation for sweeping

We consider a rectangular $n \times n$ mesh, Ω_h, where $x_i = ih_x$ $(i = 1, .., n)$, and $y_j = jh_y$ $(j = 1, ..., n)$ are the grid points. We discretize the nonlinear H-J equation (1.1) by a monotone numerical Hamiltonian $\hat{H}$,

$$\begin{cases} \hat{H}(\phi_x^-, \phi_x^+; \phi_y^-, \phi_y^+)_{ij} = f(x_i, y_j) = f_{ij}, & (x_i, y_j) \in \Omega_h \setminus \Gamma_h, \\ \phi_{ij} = g(x_i, y_j) = g_{ij}, & (x_i, y_j) \in \Gamma_h \subset \Omega_h, \end{cases} \tag{2.1}$$

where ϕ_x^- (ϕ_x^+) are backward (forward) difference quotients in the x-direction at the current point (i, j), and ϕ_y^- (ϕ_y^+) are defined similarly in the y-direction. Thus we obtain a system of nonlinear equations with as many unknowns as grid points except for those belonging to Γ. Since the number of grid points determines the number of equations, it is very involved to solve globally the system of nonlinear equations. Thus, iterative methods are desired to solve the system.

Consider a general function $G : D \subset R^n \times R^n \to R^n \times R^n$ that has components G_{ij}. The goal consists of solving a discrete system of nonlinear equations that results from (2.1) that can be written as

$$G_{i,j}(\phi_{k,l}; k = 1, ..., n; l = 1, ..., n) = 0 \qquad \text{for} \qquad i = 1, .., n;\ j = 1, ..., n \tag{2.2}$$

where the unknowns, $\phi_{k,l}$, are determined using a nonlinear Gauss-Seidel iteration procedure.

Specifically, each equation relates locally the neighboring values to the standing mesh value, $\phi_{i,j}$, as follows:

$$\begin{aligned} G_{i,j} :=& G_{i,j}(\phi_{i-p,j}, ..., \phi_{i-1,j}, \phi_{i+1,j}, ..., \phi_{i+p,j}; \phi_{i,j}; \phi_{i,j-q}, ..., \phi_{i,j-1}, \phi_{i,j+1}, ..., \phi_{i,j+q}) \\ =&0, \end{aligned} \tag{2.3}$$

where p and q are integers ≥ 1.

The standard Gauss-Seidel procedure corresponding to the increasing order of both indices consists of solving the (i, j)-th equation

$$G_{i,j}(\phi_{i-p,j}^{(k+1)}, ..., \phi_{i-1,j}^{(k+1)}, \phi_{i+1,j}^{(k)}, ..., \phi_{i+p,j}^{(k)}; \phi_{i,j}; \phi_{i,j-q}^{(k+1)}, ..., \phi_{i,j-1}^{(k+1)}, \phi_{i,j+1}^{(k)}, ..., \phi_{i,j+q}^{(k)}) = 0 \tag{2.4}$$

for the unknown $\phi_{i,j}$ and setting $\phi_{i,j}^{(k+1)} = \phi_{i,j}$. Then to obtain $\phi^{(k+1)}$ from $\phi^{(k)}$ we solve successively $n \times n$ nonlinear equations (2.4) in the increasing order of sub-indices, starting the process at an initial guess $\phi_{i,j}^{(0)}$.

The formulation of the local solver reads as the system of equations

$$\begin{aligned} & G_{ij}(\phi_{i-p,j}, ..., \phi_{i-1,j}, \phi_{i+1,j}, ..., \phi_{i+p,j}; \phi_{i,j}; \phi_{i,j-q}, ..., \phi_{i,j-1}, \phi_{i,j+1}, ..., \phi_{i,j+q}) \\ =& \phi_{ij} - F_{i,j}(\phi_{i-p,j}, ..., \phi_{i-1,j}, \phi_{i+1,j}, ..., \phi_{i+p,j}; \phi_{i,j-q}, ..., \phi_{i,j-1}, \phi_{i,j+1}, ..., \phi_{i,j+q}) \\ =& 0. \end{aligned} \tag{2.5}$$

In practice, the Gauss-Seidel iterative procedure can be expressed as the unknown $\phi_{i,j}$ solved explicitly from the above equation,

$$\phi_{i,j}^{(k+1)} = F_{i,j}(\phi_{i-p,j}^{(k+1)}, ..., \phi_{i-1,j}^{(k+1)}, \phi_{i+1,j}^{(k)}, ..., \phi_{i+p,j}^{(k)}; \phi_{i,j-q}^{(k+1)}, ..., \phi_{i,j-1}^{(k+1)}, \phi_{i,j+1}^{(k)}, ..., \phi_{i,j+q}^{(k)}), \tag{2.6}$$

where

$$\phi_{ij} = F_{i,j}(\phi_{i-p,j}, ..., \phi_{i-1,j}, \phi_{i+1,j}, ..., \phi_{i+p,j}; \phi_{i,j-q}, ..., \phi_{i,j-1}, \phi_{i,j+1}, ..., \phi_{i,j+q}) \tag{2.7}$$

is the explicit form obtained from the discretization of the numerical Hamiltonian,

$$\hat{H}(\phi_x^-, \phi_x^+; \phi_y^-, \phi_y^+)_{ij} = f_{ij}. \tag{2.8}$$

Then we cover the whole domain with four alternating direction sweepings such that a group of characteristics is covered in each direction following the causality along characteristics in a parallel way. The four sweeps are

$$(1)\ i=1:n, j=1:n; \qquad (2)\ i=n:1, j=1:n;$$

$$(3)\ i=n:1, j=n:1; \qquad (4)\ i=1:n, j=n:1.$$

The iterative procedure is finished when a stopping criterion over consecutive iterates is satisfied.

Next we detail two different discretizations to illustrate that the fast sweeping method is simple to implement.

2.2. Godunov fast sweeping schemes

We consider the two-dimensional eikonal equation

$$\begin{cases} \sqrt{\phi_x^2+\phi_y^2}=f(x,y), & (x,y)\in\Omega\subset R^2, \\ \phi(x,y)=g(x,y), & (x,y)\in\Gamma\subset\Omega. \end{cases} \tag{2.9}$$

Discretize the equation using the first-order Godunov difference scheme as proposed in [28]:

$$[(\phi_{i,j}-\phi_{i,j}^{(xmin)})^+]^2+[(\phi_{i,j}-\phi_{i,j}^{(ymin)})^+]^2=f_{i,j}^2h^2, \tag{2.10}$$

where

$$\phi_{i,j}^{(xmin)}=\min(\phi_{i-1,j},\phi_{i+1,j}), \quad \phi_{i,j}^{(ymin)}=\min(\phi_{i,j-1},\phi_{i,j+1}) \ \text{ and } \ (x)^+=\max(x,0).$$

We have

$$[(\phi_{i,j}^{new}-\phi_{i,j}^{(xmin)})^+]^2+[(\phi_{i,j}^{new}-\phi_{i,j}^{(ymin)})^+]^2=f_{i,j}^2h^2. \tag{2.11}$$

The solution for equation (2.11) is:

$$\phi_{i,j}^{new}=\begin{cases} \min(\phi_{i,j}^{(xmin)},\phi_{i,j}^{(ymin)})+f_{i,j}h, & \text{if } |\phi_{i,j}^{(xmin)}-\phi_{i,j}^{(ymin)}|\geq f_{i,j}h, \\ \dfrac{1}{2}\left(\phi_{i,j}^{(xmin)}+\phi_{i,j}^{(ymin)}+\sqrt{2f_{i,j}^2h^2-(\phi_{i,j}^{(xmin)}-\phi_{i,j}^{(ymin)})^2}\right), & \text{otherwise.} \end{cases} \tag{2.12}$$

A first-order Godunov sweeping method consists of applying this formula in each sweep of Gauss-Seidel iterations. To stop the iterative procedure one may use the following absolute stopping criterion [28]

$$||\phi^{new}-\phi^{old}||_{L^1}\leq\delta, \tag{2.13}$$

where δ is a given convergence threshold value, such as $\delta=10^{-6}$

In the first-order sweeping method, the correction $\phi_{i,j}^{new}=\min(\phi_{i,j}^{old},\bar{\phi})$, where $\bar{\phi}$ is the solution of (2.10), is enforced so that the solution at each grid point is monotonically decreasing from an initially assigned large value ([27,28]).

This monotone scheme based on the Godunov Hamiltonian is very efficient and easy to implement for eikonal equations. The scheme behaves robustly and converges to the optimal solution in a few iterations. However, for nonconvex Hamiltonians the Godunov numerical Hamiltonian may be very involved. In those cases, other simpler monotone numerical Hamiltonians can be used to construct fast sweeping methods. Next we describe the one proposed in [8] for general static HJ equations.

2.3. Lax-Friedrichs sweeping schemes

To discretize a general Hamiltonian $H(\phi_x, \phi_y)$, the Lax-Friedrichs monotone numerical Hamiltonian [14] can be used [8]

$$\hat{H}^{LF}(u^-, u^+; v^-, v^+) = H\left(\frac{u^- + u^+}{2}, \frac{v^- + v^+}{2}\right) - \frac{1}{2}\alpha^x(u^+ - u^-) - \frac{1}{2}\alpha^y(v^+ - v^-), \quad (2.14)$$

where

$$\alpha^x = \max_{\substack{A \le u \le B \\ C \le v \le D}} |H_1(u,v)|, \qquad \alpha^y = \max_{\substack{A \le u \le B \\ C \le v \le D}} |H_2(u,v)|. \quad (2.15)$$

Here $H_i(u,v)$ is the partial derivative of H with respect to the i-th argument, or the Lipschitz constant of H with respect to the i-th argument. $[A,B]$ is the value range for $u^\pm$, and $[C,D]$ is the value range for $v^\pm$. Then we have the following local solution formula,

$$\phi_{i,j}^{new} = \left(\frac{1}{\frac{\alpha_x}{h_x} + \frac{\alpha_y}{h_y}}\right)\left[f - H\left(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h_x}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h_y}\right) + \alpha_x \frac{\phi_{i+1,j} + \phi_{i-1,j}}{2h_x} + \alpha_y \frac{\phi_{i,j+1} + \phi_{i,j-1}}{2h_y}\right]. \quad (2.16)$$

The iterative procedure stops when expression (2.13) is satisfied for a certain given δ.

The Lax-Friedrichs sweeping scheme takes more iterations to converge than the Godunov sweeping. The main advantage of this method is that it is explicit and it is very simple to implement. However, the above two schemes are first-order accurate only.

2.4. Third-order WENO fast sweeping methods

Zhang, Zhao and Qian proposed in [27] the extension of fast sweeping methods to third-order accuracy. They approximate ϕ^{xmin} and ϕ^{ymin} to third-order accuracy by incorporating the third-order WENO approximations of partial derivatives of ϕ into the scheme.

The high-order Godunov fast sweeping scheme uses as ϕ^{xmin} and ϕ^{ymin} in (2.12) the following expressions

$$\begin{cases} \phi_{i,j}^{(xmin)} = \min(\phi_{i,j}^{old} - h \cdot (\phi_x)_{i,j}^-, \phi_{i,j}^{old} + h \cdot (\phi_x)_{i,j}^+), \\ \phi_{i,j}^{(ymin)} = \min(\phi_{i,j}^{old} - h \cdot (\phi_y)_{i,j}^-, \phi_{i,j}^{old} + h \cdot (\phi_y)_{i,j}^+), \end{cases} \quad (2.17)$$

where derivatives, $(\phi_x)_{ij}^-$, $(\phi_x)_{ij}^+$, $(\phi_y)_{ij}^-$, and $(\phi_y)_{ij}^+$, are computed through the third-order WENO scheme [27].

Following the same idea, the high-order Lax-Friedrichs sweeping scheme for static H-J equations is written as

$$\phi_{i,j}^{new} = \left(\frac{1}{\frac{\alpha_x}{h_x} + \frac{\alpha_y}{h_y}}\right)\left[f - H\left(\frac{(\phi_x)_{i,j}^- + (\phi_x)_{i,j}^+}{2}, \frac{(\phi_y)_{i,j}^- + (\phi_y)_{i,j}^+}{2}\right) + \alpha_x \frac{(\phi_x)_{i,j}^+ - (\phi_x)_{i,j}^-}{2} + \alpha_y \frac{(\phi_y)_{i,j}^+ - (\phi_y)_{i,j}^-}{2}\right] + \phi_{i,j}^{old}, \quad (2.18)$$

where derivatives $(\phi_x)_{ij}^-$, $(\phi_x)_{ij}^+$, $(\phi_y)_{ij}^-$ and $(\phi_y)_{ij}^+$ are computed by the third-order WENO scheme [27].

High-order approximations for derivatives $(\phi_x)_{i,j}^-$, $(\phi_x)_{i,j}^+$, $(\phi_y)_{i,j}^-$ and $(\phi_y)_{i,j}^+$ in (2.17) and (2.18) are computed using the newest available values for ϕ in the interpolation stencils according to the philosophy of Gauss-Seidel type iterations.

2.5. Fifth-order weighted PowerENO schemes

To obtain a fifth-order accurate sweeping scheme we propose to incorporate the fifth-order weighted PowerENO reconstruction into the sweeping framework based on the Godunov and Lax-Friderichs numerical Hamiltonians. In order to obtain fifth-order accuracy we need to approximate the derivatives $(\phi_x)_{ij}^+$, $(\phi_x)_{ij}^-$, $(\phi_y)_{ij}^+$ and $(\phi_x)_{ij}^-$ in expressions (2.17) and (2.18), respectively, by means of a fifth-order reconstruction method. We use the Weighted PowerENO reconstruction procedure that was first proposed for hyperbolic conservation laws in [21] and later for Hamilton-Jacobi equations in [22]. The Weighted PowerENO procedure for Hamilton-Jacobi equations consists of a convex combination of three parabolas as explained in the following.

To obtain the optimal accuracy for $(\phi_x)_{ij}^+$ at the left interface, i.e. when the wind "blows" from the right to the left the convex combination consists of

$$(\phi_x)_{ij}^+ = w_0^+ p_{i,j} + w_1^+ p_{i+\frac{1}{2},j} + w_2^+ p_{i+1,j},$$

where the weights w and the parabolas $p_{l,s}$ are computed as follows. Define

$$z_{i+\frac{1}{2},j} = \frac{\Delta_+ \phi_{i,j}}{h_x} = \frac{\phi_{i+1,j} - \phi_{i,j}}{h_x}.$$

Then the Weighted PowerENO parabolas [22] are expressed as

$$\begin{aligned} p_{i,j}^P(x_j) &= \frac{1}{2} z_{i-\frac{1}{2},j} - \frac{1}{2} z_{i+\frac{1}{2},j} - \frac{1}{6} P_{i,j}, \\ p_{i,j+\frac{1}{2}}(x_j) &= \frac{1}{12} z_{i-\frac{1}{2},j} + \frac{4}{3} z_{i+\frac{1}{2},j} - \frac{5}{12} z_{i+\frac{3}{2},j}, \\ p_{i+1,j}^P(x_j) &= \frac{3}{2} z_{i+\frac{1}{2},j} - \frac{1}{2} z_{i+\frac{3}{2},j} - \frac{1}{3} P_{i+1,j}, \end{aligned}$$

where $P_{i,j}$ is the result of applying a power-limiter over two neighboring second-order differences of z values as

$$P_{i,j} := \text{powermod}_p(D_{i-\frac{1}{2},j}, D_{i+\frac{1}{2},j}). \tag{2.19}$$

Here $D_{l,s} = z_{l+1,s} - 2z_{l,s} + z_{l-1,s}$ and

$$\text{powermod}_p(x,y) = \frac{\text{sign}(x) + \text{sign}(y)}{2} \text{power}_p(|x|,|y|). \tag{2.20}$$

The power$-p$ mean is defined as [21]

$$\text{power}_p(x,y) = \frac{x+y}{2}\Big(1 - \Big|\frac{x-y}{x+y}\Big|^p\Big). \tag{2.21}$$

In our calculations we will use $p = 3$ as the optimal p to achieve fifth-order accuracy [22].

The expressions for the weights w_k^+ are

$$w_k^+ = \frac{\alpha_k^+}{\alpha_0^+ + \alpha_1^+ + \alpha_2^+}, \qquad \text{where} \quad \alpha_k^+ = \frac{C_k^+}{(\epsilon + IS_k^+)^2} \qquad \text{for} \quad k = 0,1,2.$$

The optimal weights for this case are $C_0^+ = 0.6$, $C_1^+ = 0.2$ and $C_2^+ = 0.2$. The smoothness indicators are defined as

$$IS_0 = \frac{13}{12}(P_{i,j})^2 + \frac{1}{4}(2z_{i+\frac{1}{2},j} - 2z_{i-\frac{1}{2},j} + P_{i,j})^2,$$
$$IS_1 = \frac{13}{12}(z_{i-\frac{1}{2},j} - 2z_{i+\frac{1}{2},j} + z_{i+\frac{3}{2},j})^2 + \frac{1}{4}(z_{i-\frac{1}{2},j} - z_{i+\frac{3}{2},j})^2,$$
$$IS_2 = \frac{13}{12}(P_{i+1,j})^2 + \frac{1}{4}(2z_{i+\frac{3}{2},j} - 2z_{i+\frac{1}{2},j} - P_{i+1,j})^2.$$

On the other hand, the approximation for $(\phi_x)^-_{ij}$ is obtained from

$$(\phi_x)^-_{ij} = w_0^- p_{i-1,j} + w_1^- p_{i-\frac{1}{2},j} + w_2^- p_{i,j},$$

where the parabolas are

$$p^P_{i-1,j}(x_{i-1,j}) = -\frac{1}{2}z_{i-\frac{3}{2},j} + \frac{3}{2}z_{i-\frac{1}{2},j} + \frac{1}{3}P_{i-1,j},$$
$$p_{i-\frac{1}{2},j}(x_j) = -\frac{1}{6}z_{i-\frac{3}{2},j} + \frac{5}{6}z_{i-\frac{1}{2},j} + \frac{1}{3}z_{i+\frac{1}{2},j},$$
$$p^P_{i,j}(x_j) = \frac{1}{2}z_{i-\frac{1}{2},j} + \frac{1}{2}z_{i+\frac{1}{2},j} - \frac{1}{6}P_{i,j},$$

and

$$w_k^- = \frac{\alpha_k^-}{\alpha_0^- + \alpha_1^- + \alpha_2^-}, \quad \alpha_k^- = \frac{C_k^-}{(\epsilon + IS_k^-)^2} \qquad \text{for} \quad k = 0, 1, 2.$$

The optimal weights for this case are $C_0^- = 0.2$, $C_1^- = 0.2$ and $C_2^- = 0.6$. The smoothness indicators are defined as

$$IS_0 = \frac{13}{12}(P_{i-1,j})^2 + \frac{1}{4}(2z_{i-\frac{1}{2},j} - 2z_{i-\frac{3}{2},j} + P_{i-1,j})^2,$$
$$IS_1 = \frac{13}{12}(z_{i-\frac{3}{2},j} - 2z_{i-\frac{1}{2},j} + z_{i+\frac{1}{2},j})^2 + \frac{1}{4}(z_{i-\frac{3}{2},j} - z_{i+\frac{1}{2},j})^2,$$
$$IS_2 = \frac{13}{12}(P_{i,j})^2 + \frac{1}{4}(2z_{i+\frac{1}{2},j} - 2z_{i-\frac{1}{2},j} - P_{i,j})^2.$$

We proceed in a similar way in the y-direction varying index j instead of i to compute fifth-order approximations of $(\phi_y)^-_{i,j}$ and $(\phi_y)^+_{i,j}$.

3. Convergence and a Consistent Stopping Criterion

3.1. Convergence of first-order Lax-Friedrichs sweeping schemes

In this section we analyze the convergence of the first-order Lax-Friedrichs fast sweeping method in terms of nonlinear iteration theory.

Let $\phi^{(0)} = \phi^{(0)}_{i,j}$ represent a discrete function which is an initial guess for solving the nonlinear system of discrete equations (2.5) by using the associated nonlinear Jacobi iteration. We can write this iterative algorithm as

$$\phi^{(k+1)} = F(\phi^{(k)}), \tag{3.1}$$

where $\phi^{(k)} = \phi^{(k)}_{i,j}$ represents the k-th iterate function computed through the resulting explicit expression denoted by F.

The convergence of the nonlinear Jacobi iterative procedure is driven by the spectral radius $\rho(\partial_\phi F)$ of the Jacobian of F with respect to ϕ, $\partial_\phi F$. Indeed, a sufficient condition for the convergence of this iterative procedure is that

$$\sup \rho(\partial_\phi F) < 1,$$

since for any matrix norm $||\cdot||$, $\rho(\partial_\phi F) \leq ||\partial_\phi F||$ holds and the mean value theorem ensures that

$$||\phi^{(k+1)} - \phi^{(k)}|| \leq ||\partial_\phi F||\, ||\phi^{(k)} - \phi^{(k-1)}||. \tag{3.2}$$

A suitable matrix norm can be chosen such that the norm of the Jacobian of F is as close to its spectral radius as desired [13].

An analogous estimate can be derived for the Jacobian of the nonlinear Gauss-Seidel iterative method since it can be proved that under suitable conditions on the Jacobian matrices the convergence of the Jacobi iteration implies the convergence of the Gauss-Seidel iteration ([1], Corollary 5.22, page 188).

As an illustration we first examine the case of a one-dimensional Hamilton-Jacobi equation of the form

$$H(\phi_x) = f(x) \qquad 0 < x < 1 \tag{3.3}$$
$$\phi(0) = 0, \quad \phi(1) = 0. \tag{3.4}$$

Proposition. *Consider the boundary value problem* (3.3) − (3.4)*. Let* F *be the nonlinear Jacobi iterative function associated to the system of nonlinear discrete equations in one dimension based on the Lax-Friedrichs numerical Hamiltonian. Then, the spectral radius of the Jacobian of* F*,* $\rho(\partial_\phi F) < 1$.

Proof. Let $x_i = ih$ where $h = \frac{1}{n}$ and $i = 1, 2, \cdots, n$. We denote by ϕ_i the approximate solution of $\phi(x_i)$. The Lax-Friedrichs numerical Hamiltonian applied to discretize (3.3) yields the nonlinear system of equations

$$\phi_i = \frac{h}{\alpha}\left(f(x_i) - H(\frac{\phi_{i+1} - \phi_{i-1}}{2h})\right) + \frac{\phi_{i+1} + \phi_{i-1}}{2} \tag{3.5}$$

for ϕ_i, $i = 1, 2, \cdots, n-1$, $\phi_0 = 0$, and $\phi_n = 0$. Here α represents the artificial viscosity satisfying

$$\alpha \geq \left|\frac{\partial H}{\partial p}\right|. \tag{3.6}$$

The nonlinear Jacobi iteration can be written as

$$\phi_i^{(k+1)} = \frac{h}{\alpha}\left(f(x_i) - H(\frac{\phi_{i+1}^{(k)} - \phi_{i-1}^{(k)}}{2h})\right) + \frac{\phi_{i+1}^{(k)} + \phi_{i-1}^{(k)}}{2} \tag{3.7}$$

for $i = 1, 2, \cdots, n-1$; $\phi_0 = 0$, and $\phi_n = 0$.

We rewrite (3.7) as a fixed-point iteration procedure of the form

$$\phi^{(k+1)} = F(\phi^{(k)}), \tag{3.8}$$

where $F(\phi^{(k)})$ represents the right hand side of (3.7). We can write explicitly the Jacobian $\partial_\phi F$ as

$$\begin{bmatrix} 0 & b_1 & 0 & \cdots & \cdots & 0 \\ c_2 & 0 & b_2 & 0 & \cdots & 0 \\ 0 & c_3 & 0 & b_3 & \cdots & 0 \\ \vdots & & \vdots & & & \\ 0 & \cdots & c_{n-3} & 0 & b_{n-3} & 0 \\ 0 & \cdots & \cdots & c_{n-2} & 0 & b_{n-2} \\ 0 & \cdots & \cdots & 0 & c_{n-1} & 0 \end{bmatrix}$$

where

$$b_1 = \frac{1}{2}\biggl(1 - \frac{1}{\alpha}\frac{\partial H}{\partial u}(\frac{\phi_2}{2h})\biggr), \quad c_{n-1} = \frac{1}{2}\biggl(1 + \frac{\partial H}{\partial u}(\frac{\phi_{n-1}}{2h})\biggr), \tag{3.9}$$

$$b_i = \frac{1}{2}\biggl(1 - \frac{1}{\alpha}\frac{\partial H}{\partial u}(\frac{\phi_{i+1}-\phi_{i-1}}{2h})\biggr), \qquad i = 2, \cdots n-2; \tag{3.10}$$

$$c_i = \frac{1}{2}\biggl(1 + \frac{1}{\alpha}\frac{\partial H}{\partial u}(\frac{\phi_{i+1}-\phi_{i-1}}{2h})\biggr), \qquad i = 2, \cdots n-2. \tag{3.11}$$

It follows that

(i). $b_i + c_i = 1$ for $i = 2, \cdots, n-2$;

(ii). $0 < b_i < 1$ and $0 < c_i < 1$ for all i, $i = 1, \cdots, n-1$,

since b_i and c_i can be written as $b_i = (1-\beta_i)/2$ and $c_i = (1+\beta_i)/2$, where $\beta_i = \frac{1}{\alpha}(\frac{\partial H}{\partial p})_i$. Condition (3.6) ensures that $|\beta_i| < 1$. Consequently, (i) and (ii) above follow immediately. In addition, the matrix $\partial_\phi F$ with the properties (i) and (ii) is irreducible since its associated graph is strongly connected [13]. Therefore, the matrix $\partial_\phi F$ satisfies the conditions of Theorem 6.1.10, p. 224 in [13] so that $\rho(\partial_\phi F) < 1$. □

Remark. It can be proved that the spectral radius of the Jacobian of Gauss-Seidel iterative procedure is smaller than the spectral radius of the corresponding Jacobi iteration if the latter is smaller than one [13]. Thus Gauss-Seidel procedure enjoys better convergence than its Jacobi counterpart.

For the two-dimensional Lax-Friedrichs fast sweeping method, we present a more general result.

Theorem. 3.1 *The nonlinear Jacobi iteration defined from the two-dimensional Lax-Friedrichs monotone Hamiltonian can be written as*

$$\phi_{i,j}^{(k+1)} = \left(\frac{1}{\frac{\alpha_x}{h_x} + \frac{\alpha_y}{h_y}}\right)\left[f_{ij} - H\left(\frac{\phi_{i+1,j}^{(k)} - \phi_{i-1,j}^{(k)}}{2h_x}, \frac{\phi_{i,j+1}^{(k)} - \phi_{i,j-1}^{(k)}}{2h_y}\right)\right.$$
$$\left. + \alpha_x \frac{\phi_{i+1,j}^{(k)} + \phi_{i-1,j}^{(k)}}{2h_x} + \alpha_y \frac{\phi_{i,j+1}^{(k)} + \phi_{i,j-1}^{(k)}}{2h_y}\right], \tag{3.12}$$

where α_x and α_y are the artificial viscosities satisfying (2.15). *This iteration is a global contraction for a suitable matrix norm $||\cdot||$ for any discretization of the form* (2.1). *In addition, this iterative procedure converges to the unique solution ϕ^* of the nonlinear system of discrete* *equations for any initial data $\phi^{(0)}$ (different from the exact solution), and the following estimate holds*

$$||\phi^{(k)} - \phi^{(*)}|| \leq \frac{\nu^k}{1-\nu}||\phi^{(1)} - \phi^{(0)}||$$

provided $\nu < 1$, where ν is an upper bound of the spectral radius of the Jacobian of the iteration function.

Proof. We denote by F_{ij} the (i,j) component of the iteration function of the nonlinear Jacobi procedure (3.12) with $h_x = h_y = h$,

$$\begin{aligned} F_{i,j} =& \frac{h}{\alpha_x + \alpha_y}\left[f_{ij} - H\left(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h}\right)\right. \\ &\left. + \alpha_x \frac{\phi_{i+1,j} + \phi_{i-1,j}}{2h} + \alpha_y \frac{\phi_{i,j+1} + \phi_{i,j-1}}{2h}\right] \end{aligned}$$

which is a function that depends only on four arguments $\phi_{i-1,j}$, $\phi_{i+1,j}$, $\phi_{i,j-1}$, and $\phi_{i,j+1}$.

To calculate the Jacobian matrix of F, $\frac{\partial F}{\partial \phi}$, we order the components of F and the arguments in row vector form like

$$\begin{aligned} F &= (F_{11}, F_{12}, \cdots, F_{1n}, F_{21}, F_{22}, \cdots, F_{2n}, \cdots, F_{n1}, F_{n2}, \cdots, F_{nn}), \\ \phi &= (\phi_{11}, \phi_{12}, \cdots, \phi_{1n}, \phi_{21}, \phi_{22}, \cdots, \phi_{2n}, \cdots, \phi_{n1}, \phi_{n2}, \cdots, \phi_{nn}). \end{aligned}$$

The Jacobian matrix can be written as an $n^2 \times n^2$ matrix where the row corresponding to the index (i,j) is the partial derivatives of F_{ij} with respect to the variables $\phi_{11}, \phi_{12}, \cdots, \phi_{1n}, \phi_{21}, \phi_{22}, \cdots, \phi_{2n}, \cdots, \phi_{n1}, \phi_{n2}, \cdots, \phi_{nn}$.

All the entries of the row are zero except four of them that correspond to the partial derivatives of $F_{ij} = F_{ij}(\phi_{i-1,j}, \phi_{i+1,j}, \phi_{i,j-1}, \phi_{i,j+1})$,

$$a_l = \frac{\partial F_{ij}}{\partial \phi_{i-1,j}} = \frac{\lambda_a}{2}\Big(1 + \frac{H_1}{\alpha_x}\Big), \quad a_r = \frac{\partial F_{ij}}{\partial \phi_{i+1,j}} = \frac{\lambda_a}{2}\Big(1 - \frac{H_1}{\alpha_x}\Big),$$

$$b_l = \frac{\partial F_{ij}}{\partial \phi_{i,j-1}} = \frac{\lambda_b}{2}\Big(1 + \frac{H_2}{\alpha_y}\Big), \quad b_r = \frac{\partial F_{ij}}{\partial \phi_{i,j+1}} \frac{\lambda_b}{2} = \Big(1 - \frac{H_2}{\alpha_y}\Big),$$

where $\lambda_a = \frac{\alpha_x}{\alpha_x + \alpha_y}$, $\lambda_b = \frac{\alpha_y}{\alpha_x + \alpha_y}$, and

$$H_1 = H_1\Big(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h}\Big), \qquad H_2 = H_2\Big(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h}, \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h}\Big).$$

Here $H_1(u,v) = \partial H(u,v)/\partial u$ and $H_2(u,v) = \partial H(u,v)/\partial v$. The choice of the viscosities α_x and α_y ensures that a_l, a_r, b_l and b_r are positive and $a_l + a_r + b_l + b_r = 1$. For a non-empty set of rows some of these values might be zero due to the presence of either homogeneous Neumann or Dirichlet boundary condition on the internal boundary Γ. Therefore, the sums of the row entries in those cases are strictly smaller than one. On the other hand, it is easy to see that the Jacobian matrix is irreducible since every row has at least a nonzero entry. Thus, applying Theorem 6.1.10, p. 224, from [13] we have that the spectral radius of the Jacobian matrix is strictly smaller than one, i.e. $\rho(\partial_\phi(F)) < 1$. This shows that any fixed point of F is a point of attraction.

Since the spectral radius $\rho(\partial_\phi(F))$ is uniformly bounded by a number $\nu < 1$ (this bound depends on the number n and the boundary condition on the curve Γ), the Banach Fixed Point Theorem ([4], page 39) applies. Consequently there exists a unique solution of the system of the discrete equations and the stated estimate holds. □

3.2. A consistent stopping criterion for fast sweeping methods

When using a higher-order sweeping method we need a stopping criterion that behaves consistently with the iterative process. Next we propose a stopping criterion based on the above convergence analysis.

Consider the vector mean value theorem [13] that for any smooth iteration function F reads as

$$||F(\phi^k) - F(\phi^{k-1})|| \leq \sup_{0 \leq t \leq 1} ||\partial_\phi F(\phi^k + t(\phi^{k-1} - \phi^k))|| \cdot ||\phi^{k-1} - \phi^k|| \tag{3.13}$$

Considering as iteration function F the one determined by the system of discrete equations using a monotone numerical Hamiltonian we have

$$\phi^{k+1} = F(\phi^k).$$

Then, since

$$||F(\phi^k) - F(\phi^{k-1})|| = ||\phi^{k+1} - \phi^k||,$$

we obtain the following expression

$$||\phi^{k+1} - \phi^k|| \leq \sup_{0 \leq t \leq 1} ||\partial_\phi F(\phi^k + t(\phi^{k-1} - \phi^k))|| \cdot ||\phi^k - \phi^{k-1}||. \tag{3.14}$$

This expression links the distance between two consecutive iterates and the distance between the two previous ones.

As proved in the convergence Theorem the spectral radius of the Jacobian of the iteration function, $\rho(\partial_\phi F)$, is strictly smaller than one. Then if there exists an upper global bound to this spectral radius, then the iteration function is a global contraction and an appropriate norm can be chosen such that $||\partial_\phi F||$ is uniformly bounded by a number strictly smaller than one.

We consider the ratio

$$q_k = \frac{||\phi^{k+1} - \phi^k||}{||\phi^k - \phi^{k-1}||}. \tag{3.15}$$

In view of our convergence analysis a consistent stopping criterion for fast sweeping methods can be formulated as follows: the iteration stops at the first k for which $q_k \geq 1$. As soon as $q_k \geq 1$ the iteration saturates since the subsequent iterates $\phi^{(k+l)}$ $(l \geq 1)$ do not change too much due to the grid resolution and rounding errors.

This stopping criterion works properly for a suitable matrix norm that is able to estimate the norm of the Jacobian close enough to the spectral radius. To set the stopping criterion, we will use the L^1 norm as the standard for computing numerical errors in the approximation of Hamilton-Jacobi equations ([12]), though this norm might not be the optimal one to satisfy the conditions of our Theorem mentioned above.

This analysis is valid for a first-order iterative procedure. In practice, to stop the iterative procedure while using a fifth-order accurate fast sweeping method we will use a minor modification of the stopping criterion proposed above. We compute

$$q_k = \frac{||\phi^{(k+1)} - \phi^{(k)}||_{L^1} + h^p}{||\phi^{(k)} - \phi^{(k-1)}||_{L^1}} \qquad \text{for} \qquad k \geq 1 \tag{3.16}$$

starting from an initial guess $\phi^{(0)}$, where p is the order of accuracy of the method. We perform iterations until $q_k > 1$. In our case we will use $p = 5$.

The modification based on including the term h^p allows the algorithm to stop when the distance between two consecutive iterates is much smaller than the truncation error of the method.

4. Numerical Experiments

In this section we present numerical results obtained by the proposed fifth-order accurate Godunov and Lax-Friedrichs fast sweeping schemes. In these examples we compute numerical errors, convergence orders, and the number of iterations needed to converge using the stopping criterion proposed in Section 3.2. One iteration count includes four alternating sweepings. We compute the numerical errors using the L^1 and L^∞-norms.

As proposed in [27] for the implementation of the high-order Godunov fast sweeping scheme we use the first-order Godunov fast sweeping method to provide a good initial guess for the fast convergence of the Gauss-Seidel iterations. On the other hand, to ensure that the solution at each grid point is monotonically decreasing from an initially assigned large value, $\phi_{i,j}^{new} = \min(\phi_{i,j}^{old}, \bar{\phi})$ is enforced for the first order fast sweeping method [28].

As pointed out in [27] since the solution of the nonlinear HJ equations are in general not smooth, the high-order accuracy may not be achieved at singularities. Thus to observe the high-order convergence of the numerical scheme we assign exact solutions to a fixed local domain around singularities on the boundary Γ when we refine the mesh in the same way as proposed in [16, 17, 27].

When there are no other prescribed boundary conditions, we use linear extrapolation at boundary points if the information is flowing out of the boundary. In other cases we use high-order extrapolation at boundary points.

As remarked in [27] the main advantage of high order sweeping methods with respect to the first-order ones is that the computational cost to achieve the same accuracy is reduced for the high-order case.

Table 4.1 Example 1: First order approximation. Smooth solution. Godunov numerical Hamiltonian

Mesh	L^1 error	Order	L^∞ error	Order	Iter
40×40	1.46 E-1		6.33 E-2		2
80×80	8.64 E-2	0.75	3.54 E-2	0.83	2
160×160	4.66 E-2	0.89	1.86 E-2	0.92	2
320×320	2.41 E-2	0.95	9.58 E-3	0.95	2

Table 4.2 Example 1: Smooth solution. Godunov numerical Hamiltonian

Mesh	L^1 error	Order	L^∞ error	Order	Iter
40×40	3.51 E-7		2.65 E-7		34
80×80	1.33 E-8	4.71	9.49 E-9	4.81	41
160×160	4.91 E-10	4.77	5.26 E-10	4.17	63
320×320	1.48 E-11	5.05	4.12 E-12	5.98	98
640×640	4.67 E-13	4.98	1.85 E-13	5.49	162

Example 1. We consider the eikonal equation (2.9) with

$$f(x,y) = \frac{\pi}{2}\sqrt{\sin^2(\pi + \frac{\pi}{2}x) + \sin^2(\pi + \frac{\pi}{2}y)}, \tag{4.1}$$

where Γ is the point $(0,0)$, and the computational domain is $[-1,1]\times[-1,1]$. The exact solution for this problem is

$$\phi(x,y) = \cos(\pi + \frac{\pi}{2}x) + \cos(\pi + \frac{\pi}{2}y). \tag{4.2}$$

Table 4.3 Example 2: Smooth region away from the singularity. Godunov numerical Hamiltonian

Mesh	L^1 error	Order	L^∞ error	Order	Iter
80×80	4.41 E-8		8.48 E-7		32
160×160	1.95 E-9	4.49	3.85 E-8	4.45	49
320×320	6.86 E-11	4.83	1.48 E-9	4.69	74
640×640	2.14 E-12	4.99	4.90 E-11	4.92	105

Table 4.4 Example 2: Whole domain. Godunov numerical Hamiltonian

Mesh	L^1 error	Order	L^∞ error	Order	Iter
80×80	5.81 E-6		4.09 E-3		32
160×160	7.27 E-7	2.99	2.04 E-3	1.00	49
320×320	9.09 E-8	2.99	1.02 E-3	1.00	74
640×640	1.13 E-8	3.00	5.11 E-4	1.00	105

We use the Godunov numerical Hamiltonian to solve this problem. As a validation of the proposed stopping criterion we solve this example using the first-order Godunov Hamiltonian. The results are displayed in Table 4.1 where we observe good behavior of the algorithm. Table 4.2 shows results for the fifth-order Godunov fast sweeping method, where we see that fifth-order accuracy is achieved.

Example 2. In this problem we consider the eikonal equation (2.9) with $f(x, y) = 1$. The computational domain is $\Omega = [-1, 1] \times [-1, 1]$, and Γ is a circle of center $(0, 0)$ and radius 0.5. The exact solution is the distance function to the circle Γ. We compute the solution using the fifth-order version of the Godunov Hamiltonian. Since the solution has a singularity at the center of the circle we show the numerical errors and orders of convergence in the smooth region (0.15 distance away from the center) and in the whole domain in two Tables 4.3 and 4.4, respectively. We observe that the fifth-order accuracy is achieved in the smooth regions of the domain and it drops to third-order accuracy in the L^1-norm when considering the whole domain.

Example 3. The eikonal equation (2.9) with $f(x, y) = 1$. In this case the computational domain is $\Omega = [-3, 3] \times [-3, 3]$; Γ consists of two circles of equal radius 0.5 with centers located at $(-1, 0)$ and $(\sqrt{1.5}, 0)$, respectively. The exact solution is the distance function to Γ. The singular set for the solution is composed of the center of each circle and the line that is equally distant to the two circles.

We use the fifth-order fast sweeping method based on the Godunov Hamiltonian. We compute errors in the smooth region excluding a small region of 0.15 distance away from the singular set. The results for the smooth region are shown in Table 4.5, where we observe that the fifth-order accuracy is reached. Table 4.6 shows numerical errors computed in the whole domain where the scheme achieves only first-order accuracy.

Example 4. The problem consists of the eikonal equation (2.9) with

$$f(x, y) = 2\pi\sqrt{[\cos(2\pi x)\sin(2\pi y)]^2 + [\sin(2\pi x)\cos(2\pi y)]^2}. \tag{4.3}$$

and $\Gamma = \{(\frac{1}{4}, \frac{1}{4}), (\frac{3}{4}, \frac{3}{4}), (\frac{1}{4}, \frac{3}{4}), (\frac{3}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{2})\}$, five isolated points. The computational domain

Table 4.5 Example 3: Smooth region away from the singularities. Godunov numerical Hamiltonian

Mesh	L^1 error	Order	L^∞ error	Order	Iter
80×80	6.04 E-4		1.49 E-3		51
160×160	1.24 E-5	5.59	1.24 E-5	6.90	72
320×320	5.73 E-7	4.44	3.39 E-6	1.87	88
640×640	6.06 E-9	6.56	3.66 E-8	6.53	125

Table 4.6 Example 3: Whole region. Godunov numerical Hamiltonian

Mesh	L^1 error	Order	L^∞ error	Order	Iter
80×80	9.61 E-4		3.44 E-2		51
160×160	3.12 E-4	1.62	1.47 E-2	1.22	72
320×320	7.63 E-5	2.03	7.34 E-3	1.00	88
640×640	1.87 E-5	2.02	3.65 E-3	1.00	125

$\Omega = [0,1] \times [0,1]$. $\phi(x,y) = 0$ is prescribed at the boundary of the unit square, and

$$g\left(\frac{1}{4}, \frac{1}{4}\right) = g\left(\frac{3}{4}, \frac{3}{4}\right) = 1, \quad g\left(\frac{1}{4}, \frac{3}{4}\right) = g\left(\frac{3}{4}, \frac{1}{4}\right) = -1, \quad g\left(\frac{1}{2}, \frac{1}{2}\right) = 0.$$

The exact solution is

$$\phi(x,y) = \sin(2\pi x)\sin(2\pi y),$$

a smooth function.

In this example we compare results for both Godunov and Lax-Friedrichs fast sweeping methods. The initial guess for the Godunov case is generated by the first-order Godunov sweeping method. For the Lax-Friedrichs case the initial guess is a big constant value.

We apply both high-order Godunov and Lax-Friedrichs fast sweeping schemes. Table 4.7 and Table 4.8 show results for Godunov and Lax-Friedrichs fast sweeping methods, respectively. Both methods achieve fifth-order accuracy. We observe that the Godunov fast-sweeping method needs fewer iterations than the high-order Lax-Friedrichs fast sweeping method. Such phenomenon was already observed for third-order sweeping in [27].

Example 5. (Travel-time problem in elastic wave propagation). The quasi-P slowness surface is defined by the quadratic equation [15],

$$c_1\phi_x^4 + c_2\phi_x^2\phi_y^2 + c_3\phi_y^4 + c_4\phi_x^2 + c_5\phi_y^2 + 1 = 0, \tag{4.4}$$

where

$$\begin{aligned} &c_1 = a_{11}a_{44}, \quad c_2 = a_{11}a_{33} + a_{44}^2 - (a_{13} + a_{44})^2, \quad c_3 = a_{33}a_{44}, \\ &c_4 = -(a_{11} + a_{44}), \quad c_5 = -(a_{33} + a_{44}) \end{aligned}$$

and a_{ij}s are given elastic parameters.

The corresponding quasi-P wave eikonal equation is

$$\sqrt{-\frac{1}{2}(c_4\phi_x^2 + c_5\phi_y^2) + \sqrt{\frac{1}{4}(c_4\phi_x^2 + c_5\phi_y^2)^2 - (c_1\phi_x^4 + c_2\phi_x^2\phi_y^2 + c_3\phi_y^4)}} = 1, \tag{4.5}$$

Table 4.7 Example 4. Godunov numerical Hamiltonian

Mesh	L^1 error	Order	Iter
80 × 80	1.27 E-5		20
160 × 160	5.24 E-7	4.60	25
320 × 320	1.28 E-8	5.36	44
640 × 640	3.45 E-10	5.20	90

Table 4.8 Example 4. Lax-Friedrichs numerical Hamiltonian

Mesh	L^1 error	Order	Iter
80 × 80	1.06 E-5		34
160 × 160	3.06 E-7	5.11	42
320 × 320	9.27 E-9	5.04	96
640 × 640	1.09 E-10	6.40	100

Table 4.9 Example 5. Lax-Friedrichs numerical Hamiltonian, $\alpha_x = \alpha_y = 5$

Mesh	L^1 error	Order	L^∞ error	Order	Iter
40 × 40	9.70 E-6	–	3.28 E-4	–	33
80 × 80	6.31 E-7	3.94	2.62 E-5	3.65	53
160 × 160	2.62 E-8	4.59	1.89 E-6	3.79	91
320 × 320	2.20 E-9	3.58	1.77 E-7	3.42	162
640 × 640	1.01 E-10	4.45	1.53 E-8	3.53	223

which is a convex Hamilton-Jacobi equation. The elastic parameters are taken to be

$$a_{11} = 15.0638, \quad a_{33} = 10.8373, \quad a_{13} = 1.6381, \quad a_{44} = 3.1258$$

The computational domain is $[-1,1] \times [-1,1]$, and $\Gamma = \{(0,0)\}$. The quasi-P wave travel-time problem is smooth in the whole domain except at the source point. Initial values are assigned in a box with length 0.3 which includes the source point. To initialize a fixed box around the source point, we use a shooting method by solving a two-point boundary value problem; see [16] for details. We apply the fifth-order Lax-Friedrichs fast sweeping method to this problem. Table 4.9 shows errors and convergence rates for this problem. We observe that the convergence is not uniform when refining the grid and the proposed algorithm does not achieve ideal fifth-order accuracy; this might be due to the strong anisotropy present in the model.

5. Concluding Remarks

We propose a fifth-order weighted PowerENO sweeping scheme for static Hamilton-Jacobi equations with convex Hamiltonians. To design an effective stopping criterion for the higher order sweeping scheme, we analyze the convergence of the first-order Lax-Friedrichs sweeping scheme by using the theory of nonlinear iteration. The resulting stopping criterion is based on ratios of three consecutive iterations. Numerical examples validate the fifth-order accuracy of the new high-order sweeping scheme and the effectiveness of the new stopping criterion.

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