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# LOCALLY STABILIZED FINITE ELEMENT METHOD FOR STOKES PROBLEM WITH NONLINEAR SLIP BOUNDARY CONDITIONS*

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**Abstract**

Based on the low-order conforming finite element subspace $(V_h, M_h)$ such as the $P_1$-$P_0$ triangle element or the $Q_1$-$P_0$ quadrilateral element, the locally stabilized finite element method for the Stokes problem with nonlinear slip boundary conditions is investigated in this paper. For this class of nonlinear slip boundary conditions including the subdifferential property, the weak variational formulation associated with the Stokes problem is an variational inequality. Since $(V_h, M_h)$ does not satisfy the discrete inf-sup conditions, a macroelement condition is introduced for constructing the locally stabilized formulation such that the stability of $(V_h, M_h)$ is established. Under these conditions, we obtain the $H^1$ and $L^2$ error estimates for the numerical solutions.

*Mathematics subject classification:* 35Q30.
*Key words:* Stokes Problem, Nonlinear Slip Boundary, Variational Inequality, Local Stabilized Finite Element Method, Error Estimate.

## 1. Introduction

Numerical simulation for the incompressible flow is the fundamental and significant problem in computational mathematics and computational fluid mechanics. It is well known that the mathematical model of viscous incompressible fluid with homogeneous boundary conditions is the Navier-Stokes equations which can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \nabla p = f & \text{in } Q_T, \\ \text{div} u = 0 & \text{in } Q_T \\ u(0) = u_0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0, T], \end{cases}$$

where $Q_T = (0, T] \times \Omega$, $0 < T \leq +\infty$, $\Omega \subset \mathbb{R}^n, n = 2, 3$, is a bounded convex domain; $u(t, x)$ and $f(t, x)$ are vector functions representing the flow velocity and the external force, respectively; $p(t, x)$ is a scalar function representing the pressure. The viscous coefficient $\mu > 0$ is a positive constant. The solenoidal condition means that the fluid is incompressible.

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Note that the velocity $u$ and the pressure $p$ are coupled by the solenoidal condition $\mathrm{div} u = 0$ which makes that it is difficult to solve the Navier-Stokes equations. Some popular techniques to overcome this difficulty are to relax the solenoidal condition in an appropriate way which leads to a pesudo-compressible system, such as the penalty method, the artificial compressible method, the pressure stabilized method and the projection method, see, e.g., [1,2,10-14,18-21].

In this paper, we will consider Stokes problem

$$\begin{cases} -\mu\Delta u + \nabla p = f & \text{in } \Omega, \\ \mathrm{div} u = 0 & \text{in } \Omega \end{cases} \tag{1.1}$$

with the nonlinear slip boundary conditions

$$\begin{cases} u = 0, & \text{on } \Gamma, \\ u_n = 0, \quad -\sigma_\tau(u) \in g\partial|u_\tau| & \text{on } S, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded convex domain; $\Gamma \cap S = \emptyset, \overline{\Gamma \cup S} = \partial\Omega$; $g$ is a scalar function; $u_n = u \cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, with $n$ the unit vector of the external normal to $S$; $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of $p$, is the tangential components of the stress vector $\sigma$ defined by

$$\sigma_i = \sigma_i(u,p) = (\mu e_{ij}(u) - p\delta_{ij})n_j.$$

Here

$$e_{ij}(u) = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}, i, j = 1, 2.$$

The set $\partial\psi(a)$ denotes a subdifferential of the function $\psi$ at the point $a$, whose definition will be given in next section.

The boundary conditions (1.2) are introduced by Fujita in [4], who investigated some hydrodynamics problems under nonlinear boundary conditions, such as leak and slip boundary involving subdifferential property. These types of boundary conditions appear in the modeling of blood flow in a vein of an arterial sclerosis patient and in that of avalanche of water and rocks. Fujita in [5] showed the existence and uniqueness of weak solution to the Stokes problem with slip boundary conditions (1.2). Subsequently, Saito in [17] showed the regularity of the weak solution by using Yosida's regularized method and finite difference quotients method. Other theoretical results about the Stokes problems with nonlinear subdifferential boundary conditions can be found in [6-8,16]. We remark that the steady homogeneous and inhomogeneous Stokes system with linear slip boundary conditions without subdifferential property have recently been studied from the theoretical view point by Veiga in [22-24].

The aim of this paper is to extend the locally pressure stabilized finite element method, which is introduced by Kechkar & Silvester in [14] and developed by He *et al.* for the Navier-Stokes equations in [10-13], and to the problem (1.1)-(1.2). This method bases on the lower order conforming finite element subspace $(V_h, M_h)$ such as $P_1$-$P_0$ triangle element (linear velocity, constant pressure) or the $Q_1$-$P_0$ quadrilateral element (bilinear velocity, constant pressure). Since $(V_h, M_h)$ does not satisfy the discrete inf-sup conditions, a macroelement condition is introduced for constructing the locally stabilized formulation such that the stability of $(V_h, M_h)$ is established. Under these conditions, we show that if the true solution $(u,p) \in H^2(\Omega)^2 \cap V \times H^1(\Omega) \cap M$, then the following $H^1$ and $L^2$ error estimates hold:

$$\|u - u_h\|_V + \|p - p_h\| \le ch^{\frac{1}{2}}, \tag{1.3}$$

$$\|u - u_h\| \le ch^{\frac{3}{2}}, \tag{1.4}$$

which are not optimal and are similar to the error estimates for the flow of Bingham fluid (see, e.g., [9,15]).

This paper is organized as follows: in next section, we will introduce some function spaces and describe the well-posedness of the weak solution to the problem (1.1)-(1.2). The locally stabilized finite element method and the relevant error estimates will be given in the last two sections.

## 2. Stokes Problem with Nonlinear Slip Boundary Conditions

Firstly, we give the definition of the subdifferential property (see, e.g., [3]). Let $\psi : \mathbb{R}^2 \to \overline{\mathbb{R}} = (-\infty, +\infty]$ be a given function possessing the properties of convexity and weak semi-continuity from below ($\psi$ is not identical with $+\infty$). We say that the set $\partial\psi(a)$ is a subdifferential of the function $\psi$ at the point $a$ if

$$\partial\psi(a) = \Big\{ b \in \mathbb{R}^2 : \psi(h) - \psi(a) \geq b \cdot (h-a), \quad \forall\, h \in \mathbb{R}^2 \Big\}.$$

We introduce some spaces which are usually used in this paper. Denote

$$V = \Big\{ u \in H^1(\Omega)^2,\ u|_\Gamma = 0,\ u\cdot n|_S = 0 \Big\}, \quad V_0 = H_0^1(\Omega)^2,$$
$$H = \Big\{ u \in L^2(\Omega)^2,\ \mathrm{div} u = 0,\ u \cdot n|_{\partial\Omega} = 0 \Big\}, \quad M = L_0^2(\Omega) = \Big\{ q \in L^2(\Omega), \int_\Omega q \mathrm{d}x = 0 \Big\}.$$

Let $||\cdot||_k$ be the norm in Hilbert space $H^k(\Omega)^2$. Let $(\cdot,\cdot)$ and $||\cdot||$ be the inner product and the norm in $L^2(\Omega)^2$. Then we can equip the inner product and the norm in $V$ by $(\nabla\cdot, \nabla\cdot)$ and $||\cdot||_V = ||\nabla\cdot||$, respectively, because $||\nabla\cdot||$ is equivalent to $||\cdot||_1$. Let $\mathbb{X}$ be the Banach space. Denote $\mathbb{X}'$ the dual space of $\mathbb{X}$ and $<\cdot,\cdot>$ be the dual pairing in $\mathbb{X}\times\mathbb{X}'$. Introduce the following bilinear forms

$$\begin{cases} a(u,v) = \mu(\nabla u, \nabla v) & \forall\, u, v \in V, \\ b(v,p) = (p, \mathrm{div} v) & \forall\, v \in V, p \in M. \end{cases}$$

The weak formulation associated with problem (1.1)-(1.2) is the following variational inequality:

$$\begin{cases} \text{Find } (u,p) \in V \times M \text{ such that} \\ a(u, v-u) + j(v_\tau) - j(u_\tau) - b(v-u, p) \geq (f, v-u) \qquad \forall\, v \in V, \\ b(u,q) = 0 \qquad \forall\, q \in M, \end{cases} \tag{2.1}$$

where $j(\eta) = \int_S g|\eta| \mathrm{d}s$. For the variational inequality (2.1) and problem (1.1)-(1.2), we have

**Theorem 2.1.** *If* $(u,p)$ *is the solution of* (1.1)-(1.2)*, then it satisfies the variational inequality* (2.1)*. Conversely, if the solution* $(u,p)$ *of the variational inequality* (2.1) *is sufficiently smooth, then it also satisfies* (1.1)-(1.2).

*Proof.* If $(u,p)$ satisfies the problem (1.1)-(1.2) for all $v \in V$, then multiplying the first equation in (1.1) by $v-u$ and integrating over $\Omega$ yield

$$a(u, v-u) - b(v-u, p) - \int_S \sigma_n \cdot (v-u) \mathrm{d}s = (f, v-u).$$

Since

$$\sigma_n = \sigma^{ij} n_j, \qquad v - u = (v_n - u_n)n + (v_\tau - u_\tau),$$

we have

$$\begin{aligned}\int_S \sigma_n \cdot (v-u)\mathrm{d}s &= \int_S \sigma^{ij} n_j (v_n - u_n) n_i + \sigma_n \cdot (v_\tau - u_\tau)\mathrm{d}s \\ &= \int_S \sigma^{ij} n_j n_i (v_n - u_n) + \sigma_n \cdot (v_\tau - u_\tau)\mathrm{d}s \\ &= \int_S (\sigma_n \cdot n)(v_n - u_n)\mathrm{d}s + \sigma_n \cdot (v_\tau - u_\tau)\mathrm{d}s. \end{aligned} \tag{2.2}$$

Observe that $u_n = v_n = 0$ on $S$. Thus we have

$$\int_S \sigma_n \cdot (v-u)\mathrm{d}s = \int_S \sigma_n \cdot (v_\tau - u_\tau)\mathrm{d}s. \tag{2.3}$$

From the definition of the differential, we obtain

$$g|v_\tau| - g|u_\tau| \geq -\sigma_n \cdot (v_\tau - u_\tau) \quad \text{on } S, \tag{2.4}$$

which gives the variational inequality (2.1). Next, we show that if the solution $(u,p)$ is sufficiently smooth, then it also satisfies (1.1)-(1.2). For all $w \in C_0^\infty(\Omega)$, let $v = u \pm w$ in (2.1). Then we have

$$a(u,w) - b(w,p) = (f,w).$$

Integrating by parts for the above equation gives the first equation in (1.1). Using integration by parts again in (2.1), we have

$$(-\mu\Delta u + \nabla p - f, v-u) + \int_S \sigma_n \cdot \tau(v_\tau - u_\tau)\mathrm{d}s + j(v_\tau) - j(u_\tau) \geq 0. \tag{2.5}$$

According to the first equation in (1.1), we obtain

$$\int_S g|v_\tau| - g|u_\tau|\mathrm{d}s \geq -\int_S \sigma_n \cdot (v_\tau - u_\tau)\mathrm{d}s, \tag{2.6}$$

which gives the nonslip boundary condition (1.2). □

Define the bilinear form $B : (V,M) \times (V,M) \longrightarrow \mathbb{R}$ by

$$B(u,p;v,q) = a(u,v) - b(v,p) + b(u,q). \tag{2.7}$$

It is well-known that for all $(u,p),(v,q) \in (V,M)$, the bilinear form $B$ satisfies the following stability property:

$$B(u,p;u,p) = \mu\|u\|_V^2, \tag{2.8a}$$

$$|B(u,p;v,q)| \leq \gamma_0\Big(\|u\|_V + \|p\|)(\|v\|_V + \|q\|\Big), \tag{2.8b}$$

$$\alpha_0(\|u\|_V + \|p\|) \leq \sup_{(v,q)\in(V,M)} \frac{B(u,p;v,q)}{\|v\|_V + \|q\|}, \tag{2.8c}$$

where $\gamma_0 > 0$ and $\alpha_0 > 0$ are some constants. Introduce the operators $J : (V,M) \longrightarrow \mathbb{R}$ and $F : (V,M) \longrightarrow \mathbb{R}$ by

$$J(u,p) = j(u), \qquad (F,(v,q)) = (f,v).$$

Under above notions, the variational inequality (2.1) reads as follows:

$$\left\{\begin{array}{l}\text{Find } (u,p) \in (V,M) \text{ such that} \\ B(u,p;v-u,q-p) + J(v_\tau,q) - J(u_\tau,p) \geq (F,(v-u,q-p)) \quad \forall\ (u,q) \in (V,M).\end{array}\right. \tag{2.9}$$

## 3. Locally Stabilized Finite Element Approximation

In this section,we will give the locally stabilized finite element method for problem (1.1)-(1.2). Let $\tau_h$ be a family of regular triangular partition (or quadrilateral partition) of $\Omega$ into triangles (or quadrilaterals) of diameter not great than $0 < h < 1$. Moreover, assume that $\tau_h$ is regular, i.e., there exists two positive constant $\sigma$ and $\omega$ with $\sigma > 1$ and $0 < \omega < 1$ such that

$$\begin{aligned} &h_K \le \sigma\rho_K, \qquad \forall\ K \in \tau_h, \\ &|\cos\theta_{iK}| \le \omega, \quad i = 1, 2, 3, 4, \quad \forall\ K \in \tau_h, \end{aligned}$$

where $h_K$ is the diameter of element $K$, $\rho_K$ is the diameter of the inscribed circle of element $K$, and $\theta_{iK}$ are the angles of $K$ in the case a quadrilateral partitioning. The mesh parameter $h$ is given by $h = \max\{h_K\}$, and the set of all interelement boundaries will be denoted by $\Gamma_h$.

Introduce the finite element space:

$$R_1(K) = \begin{cases} P_1(K) & \text{if } K \text{ is triangular}, \\ Q_1(K) & \text{if } K \text{ is quadrilateral}. \end{cases}$$

Then the finite element subspaces of $V$ and $M$ in this paper are defined by

$$\begin{aligned} V_h &= \Big\{v \in V \ :\ v|_K \in R_1(K) \quad \forall\ K \in \tau_h\Big\}, \\ P_h &= \Big\{q \in M \ :\ q|_K \in P_0(K) \quad \forall\ K \in \tau_h\Big\}. \end{aligned}$$

Note that the finite element spaces $V_h$ and $P_h$ are not stable in the standard Babuska-Brezzi sense. In order to define a locally stabilized formulation of the problem (2.1), we introduce the macroelement partitioning $\Lambda_h$ in [14]. Given any subdivision $\tau_h$, a macroelement partitioning $\Lambda_h$ may be defined such that each macroelement $M$ is a connected set of adjoining elements from $\tau_h$. Every element $K$ must lie in exactly one macroelement, which implies that macroelement do not overlap. For each $M$, the set of interelement edges, which are strictly in the interior of $M$, will be denoted by $\Gamma_M$, and the length of an edge $e \in \Gamma_M$ is denoted by $h_e$.

With these addition definition, we can define the locally stabilized finite element formulation of (2.1) as follows:

$$\begin{cases} \text{Find } (u_h, p_h) \in V_h \times M_h \text{ such that} \\ a(u_h, v_h - u_h) - b(v_h - u_h, p_h) + j(v_{h\tau}) - j(u_{h\tau}) \ge (f, v_h - u_h) \quad \forall\ v_h \in V_h, \\ b(u_h, q_h) + \beta C_h(p_h, q_h) = 0 \quad \forall\ q_h \in M_h, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} M_h &= \Big\{q_h \in P_h \cap L_0^2(M) \quad \forall\ M \in \Lambda_h\Big\}, \\ C_h(p_h, q_h) &= \sum_{M \in \Lambda_h} \sum_{e \in \Gamma_M} h_e \int_e [p_h]_e [q_h]_e \qquad \forall\ p_h, q_h \in M_h, \end{aligned}$$

and $[\cdot]_e$ is the jump operator across $e \in \Gamma_M$ and $\beta > 0$ is the locally stabilized parameter. In [14], Kechkar & Silvester proved that there exist two positive constants $\alpha_1$ and $\alpha_2$, independent of $h$, such that

$$|C_h(p_h, q_h)| \le \alpha_1 \|p_h\| \|q_h\| \qquad \forall\ p_h, q_h \in M_h \tag{3.2}$$

$$C_h(p_h, p_h) \ge \alpha_2 \|p_h\|^2 \qquad \forall\ p_h \in M_h. \tag{3.3}$$

If we denote

$$B_h(u_h, p_h; v_h - u_h, q_h) = a(u_h, v_h) - b(v_h, p_h) + b(u_h, q_h) + \beta C_h(p_h, q_h),$$

then the locally stabilized formulation (3.1) can be written as

$$\begin{cases} \text{Find } (u_h, p_h) \in V_h \times M_h \text{ such that for all } (v_h, q_h) \in V_h \times M_h, \\ B_h(u_h, p_h; v_h - u_h, q_h - p_h) + J(v_{h\tau}, q_h) - J(u_{h\tau}, p_h) \geq (F, (v_h - u_h, q_h - p_h)). \end{cases} \tag{3.4}$$

A general framework for analyzing the locally stabilized formulation (3.1) or (3.4) can be developed using the notion of equivalence class of macroelements. As in Stenberg in [19], each equivalence class, denoted by $\mathcal{E}_{\hat{M}}$, containing macroelements which are topologically equivalent to a reference macroelement $\hat{M}$.

The following stability theorem of these mixed methods for macroelement partitioning defined above was established by Kechkar and Silvester [14].

**Theorem 3.1.** *Given a stabilization parameter $\beta \geq \beta_0 > 0$, suppose that every macroelement $M \in \Lambda_h$ belongs to one of the equivalence classes $\mathcal{E}_{\widehat{M}}$, and that the following macroelement connectivity condition is valid: for any two neighboring macroelement $M_2$ and $M_2$ with $\int_{M_1 \bigcap M_2} ds \neq 0$, there exists $v_h \in V_h$ such that*

$$supp\ v_h \subset M_1 \cup M_2 \quad \text{and} \quad \int_{\mathrm{M_1 \cap M_2}} \mathrm{v_h \cdot n} \neq 0.$$

*Then*

$$\begin{aligned} &|B_h(u_h, p_h; v_h, q_h)| \leq \gamma_1 \Big( \|u_h\|_V + \|p_h\|)(\|v_h\|_V + \|q_h\| \Big) && \forall\ (u_h, p_h), (v_h, q_h) \in (V_h, M_h), \\ &\gamma_2(\|u_h\|_V + \|p_h\|) \leq \sup_{(v_h, q_h) \in (V_h, M_h)} \frac{B_h(u_h, p_h; v_h, q_h)}{\|v_h\|_V + \|q_h\|} && \forall\ (u_h, p_h) \in (V_h, M_h), \\ &|C_h(p, q_h)| = 0, && \forall\ p \in H^1(\Omega) \cap M, q_h \in M_h, \end{aligned}$$

*where $\gamma_1 > 0, \gamma_2 > 0$ are two constants independent of $h$ and $\beta$, $\beta_0$ is a fixed positive constant and $n$ is the outnormal vector.*

Throughout this paper, we will assume that $\beta \geq \beta_0 > 0$. For the existence and uniqueness of the solution to the discrete problem (3.1) or (3.4), we have

**Theorem 3.2.** *If $f \in H$ and $g \in L^2(S)$, then the discrete problem* (3.1) *or* (3.4) *admits a unique solution $(u_h, p_h) \in (V_h, M_h)$.*

*Proof.* By the definition of $B_h$, using (3.3) we have

$$B_h(v_h, q_h; v_h, q_h) = a(v_h, v_h) + \beta C_h(q_h, q_h) \geq \mu \|v_h\|_V^2 + \beta \alpha_2 \|q_h\|^2. \tag{3.5}$$

Hence $B_h(v_h, q_h; v_h, q_h)$ is coercive in $(V_h, M_h)$. Consequently, by the existence theorem of the solution to elliptic variational inequality of the second kind in finite dimensional space (see, e.g., [9]), we conclude that the discrete problem (3.1) or (3.4) admits a unique solution $(u_h, p_h) \in (V_h, M_h)$. □

## 4. Error Estimates

In order to obtain the error estimates, we define the Galerkin projection operators $(R_h, Q_h) : (V, M) \to (V_h, M_h)$, by

$$B_h(R_h v, Q_h q; v_h, q_h) = B_h(v, q; v_h, q_h) \quad \forall\ (v_h, q_h) \in (V_h, M_h)$$

for each $(v, q) \in (V, M)$. Using Theorem 3.1, He *et al.* [10,11] proved the following approximate property:

$$\begin{aligned} &\|v - R_h v\| + h\|v - R_h v\|_V + h\|q - Q_h q\| \\ \le & Ch^2(\|v\|_2 + \|q\|_1) \quad \forall\ v \in H^2(\Omega)^2, q \in H^1(\Omega). \end{aligned} \tag{4.1}$$

**Theorem 4.1.** *Let $(u, p) \in V \times M$ and $(u_h, p_h) \in V_h \times M_h$ be the weak solution of (2.1) and (3.1), respectively. Furthermore, if $u \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$, then we have the error estimate*

$$\|u - u_h\|_V + \|p - p_h\| \le ch^{\frac{1}{2}}, \tag{4.2}$$

*where $c > 0$ is independent of $h$.*

*Proof.* By the triangular inequality, we have

$$\begin{aligned} &\mu\|u - u_h\|_V^2 + \beta\alpha_2\|p - p_h\|^2 \\ \le & 2\mu\|u - R_h u\|_V^2 + 2\mu\|R_h u - u_h\|_V^2 + 2\beta\alpha_2\|p - Q_h p\|^2 + 2\beta\alpha_2\|Q_h p - p_h\|^2. \end{aligned} \tag{4.3}$$

It follows from (3.5), that

$$\begin{aligned} &\mu\|u_h - R_h u\|_V^2 + \beta\alpha_2\|p_h - Q_h p\|^2 \\ \le & B_h(u_h - R_h u, p_h - Q_h p; u_h - R_h u, p_h - Q_h p) \\ = & B_h(u_h, p_h; u_h - R_h u, p_h - Q_h p) - B_h(R_h u, Q_h p; u_h - R_h u, p_h - Q_h p) \\ = & a(u_h, u_h - R_h u) - b(u_h - R_h u, p_h) + b(u_h, p_h - Q_h p) \\ & \quad + \beta C_h(p_h, p_h - Q_h p) - B_h(R_h u, Q_h p; u_h - R_h u, p_h - Q_h p) \\ \le & (f, u_h - R_h u) + j((R_h u)_\tau) - j(u_{h\tau}) - B_h(R_h u, Q_h p; u_h - R_h u, p_h - Q_h p). \end{aligned} \tag{4.4}$$

Setting $v = u_h$ and $v = 2u - R_h u$ in (2.1), gives

$$\begin{aligned} &a(u, u_h - u) - b(u_h - u, p) + j(u_{h\tau}) - j(u_\tau) \ge (f, u_h - u) \\ &a(u, u - R_h u) - b(u - R_h u, p) + j((2u - R_h u)_\tau) - j(u_\tau) \ge (f, u - R_h u), \end{aligned}$$

which yields

$$a(u, u_h - R_h u) - b(u_h - R_h u, p) + j((2u - R_h u)_\tau) + j(u_{h\tau}) - 2j(u_\tau) \ge (f, u_h - R_h u).$$

Substituting the above into (4.4) yields

$$\begin{aligned}
&\mu\|u_h - R_h u\|_V^2 + \beta\alpha_2\|p_h - Q_h p\|^2\\
\le& a(u, u_h - R_h u) - b(u_h - R_h u, p) + j((2u - R_h u)_\tau) + j((R_h u)_\tau) - 2j(u_\tau)\\
&- a(R_h u, u_h - R_h u) + b(u_h - R_h u, Q_h p) - b(R_h u, p_h - Q_h p) - \beta C_h(Q_h p, p_h - Q_h p)\\
=& a(u - R_h u, u_h - R_h u) - b(u_h - R_h u, p - Q_h p) + b(u - R_h u, p_h - Q_h p)\\
&+ j((2u - R_h u)_\tau) + j((R_h u)_\tau) - 2j(u_\tau) - \beta C_h(p - Q_h p, p_h - Q_h p)\\
\le& \mu\|u - R_h u\|_V\|u_h - R_h u\|_V + \|u_h - R_h u\|_V\|p - Q_h p\| + \|u - R_h u\|_V\|p_h - Q_h p\|\\
&+ \beta\alpha_1\|p - Q_h p\|\|p_h - Q_h p\| + c\|u - R_h u\|_V\\
\le& \frac{\mu}{4}\|u_h - R_h u\|_V^2 + \mu\|u - R_h u\|_V^2 + \frac{\mu}{4}\|u_h - R_h u\|_V^2 + \frac{1}{\mu}\|p - Q_h p\|^2\\
&+ \frac{\beta\alpha_2}{4}\|p_h - Q_h p\|^2 + \frac{1}{\beta\alpha_2}\|u - R_h u\|_V^2 + \frac{\beta\alpha_2}{4}\|p_h - Q_h p\|^2\\
&+ \frac{\beta\alpha_1^2}{\alpha_2}\|p - Q_h p\|^2 + c\|u - R_h u\|_V.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{\mu}{2}\|u_h - R_h u\|_V^2 + \frac{\beta\alpha_2}{2}\|p_h - Q_h p\|^2\\
\le& \mu\|u - R_h u\|_V^2 + \frac{1}{\mu}\|p - Q_h p\|^2 + \frac{1}{\beta\alpha_2}\|u - R_h u\|_V^2 + \frac{\beta\alpha_1^2}{\alpha_2}\|p - Q_h p\|^2 + c\|u - R_h u\|_V.
\end{aligned}$$

Substituting the above inequality into (4.3) yields

$$\begin{aligned}
&\mu\|u - u_h\|_V^2 + \beta\alpha_2\|p - p_h\|^2\\
\le& 6\mu\|u - R_h u\|_V^2 + \frac{4}{\mu}\|p - Q_h p\|^2 + \frac{4}{\beta\alpha_2}\|u - R_h u\|_V^2\\
&+ \frac{4\beta\alpha_1^2}{\alpha_2}\|p - Q_h p\|^2 + 2\beta\alpha_2\|p - Q_h p\|^2 + 4c\|u - R_h u\|_V,
\end{aligned}$$

which together with the approximation property (4.1) yields

$$\mu\|u - u_h\|_V^2 + \beta\alpha_2\|p - p_h\|^2 \le ch^2 + ch \le ch.$$

This proves the desired result (4.2). □

**Remark 4.1.** We remark that the error estimate (4.2) is not optimal, which is similar to the $H^1$ error estimate for elliptic variational inequality of the second kind, see, e.g., [9,15]. The reason is that $|j(u_\tau) - j(v_\tau)| \le c\|u - v\|_V$ for some positive constant $c > 0$.

Next, we will give the $L^2$ error estimate $\|u - u_h\|$ by the Aubin-Nitsche's technique. To this end, we need the following regularity assumptions about the homogeneous Stokes problem with linear slip boundary conditions.

(**A**) *Given $u$ and $u_h$ the solutions of* (2.1) *and* (3.1), *respectively. We assume that the following linear stokes problem:*

$$\begin{cases} -\mu\triangle w + \nabla\pi = u - u_h & \text{in } \Omega,\\ divw = 0 & \text{in } \Omega,\\ w = 0 & \text{on } \Gamma,\\ w_n = 0, \quad -\sigma_\tau w = 0 & \text{on } S \end{cases} \tag{4.5}$$

*admits a unique solution* $(w,\pi) \in H^2(\Omega)^2 \cap V \times H^1(\Omega) \cap M$ *such that*

$$\|w\|_2 + \|\pi\|_1 \le c\|u - u_h\|, \tag{4.6}$$

*where* $c > 0$ *is independent of* $h$.

For results about the problem (4.5), we refer the reader to [22,24]. The weak variational formulation associated with (4.5) is

$$\begin{cases} \text{Find}(w,\pi) \in (V, M) \quad \text{such that} \\ a(w,v) - b(v,\pi) = (u - u_h, v) \qquad \forall\, v \in V, \\ b(w,q) = 0 \qquad \forall\, q \in M. \end{cases} \tag{4.7}$$

Let $w_h \in \widetilde{V_h} \subset V_0$ and $\pi_h \in M_h$ be the locally stabilized finite-element approximation solution of (4.7) which satisfies the following problem:

$$\begin{cases} a(w_h, v_h) - b(v_h, \pi_h) = (u - u_h, v_h) & \forall\, v_h \in V_h, \\ b(w_h, q_h) + \beta C_h(\pi_h, q_h) = 0 & \forall\, q_h \in M_h. \end{cases} \tag{4.8}$$

Then the following error estimate holds:

$$\|w - w_h\|_V + \|\pi - \pi_h\| \le ch\|u - u_h\|, \tag{4.9}$$

where $c > 0$ is independent of $h$.

**Theorem 4.2.** *Under the assumption (A), let* $(u,p) \in (V, M)$ *and* $(u_h, p_h) \in (V_h, M_h)$ *be the weak solution of* (2.1) *and* (3.1), *respectively. If* $u \in H^2(\Omega)^2$ *and* $p \in H^1(\Omega)$, *then we have the* $L^2$ *error estimate*

$$\|u - u_h\| \le ch^{\frac{3}{2}}, \tag{4.10}$$

*where* $c > 0$ *is independent of* $h$.

*Proof.* Setting $v = u - u_h$ in (4.7) yields

$$\begin{aligned} \|u - u_h\|^2 &= a(w, u - u_h) - b(u - u_h, \pi) \\ &= a(w - w_h, u - u_h) + a(w_h, u - u_h) - b(u - u_h, \pi - \pi_h) - b(u - u_h, \pi_h). \end{aligned}$$

Since $b(u - u_h, \pi_h) = \beta C_h(p_h, \pi_h)$ and $C_h(p, \pi_h) = 0$, we have

$$\begin{aligned} &\|u - u_h\|^2 \\ =&a(w - w_h, u - u_h) + a(w_h, u - u_h) - b(u - u_h, \pi - \pi_h) + \beta C_h(p - p_h, \pi_h - \pi). \end{aligned} \tag{4.11}$$

On the other hand, for $w \in V$ and $w_h \in \widetilde{V_h}$, setting $v = u \pm w$ in (2.1) and $v_h = u_h \pm w_h$ in (3.1) yields

$$\begin{aligned} &a(u, w) - b(w, p) = (f, w) &&\forall\, w \in V_0 \\ &a(u_h, w_h) - b(w_h, p_h) = (f, w_h) &&\forall\, w_h \in \widetilde{V_h}. \end{aligned}$$

Consequently,

$$a(u - u_h, w_h) = b(w_h, p - p_h) = b(w_h - w, p - p_h).$$

Substituting the above into (4.10) and using (4.9), we have

$$\begin{aligned}\|u-u_h\|^2 &= a(w-w_h,u-u_h)+b(w_h-w,p-p_h)\\ &\quad -b(u-u_h,\pi-\pi_h)+\beta C_h(p-p_h,\pi_h-\pi)\\ &\le \mu\|w-w_h\|_V\|u-u_h\|_V+\|w-w_h\|_V\|p-p_h\|\\ &\quad +\|u-u_h\|_V\|\pi-\pi_h\|+\beta\alpha_1\|p-p_h\|\|\pi-\pi_h\|\\ &\le ch^{\frac{1}{2}}\|w-w_h\|_V+ch^{\frac{1}{2}}\|\pi-\pi_h\|\\ &\le ch^{\frac{3}{2}}||u-u_h||.\end{aligned}$$

This completes the proof of the theorem. □

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