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ON EXTRAPOLATION CASCADIC MULTIGRID METHOD*

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Abstract

Based on an asymptotic expansion of (bi)linear finite elements, a new extrapolation formula and extrapolation cascadic multigrid method (EXCMG) are proposed. The key ingredients of the proposed methods are some new extrapolations and quadratic interpolations, which are used to provide better initial values on the refined grid. In the case of triple grids, the errors of the new initial values are analyzed in detail. The numerical experiments show that EXCMG has higher accuracy and efficiency.

Mathematics subject classification: 65N30.
Key words: Cascadic multigrid, Finite element, New extrapolation, Error analysis.

1. Introduction

To solve a linear system of equations derived by the finite difference method or finite element method for PDEs, it is expected that the solution of an N-order system can be obtained by using $\mathcal{O}(N)$. Multigrid method (MG) was first realized this purpose and was then become one of the most effective algorithms. There are two important types of MG methods:

(i) (Classical) Multigrid Method. The basic idea of MG was early proposed by Fedorenko [1] in 1964, and was re-discovered by Brandt [2] in 1977. Later on, MG has been gradually completed by Bank-Dupont, Braess-Hackbush, McCormick, Bramble-Pasciak-Xu et al., see, e.g., [3,4]. In MG, three operators between different levels of grid, i.e., the interpolation, restriction and iteration, are used. There are V-cycle and W-cycle algorithms.

(ii) Cascadic Multigrid Method (CMG). CMG was proposed by Borneman-Deunfhard [5] in 1996 and Shaidurov [6,7] (also in 1996, called Cascadic CG; 1999, discussed the domain with re-entrant corners). Shi and Xu et al. [8–10] made further analysis and extensions in 1998 and 1999. Later on, CMG has been generalized to the nonconforming elements, finite volume method, nonlinear and parabolic problems and so on, see, e.g., [11–19]. In CMG, only the interpolation and iteration from coarse grids to refined grids are used. Compare with the classical CG method, the code for CMG is simpler.

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Let Ω be a planar polygon with the boundary Γ, we consider an elliptic problem to find $u \in H_0^1$ such that

$$A(u,v) \equiv \int_\Omega \nabla u \nabla v dx = (f,v), \quad \forall v \in H_0^1 = \Big\{ u: \ u \in H^1(\Omega), \ u = 0 \ on \ \Gamma \Big\}, \tag{1.1}$$

where the bilinear form $A(u,v)$ is bounded and H_0^1-coercive, $A(u,u) \geq \nu ||u||_1^2$.

Subdivide Ω into a sequence of triangular or rectangular grids $Z_l, l = 0, 1, 2, \cdots, L$ with step-length $h_l = h_0/2^l$. Denote by $V_l \subset H_0^1$ the (bi)linear finite element subspace on the grid Z_l and by $U^l \in V_l$ the corresponding finite element solution satisfying

$$A(U^l, v) = (f, v), \quad v \in V_l, \quad l = 0, 1, \cdots, L, \tag{1.2}$$

which leads to a linear system of equation

$$K_l U^l = b_l, \quad on \ \ Z_l, \quad l = 0, 1, \cdots, L. \tag{1.3}$$

Define the linear interpolation $u^l = I_1 u$ of u, and let the error $e^l = U^l - u^l$ on Z_l. The energy error is defined by $||e^l||_{K_l} = (K_l e^l, e^l)^{1/2}$. It is known that the norms $||e^l||_{K_l}$ and $||e^l||_1$ are equivalent.

Algorithm 1.1 Assume that the exact solution $\bar{U}^0$ on Z_0 given, CMG consists of three steps, $l = 1, \cdots, L$:
Step 1. Take the linear interpolation $I_1 \bar{U}^{l-1}$ to define the initial value $U^{l,0} = I_1 \bar{U}^{l-1}$ on Z_l;
Step 2. Use the operator S_l to get the iteration solution $\bar{U}^l = S_l^{m_l} U^{l,0}$;
Step 3. Come back to steps 1 and 2 if $l < L$, until get the final solution $\bar{U}^L$ on Z_L.

In CMG, the errors are often measured by K_l-norm, $||u - U^l||_{K_l} = \mathcal{O}(h_l)$. Since the gradient $D(U^l - u^l) = \mathcal{O}(h_l^2)$ is superconvergent on the (piecewise) uniform grid, the iteration error

$$||u - U^l||_{K_l} \leq ||u - u^l||_{K_l} + ||u^l - U^l||_{K_l} = \mathcal{O}(h_l)$$

is easily attained. Thus CGM is efficient in the energy norm and applicable to many problems, for example, the elastic system.

However, CMG is not of the optimal convergence in L^2-norm. By the embedding theorem, we can only get

$$||u - U^l||_0 \leq C ||u - U^l||_1 = \mathcal{O}(h_l),$$

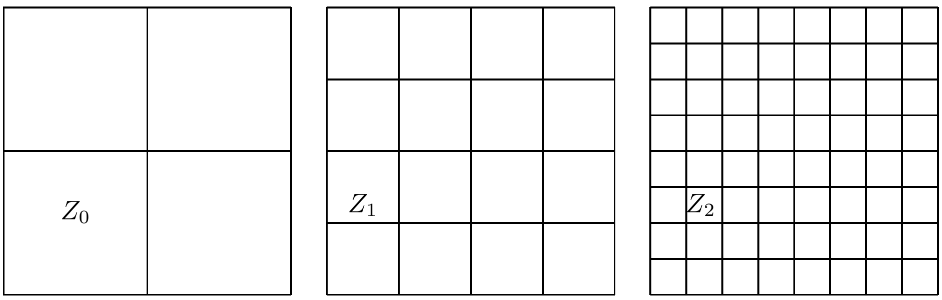

Fig. 1.1. Triple grids: $K_0 U^0 = b_0$ *in* Z_0; $K_1 U^1 = b_1$ *in* Z_1; $K_2 U^2 = b_2$ *in* Z_2.

rather than $||u - U^l||_0 = \mathcal{O}(h_l^2)$.

We found that the linear interpolation, as an important operator in CMG, yields a serious problem, i.e., not only the original nodal errors remain on the refined grids, but also reproduces the larger errors at mid-points, which can not be cancelled by several iterations so will be brought to the refined grids.

To overcome the shortcoming, we shall supplement two important techniques: 1. Proposing new extrapolation formulas to improve the nodal accuracy; 2. Using quadratic interpolation, the larger deviation at mid-point is cancelled. Our new algorithm [20–22] (2007) is

EXCMG= new extrapolation + quadratic interpolation + CG.

The numerical experiments and analysis show that EXCMG is the higher accuracy in both K-norm and discrete L^2-norm.

2. Superconvergence and Extrapolation of FEM

In 1978, Marchuk-Shaidurov [23] systematically studied the extrapolation of finite difference method. They obtained the asymptotic expansion of nodal error based on the maximum principles, where higher regularities are required.

In early 1980's, the extrapolation of FEM was discussed by H.C. Huang and Q. Lin et al. In 1983, the first complete proof for triangular linear element was provided by Lin et al. [24, 25]. Based on the regularized Green's function, the regularity requirement of solution is decreased by order two [26, 27].

On the other hand, superconvergence of FEM is first proved by Douglas-Dupont-Wheeler [29, 30] (1973 and 1974, Quasi-Projection Method), and soon by Zlamal-Lesaint [31] (1977) and Chen [32, 33] (1978, Element Analysis Method). Since then, both superconvergence and extrapolation of FEM have been systematically studied by Chinese scholars, and generalized to the general domain, singular solution and nonlinear problems and so on, see, e.g., the review papers [34, 35] and books [28, 36, 37].

At that time, Chinese scholars have studied the high accuracy and the posterior error estimates. In recent years, we found new extrapolation formulas and proposed EXCMG, which maybe regarded as the third important application of the extrapolation.

2.1. Classical extrapolation formulas

We begin with one-dimensional linear element on the uniform grid.

Consider the triple grids Z_l with step-length $h_l = h_0/2^l,\ l = 0, 1, 2$ and corresponding linear finite element solutions U^l:

Assuming $u \in C^4$ and denoting the error $e^l = U^l - u$ and nodal value $e_j^l = e^l(x_j)$. There is an asymptotic expansion

$$e_j^l = (U^l - u)(x_j) = A(x_j)h_l^2 + r_j^l, \quad r_j^l = B(x_j)h_l^4, \quad l = 0, 1, 2, \tag{2.1}$$

where $A(x) \in C^2$ is a smooth function independent of h and $B(x)$ is bounded.

Assume that U^0 and U^1 are given. The classical extrapolation [26, 28] gives

$$EU_j^1 \equiv U_j^1 + (U_j^1 - U_j^0)/3 = u_j + \mathcal{O}(h_1^4). \tag{2.2}$$

Chen and Lin [26] proposed the extrapolation formula at mid-point $x_{j+1/2} = (x_j + x_{j+1})/2$ as follwos

$$EU^1_{j+1/2} \equiv U^1_{j+1/2} + \frac{1}{6}\Big((U^1 - U^0)_{j+1} + (U^1 - U^0)_j\Big) = u_{j+1/2} + \mathcal{O}(h_1^4). \tag{2.3}$$

2.2. New extrapolation formulas

Taking the extrapolation values (2.2) and (2.3) as the new initial values of U^2_j, then $U^2_{j+1/2}$ will be exact. For this we propose a new combination close to finite element solution U^2, rather than the exact solution u.

At the nodes $x_k \in Z_0,\ k = j, j+1$, letting the constant c satisfy

$$ce^0_k + (1-c)e^1_k = e^2_k + \mathcal{O}(h_1^4), \quad i.e.\ \ c = -1/4,$$

we get a new extrapolation formula at the node (see [20])

$$W^2_k \equiv U^1_k + \frac{1}{4}(U^1_k - U^0_k) = U^2_k + \mathcal{O}(h_1^4), \quad k = j, j+1. \tag{2.4}$$

Based on the asymptotic expansion (4) at the nodes $x_j,\ x_{j+1/2},\ x_{j+1}$,

$$U^1_k - U^0_k = -(3/4)A_k h_0^2 + \mathcal{O}(h_1^4), \quad k = j, j+1,$$
$$(U^2 - U^1)_{j+1/2} = -(3/16)A_{j+1/2}h_0^2 + \mathcal{O}(h_1^4),$$

and the center difference $A_{j+1} + A_j - 2A_{j+1/2} = \mathcal{O}(h^2)$, we have

$$A_k = \frac{-2}{3h^2}[(U^1 - U^0)_j + (U^1 - U^0)_{j+1}] + \mathcal{O}(h_1^2).$$

Consequently, a new extrapolation formula at mid-point is obtanied (see [20]):

$$W^2_{j+1/2} \equiv U^1_{j+1/2} + \frac{1}{8}\Big((U^1 - U^0)_j + (U^1 - U^0)_{j+1}\Big) = U^2_{j+1/2} + \mathcal{O}(h_2^4). \tag{2.5}$$

To define other two values $W^2_{j+1/4}$ and $W^2_{j+3/4}$, we use the quadratic interpolation

$$f(t) = \frac{1}{2}(t^2 - t)f_{-1} + (1 - t^2)f_0 + \frac{1}{2}(t^2 + t)f_1 \quad \text{in} \quad (-1, 1)$$

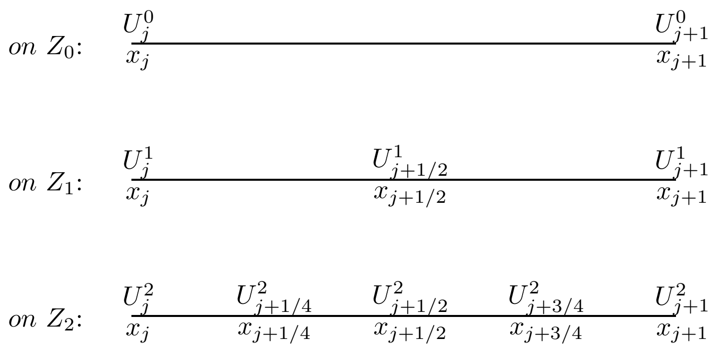

Fig. 2.1. Triple grids: U^0_k onZ_0; U^1_k on Z_1; U^2_k on Z_2.

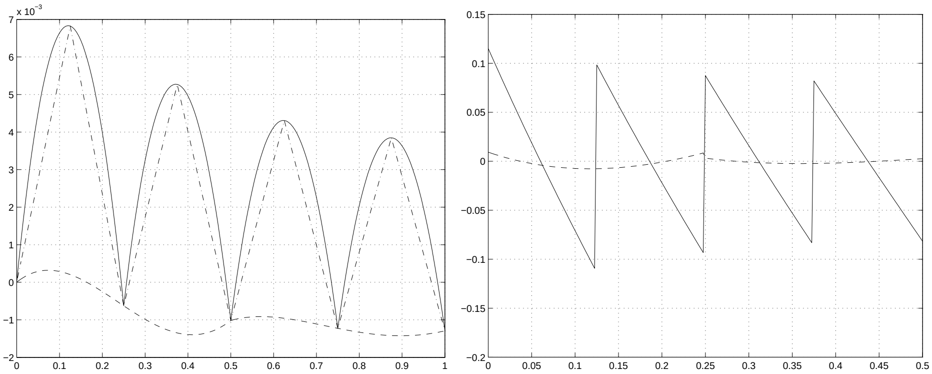

Fig. 2.2. Errors e, De of linear element (real lines), and the errors of quadratic interpolation I_2U (dot lines)

and get the values at two quarter points

$$f(-\frac{1}{2}) = \frac{1}{8}(3f_0 + 6f_1 - f_2), \quad f(\frac{1}{2}) = \frac{1}{8}(-f_0 + 6f_1 + 3f_2).$$

Summarizing these results we get five initial values $W^2 = F(U^0, U^1)$ of U^2 on Z_2:

$$W_k^2 = U_k^1 + (U_k^1 - U_k^0)/4, \qquad k = j,\ j+1, \tag{2.6a}$$

$$W_{j+1/2}^2 = U_{j+1/2}^1 + \frac{1}{8}\Big((U^1 - U^0)_j + (U^1 - U^0)_{j+1}\Big), \tag{2.6b}$$

$$W_{j+1/4}^2 = \frac{1}{16}\Big((9U_j^1 + 12U_{j+1/2}^1 - U_{j+1}^1) - (3U_j^0 + U_{j+1}^0)\Big), \tag{2.6c}$$

$$W_{j+3/4}^2 = \frac{1}{16}\Big((9U_{j+1}^1 + 12U_{j+1/2}^1 - U_j^1) - (3U_{j+1}^0 + U_j^0)\Big). \tag{2.6d}$$

2.3. Superconvergnce for gradient

It is proved that the gradient of linear finite element U is superconvergent at mid-points $\bar{x}_j$ (or at the center for rectangular element)

$$(DU - Du)(\bar{x}_j) = \mathcal{O}(h^2)||u||_{3,\infty}. \tag{2.7}$$

Let $\tau_j = (x_{j-1}, x_j), \tau_{j+1} = (x_j, x_{j+1})$ be two adjacent elementsthen the averaging gradient at x_j. Then

$$\bar{D}U(x_j) \equiv \frac{1}{2}\Big(DU^0(x_j - 0; \tau_j) + DU^0(x_j + 0; \tau_{j+1})\Big) = Du(x_j) + \mathcal{O}(h^2)$$

is also superconvergent. For the rectangular or triangular elements, the averaging gradient at inner node or side-midpoint is also superconvergent, see, e.g., [32–35].

Further, the averaging gradient $\bar{D}U$ at inner node has similar asymptotic expansion

$$\bar{D}U(x_j) - Du(x_j) = C(x_j)h^2 + \mathcal{O}(h^3)||u||_{4,\infty}. \tag{2.8}$$

Then its extrapolation is valid.

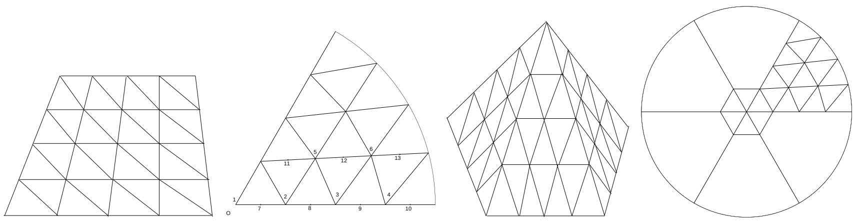

Fig. 2.3. The piecewise (almost) uniform grid (PC-grid).

Besides, using the quadratic interpolation I_2U of U, we have uniformly superconvergence estimate

$$\max_{x\in\Omega}|D(I_2U-u)(x)|\le Ch^2||u||_{3,\infty}. \tag{2.9}$$

Example 1. Consider an one-dimensional problem

$$\begin{aligned}&-u''+u=1, && 0<x<1,\\ &u(0)=u'(1)=0, && u=1-\cos h(1-x)/\cos h1.\end{aligned}$$

We see that in the left of Fig. 2.2, the error $e(x)$ is four piece arches (real line), and the errors at mid-points (as a new nodes) are larger, while the error I_2U-u is uniformly smaller (dot line). In the right of Fig. 2.2, the error $De(x)$ is four piece saw-teeth (real line), which is superconvergent at mid-points, while the error $D(I_2U-u)$ is uniformly superconvergent (dot line).

2.4. Various generalization

The classical results mentioned above are already generalized to various cases.

(i) **The weaker regularity** $u\in H^k$. We introduce the discrete L^2-norm and energy norm

$$||v||=\left(\sum_{x_j\in Z_h}|v(x_j)|^2h_j^2\right)^{1/2},\qquad ||v||_K=(Kv,v),\quad v\in V_h. \tag{2.10}$$

The extrapolation error(on uniform grid) is of high order in discrete L^2-norm [32, 34]

$$||EU-u||\le Ch^k||u||_{k,\Omega},\quad k=3,4. \tag{2.11}$$

(ii) **The general domain** can be simulated by piecewise (almost) uniform grid (PC-grid) [25, 27], see, Fig. 2.3. It is proved that the extrapolation error on PC-grid [25, 27, 34–37] is of high order

$$||EU-u||\le Ch^3|\ln h|\ ||u||_{3,\Omega}, \tag{2.12}$$

where the factor $|\ln h|$ is brought by the embedding theorem on the coarse grid lines.

(iii) **The singular solutions**. Subdivide the radial direction by geometrical series (see, Fig. 2.4)

$$r_j=C(jh)^\lambda,\quad j=0,1,\cdots,N,\quad 1\le\lambda\le4.$$

If taking $\lambda>1$, then (2.12) is still valid [36, 37].

(iv) **For (strongly or weakly) nonlinear problems**, all results mentioned above are automatically valid [36, 37].

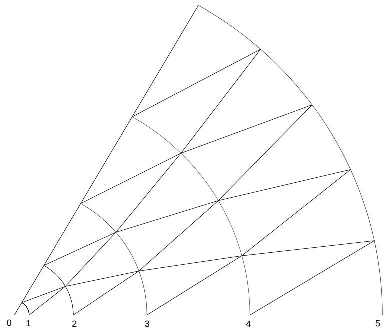

Fig. 2.4. The λ-graded grids to singularity O with $\lambda = 2$.

3. EXCMG Algorithm

Algorithm 3.1. Assume that two exact solutions $\bar{U}^i$ on Z_i, $i = 0, 1$, given, EXCMG consists of the following four steps, $l = 2, 3, \cdots, L$:
Step 1. Use $\bar{U}^{l-i}$ on Z_{l-i}, $i = 1, 2$ to make new extrapolation $\hat{U}^{l-1} = F(\bar{U}^{l-2}, \bar{U}^{l-1})$ on Z_{l-1} (see the formulas (9));
Step 2. Use quadratic interpolation $I_2\hat{U}^{l-1}$ to get the initial value $U^{l,0} = I_2\hat{U}^{l-1}$ on Z_l;
Step 3. Use the operator $S_l^{m_l}$ to get the iteration solution $\bar{U}^l = S_l^{m_l} U^{l,0}$ on Z_l;
Step 4. Come back to steps 1-3 if $l < L$, until get the final iterative solution $\bar{U}_L$ on the finest grid Z_L.

It is noted that, once the exacter solutions $\bar{U}^{L-1}$ and $\bar{U}^L$ are obtained, then the classical extrapolation and superconvergence techniques can be used to further improve the accuracy or provide the exacter posterior error estimates. This is the other advantage of EXCMG.

4. Three Important Properties

4.1. A good property of quadratic interpolation

Assume that $f(x) \in C^4(J)$ is a smooth function in an interval J. The quadratic interpolation $I_2 f$ of u in an element $(-h, h)$ has the remainder

$$R(x) = f - I_2 f = \frac{1}{6}x(x^2 - h^2)f'''(0) + \mathcal{O}(h^4),\ R(\pm h) = R(0) = 0,$$

where $R(\pm h/2) = \pm(h^3/16)f'''(0) + \mathcal{O}(h^4)$ are of only third order accuracy, but their main parts $\pm(h^3/16)f'''(0)$ have formed almost the anti-symmetric highest frequency oscillations in a whole interval J. The numerical experiments show that this highest frequency oscillation can be cancelled by twice CG-iterations [20–22]. Thus, in theoretical analysis, they can be omitted, while the initial values W_k^2 are of high order errors [21]

$$E^2_{j+i/4} = W^2_{j+i/4} - U^2_{j+i/4} = \mathcal{O}(h_2^4) \ \ on\ Z_2, \quad i = 0,\ 1,\ 2,\ 3,\ 4.$$

Below we shall discuss CG-iteration and change the notation as

$$U_n^m,\ \ U_n\ \ on\ \ Z_n,$$

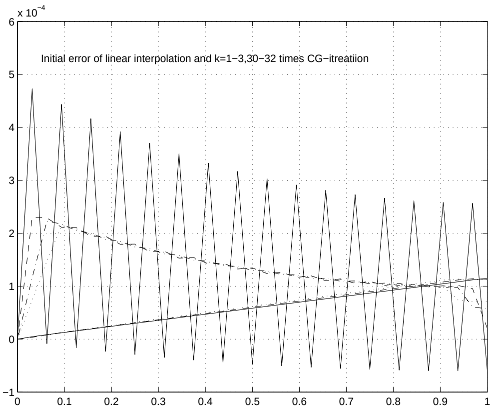

Fig. 4.1. Linear interpolation: the errors e_{32}^0 and $e_{32}^k,\ k=1\sim 3, 29\sim 31$.

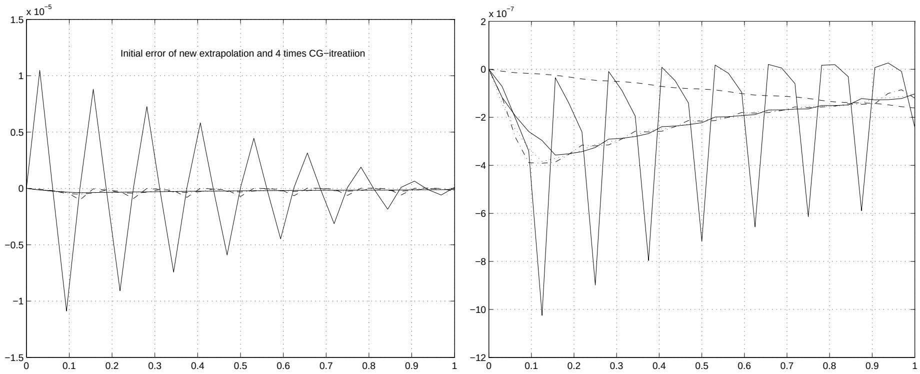

Fig. 4.2. New extrapolation errors e_{32}^0 (left, scale 10^{-5}) and $e_{32}^k,\ k=1\sim 4$ (right, scale 10^{-7}), e_{32}^{31} is the dot straight-line.

where m is the iteration times on the grid Z_n.

Example 2. As in Example 1, taking step-lengths $h=1/m,\ m=8,\ 16,\ 32$, and computing the exact linear elements U_m on the grids Z_m, we investigate k-th iteration solution U_{32}^k and its error

$$e_{32}^k = U_{32}^k - U_{32} \quad on \ \ Z_{32}, \quad k=0,1,\cdots,31.$$

We compare two algorithms as follows:

A. Linear interpolation to get $U_{32}^0 = I_1(U_{16}^1)$ by U_{16}^1.

We see in Fig. 4.1 that the initial error e^0 is high frequency oscillation (broken line, $||e^0|| = 10^{-4}$), but its center line is far from x-axis. Thus the oscillation can be smoothed by several iterations, but their contraction is very slow.

B. New extrapolation + quadratic interpolation $U_{32}^0 = F(U_8, U_{16})$.

We see in Fig. 4.2 that a dramatic change appears:

1). The initial error $e^0 \approx 10^{-5}$ is already reduced, but the errors at quarter points are larger, and form a high frequency oscillation with center-line close to x-axis;

2). The oscillation is almost cancelled by twice iterations and attains the accuracy 10^{-7}.

Table 4.1: The contraction factor ρ_i of CG-iterations on i^{th} grid Z_i, $j = i_0$.

i^{th} grids	Z_0	Z_1	Z_2	...	Z_{j-2}	Z_{j-1}	Z_j	Z_{j+1}	Z_{j+2}	...	Z_L
iterat. sol.	U_0	U_1	U_2	...	U_{j-2}	U_{j-1}	U_j	U_{j+1}	U_{j+2}	...	U_L
contrac. ρ_i	≈ 0	≈ 0	$e^{-8^{(j-2)}}$	...	e^{-64}	e^{-8}	e^{-1}	$e^{-1/8}$	$e^{-1/64}$	...	≈ 1

4.2. CG-estimates in L^2-norm and its valve-value

Assume that v^* is the exact solution of linear system $Kv = b$, for any initial value v_0 given, we get a series of CG-iteration, $v^0,\ v^1,\ v^2, \cdots, v^m$, and their errors $e^m = v^m - v^*$. It is well known that there are two classical estimates in K-norm

$$||e^m||_K \le ||e^0||_K, \quad ||e^m||_K \le 2(\frac{1-\nu}{1+\nu})^m ||e^0||_K, \quad \nu = 1/\sqrt{cond(K)} << 1.$$

Assume that $\nu = ch$. For example, $c \approx 1/2$ for five-point difference scheme for $-\Delta u = f, u|_\Gamma = 0$ in a square, and $c \ge 1/2$ for bilinear finite elements and other boundary value conditions. To simplify the analysis we take $c = 1/2$.

As $(1-t)/(1+t) \le e^{-2t}, 0 < t < 1$, we have a simple estimate

$$||e^m||_K \le 2e^{-mh}||e^0||_K, \quad m = 0, 1, 2, \cdots. \tag{4.1}$$

Denoting $e^m = S^m e^0$, we have got two similar results in the discrete L^2-norm as follows [21,22].

Theorem 4.1 (Monotonicity) *The CG-iteration operator S is monotonously decreasing*

$$||S^m e^0|| \le ||e^0||, \quad m = 1, 2, \cdots. \tag{4.2}$$

Theorem 4.2 (Convergence) *The CG-iterations have the convergence estimates*

$$||S^m e^0|| \le 2e^{-mh}||e^0||, \quad m = 1, 2, \cdots. \tag{4.3}$$

In EXCMG, we take CG-iteration times m_i in i-th grid Z_i as

$$m_i = m_L \beta^{L-i}, \quad i = 2, 3, \cdots, L, \quad \beta = 4, \tag{4.4}$$

where $m_L = 2^s$ is CG-iteration times on the finest grid Z_L.

Taking the initial step-length $h_0 = 2^{-k}$, and $h_i = h_0/2^i = 2^{-k-i}$ on Z_i, by Theorem 4.2 we have

$$||S_i^{m_i}|| \le 2e^{-p_i}, \quad p_i = m_i h_i = 2^{2L-3i+s-k}, \quad p_{i+1} = p_i/8,$$

We define a valve-value by

$$i_0 = (2L + s - k)/3, \quad \text{always assuming} \quad i_0 \ge 3,$$

(obviously if L is large enough, $i_0 \approx 2L/3$). Then

$$||S_i^{m_i}|| \le 2e^{-p_i} = 2\rho_i, \quad \rho_i = \exp(-2^{3(i_0-i)}) = \exp(-8^{i_0-i}).$$

We see that the valve-value $i_0 \ge 3$ is a critical point. When $i < i_0$, $||S_i^{m_i} e^0||$ contracts fast; When $i > i_0$, CG-iterations are of the smoothing effects. Thus EXCMG exhibits the convergence feature as follows.

Under the condition of (4.4) the most important contractions appear in the levels of $i \le i_0 - 1$. When $i < i_0 - 2$, the iteration time m_i can greatly decrease, also see [10]. When $i > i_0$, the CG-iterations are of only smoothing effects. Thus the term $S_{i_0}^{m_{i_0}} U_{i_0}$ iterated by CG will bring about the main error of EXCMG.

In Section 5.1, the Poisson equation in a square and the grids Z_n of $n \times n$ square elements, $n = 32, 64, 128, \cdots, 2048$, are discussed. Taking the coarse steplength $h_0 = 2^{-5}$ on Z_{32}, $L = 6$ and $m_L = 2^2$, we have the valve-value $i_0 = (2L + s - k)/3 = 3$, i.e. the error on the third level (i.e. Z_{128}) is greatly contracted, the main errors come from U_{256}, while CG-iterations on $Z_n, n = 512, 1024, 2048$, are of the smoothing effects and their highest accuracy will be slowly decreased.

4.3. Convergence of EXCMG in discrete L^2-norm

Assume that $u \in H^\alpha(\Omega),\ 2 \le \alpha \le 4$, we have the initial remainders in (4)

$$||r_i|| \le CMh_i^\alpha,\ i = 0, 1, 2, \quad M = ||u||_{\alpha,\Omega},$$

using the new extrapolation and quadratic interpolation, the error $r^2 = W_2 - U_2$ on Z_2 is of the optimal order

$$||r_2|| \le CMh_2^\alpha,$$

which will be further contracted by CG-iterations m_2 times. When $L < 20$, we have proved the following convergence result independent of k, s.

Theorem 4.3 ([22]) *Assume that $U_L \in V_L$ are (bi-)linear finite element solutions on Z_L. Then the final iteration solution U_L^* of EXCM has the error estimate*

$$||U_L - U_L^*|| \le CMh_L^\beta, \quad \beta = (4\alpha + 1)/6 < \alpha, \tag{4.5}$$

where the constant C is independent of h, α, and $\beta = 2.80,\ 2.17,\ 1.50$ for $\alpha = 4,\ 3,\ 2$ respectively.

Remark 1. This index β is not optimal, later on which can be improved to $\beta = (2\alpha+1)/3$, i.e. $\beta = 3,\ 2.33,\ 1.67$, for $\alpha = 4,\ 3,\ 2$, respectively. In fact the optimal index β is also dependent in L, k, s. When L is not large, decreasing k and increasing s, the index β will suitably grows. The numerical experiments show that EXCMG is of better order $\beta \approx \alpha$, however, this optimal convergence can not be proved yet.

5. Numerical Experiments in a Square

5.1. The smooth solution $u \in C^4$

Consider an elliptic problem

$$-\Delta u = f \qquad \text{in } \Omega = [0, 1] \times [0, 1], \tag{5.1a}$$

$$u = 0 \qquad \text{on } \Gamma_1 = \{x = 0, 0 < y < 1\} \cup \{y = 0, 0 < x < 1\}, \tag{5.1b}$$

$$D_n u = 0 \qquad \text{on } \Gamma_2 = \{x = 1, 0 < y < 1\} \cup \{y = 1, 0 < x < 1\}, \tag{5.1c}$$

and the exact solution $u = \sin(\pi x)(e^y - 1 - ey)$.

Subdivide Ω into $m \times m$ uniform square grids Z_m with $h_m = 1/m$. Denote U_m— the exact bilinear finite element solution on Z_m, U_m^k— the k-th CG-iteration solution, u_I— the bilinear interpolation of u on Z_m, $e_m = U_m - u_I$— the exact error, $e_m^k = U_m^k - u_I$— the k^{th}-iteration error.

Define the discrete L^2-norm and energy norm

$$||v|| = \left(\sum_{x \in Z_m} |v(x)|^2 h_m^2 \right)^{1/2}, \quad ||Dv|| = \sqrt{(K_m v, v)}, \quad \text{on} \quad Z_m.$$

Note that the discrete energy norm $||De_m|| = \mathcal{O}(h_m^2)$ is superconvergent. We shall investigate whether their error ratio $||e_m^k||/||e_{2m}^{k'}||$ is close to 4 in order to judge the accuracy of the iteration errors.

We have calculated the exact finite elements and the errors: U_{32} on Z_{32}: $||e_{32}|| = 1.006795 \times 10^{-4}$; U_{64} on Z_{64}: $||e_{64}|| = 2.516926 \times 10^{-5}$, their error ratio $||e_{32}||/||e_{64}|| = 4.0001$. By EXCMG, we get the new initial value U_{128}^0, whose error $||e_{128}^0|| = 6.305631 \times 10^{-6}$ is very small and the error ratio $||e_{64}||/||e_{128}^0|| = 3.9916$.

Taking different iteration times k from Z_{128} to Z_{2048}, the results computed are listed in Table 5.1.

We see that: 1). All three errors $||e_{2048}^k||$ attain the accuracy 10^{-8}; 2). but these errors have 1, 1 and 3 digits exact, respectively; 3). So the classical extrapolation leads to smaller errors $10^{-9} \sim 10^{-11}$, respectively.

The final grid Z_{2048} contains 4×10^6 unknowns, which was solved in PC with memory 2G, and CPU times are about 10 minutes.

In Table 5.2 we see that: 1). For EXCMG, the errors $||De^k||$(superconvergent) and $||e^k||$ are very small, about 10^{-8}, and the ratio of errors is very close to 4, so the extrapolation is effective. 2). For CMG, the errors $||De^k||$ are smaller(in comparison with $||u^m - u||_1 = \mathcal{O}(h_m)$, but which grow on the fine grids. Besides, $||e_{2048}^k||$=1.67e-6 does not attain the optimal error 2.4×10^{-8}. It means that CMG is not convergent in the discrete L^2-norm.

5.2. The non-smooth solution $u \in H^3$

Consider a non-smooth solution $u = xy\ln(x^2 + y^2) \in H^{3-\epsilon}(\Omega)$ satisfying

$$\triangle u = 8xy/(x^2 + y^2) \quad \text{in} \quad \Omega = (0,1) \times (0,1), \tag{5.2a}$$

$$u = 0 \quad \text{on} \quad \Gamma_1, \tag{5.2b}$$

$$u = g(x,y) \quad \text{on} \quad \Gamma_2, \tag{5.2c}$$

Table 5.1: EXCMG. Different times k, the error $||e_n^k||$ and error ratio.

j	Z_m	k=10,10,10,10,10	k=64,32,16,8,4	k=1024,256,64,16,4
2	128	6.293601e-6(3.9992)	6.293455e-6(3.9993)	6.292382e-6(4.0000)
3	256	1.574828e-6(3.9964)	1.574527e-6(3.9970)	1.573159e-6(3.9998)
4	512	3.952029e-7(3.9849)	3.947916e-7(3.9882)	3.933426e-7(3.9995)
5	1024	1.004977e-7(3.9325)	9.987299e-8(3.9529)	9.838661e-8(3.9979)
6	2048	2.751062e-8(3.6531)	2.620189e-8(3.8117)	2.464734e-8(3.9918)
Ext.	Ee_{2048}^k	7.6373e-9(good)	2.6034e-9(better)	7.2075e-11(best)

Table 5.2: Comparison of CMG and EXCMG, iteration times $k = 1024, 256, 64, 16, 4$.

Z_m	CMG-$\|\|De^k\|\|$	EXCMG-$\|\|De^k\|\|$	CMG-$\|\|e^k\|\|$	EXCMG-$\|\|e^k\|\|$
128	1.3995e-5	1.3995e-5	6.2924e-6(4.7493)	6.2924e-6 (4.0000)
256	3.5118e-6(3.9851)	3.4989e-6(3.9998)	1.4524e-6(4.3323)	1.5732e-6 (3.9998)
512	2.7471e-6(1.2784)	8.7484e-7(3.9994)	9.4508e-7(1.5368)	3.9334e-7 (3.9992)
1024	4.1586e-6(0.6606)	2.1882e-7(3.9980)	1.5250e-6(0.6187)	9.8387e-8 (3.9979)
2048	4.7135e-6(0.8823)	5.4823e-8(3.9914)	1.6714e-6(0.9124)	2.4647e-8 (3.9918)
Ext.			1.7205e-6(poor)	7.2075e-11(good)

Table 5.3: The errors $\|De^k\|$ and error ratio.

Z_m	k(m)	linear interp.CMG	quadratic interp.CMG	EXCMG
128	1024	3.0215e-5 (3.6786)	3.0215e-5 (3.6786)	3.0215e-5
256	256	8.0130e-6 (3.7708)	8.1189e-6(3.7216)	8.1155e-6(3.7231)
512	64	1.1514e-5(0.6959)	4.3685e-6 (1.8585)	2.1606e-6(3.7561)
1024	16	1.6533e-5 (0.6964)	4.0982e-6 (1.0660)	5.7320e-7(3.7694)
$\|\|\bar{D}e^k_{1024}\|\|$	super.	1.6206e-5 (good)	3.0179e-6 (better)	1.3070e-6(best)

where $\mathcal{O}$ is only singularity point of u. Assume that U_{32}, U_{64} given, three algorithms: the linear or quadratic interpolations are used in CMG, and EXCMG. The results computed are listed in Table 5.3.

Denote by $\bar{D}U_m(x_j)$ the averaging gradient of 4 elements around the inner node x_j. While $\bar{D}e_m(x_j) = \bar{D}U_m(x_j) - Du(x_j)$ is the practical error we need.

5.3. The singular solution $u \in H^2$

Consider a singular solution $u = xy/\sqrt{x^2+y^2} \in H^{2-\epsilon}(\Omega)$ satisfying

$$-\Delta u = \frac{3xy}{(\sqrt{x^2+y^2})^3} \quad in \;\; \Omega, \quad u = 0 \;on\;\; \Gamma_1, \quad u = g(x,y) \;\; on \;\; \Gamma_2, \tag{5.3}$$

We see that: 1). Three errors $||De^k||$ are close each other and the error ratios are close to 2; 2). Three errors $||\bar{D}e_m||$ are very good and the error ratios are about 3.25. It seems that their classical extrapolation has still improved the accuracy. This is surprising.

The results of EXCMG show that both classical and new extrapolations are still effective, which is quite unexpected.

Table 5.4: The errors $\|e^k\|$ and error ratio.

Z_m	k(m)	linear interp.CMG	quadratic interp.CMG	EXCMG
128	1024	3.2951e-6 (3.9685)	3.2951e-6 (3.9685)	3.2951e-6
256	256	8.0794e-7 (4.0785)	8.2879e-7(3.9758)	8.2705e-7(3.9842)
512	64	1.8544e-6(0.4357)	7.4596e-7 (1.1110)	2.0720e-7(3.9915)
1024	16	2.5402e-6 (0.7300)	7.4528e-7 (1.0009)	5.1890e-8(3.9931)
$\|\|Ee^k_{1024}\|\|$	Ext.	2.7683e-6(poor)	7.4410e-7 (good)	1.2232e-10(best)

Table 5.5: The errors $\|De^k\|$ and error ratio.

Z_m	k(m)	linear interp.CMG	quadratic interp.CMG	EXCMG
128	1024	5.7544e-4 (1.9997)	5.7544e-4 (1.9997)	5.7543e-4
256	256	2.8773e-4 (1.9999)	2.8773e-4 (1.9999)	2.8773e-4(1.9999)
512	64	1.4409e-4 (1.9969)	1.4404e-4 (1.9975)	1.4387e-4(2.0000)
1024	16	7.4272e-5 (1.9401)	7.3554e-5 (1.9583)	7.1978e-5(1.9988)
$\|\|\bar{D}e_{512}\|\|$	super.	4.3858e-4(good)	4.3846e-4(good)	4.3843e-4(good)
$\|\|\bar{D}e_{1024}\|\|$	super.	1.3616e-4(3.2210)	1.3530e-4(3.2201)	1.3477e-4(3.2532)
$\|\|E(\bar{D}e)\|\|$	Ext.	4.5942e-5(good)	4.088e-5(good)	3.8455e-5(good)

Table 5.6: The errors $\|e^k\|$ and error ratio.

Z_m	k(m)	linear interp.CMG	quadratic interp.CMG	EXCMG
128	1024	5.7980e-6 (3.6713)	5.7980e-6 (3.6713)	5.7980e-6
256	256	1.5627e-6 (3.7101)	1.5625e-6 (3.7106)	1.5570e-6 (3.7238)
512	64	1.0389e-6 (1.5043)	8.8791e-7 (1.7598)	4.1516e-7 (3.7504)
1024	16	1.4672e-6 (0.7081)	8.4716e-7 (1.0481)	1.1226e-7 (3.6983)
$\|\|Ee_{1024}\|\|$	Ext.	1.6436e-6(poor)	8.4070e-7(poor)	6.9561e-9(better)

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