Journal of Computational Mathematics
Vol.29, No.6, 2011, 698–719.

http://www.global-sci.org/jcm
doi:10.4208/jcm.1110-m11si06

SUPERGEOMETRIC CONVERGENCE OF SPECTRAL COLLOCATION METHODS FOR WEAKLY SINGULAR VOLTERRA AND FREDHOLM INTEGRAL EQUATIONS WITH SMOOTH SOLUTIONS*

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Abstract

A spectral collocation method is proposed to solve Volterra or Fredholm integral equations with weakly singular kernels and corresponding integro-differential equations by virtue of some identities. For a class of functions that satisfy certain regularity conditions on a bounded domain, we obtain geometric or supergeometric convergence rate for both types of equations. Numerical results confirm our theoretical analysis.

Mathematics subject classification: 45G20, 46A16, 46A22, 65M70.
Key words: Weakly singular kernel, Integro-differential equations, Collocation method.

1. Introduction

Given $0 < \mu < 1$, we consider two classes of linear Volterra type integral equations of the form

$$y(t) - \int_0^t (t-s)^{-\mu} y(s)ds = b(t), \qquad t \in (0,T], \tag{1.1}$$

and of the form (Fredholm)

$$y(t) - \int_0^T |t-s|^{-\mu} y(s)ds = b(t), \qquad t \in (0,T], \tag{1.2}$$

where $y(t)$ is the unknown function and $b(t)$ is a sufficiently smooth function. We also consider the corresponding integro-differential equations

$$y'(t) = a(t)y(t) + \int_0^t (t-s)^{-\mu} y(s)ds + b(t), \quad y(0) = y_0, \quad t \in (0,T], \tag{1.3}$$

and

$$y'(t) = a(t)y(t) + \int_0^T |t-s|^{-\mu} y(s)ds + b(t), \quad y(0) = y_0, \quad t \in (0,T], \tag{1.4}$$

* Received March 23, 2011 / Revised version received June 25, 2011 / Accepted June 30, 2011 /
Published online November 15, 2011 /

where $y(t)$ is the unknown function, $a(t)$ is an analytic function, and $b(t)$ is a sufficiently smooth function. We assume that each of these equations possesses a unique solution [9, 21].

Numerical approximation of integral equations with singular kernel has caught numerous attentions, among which extensive studies can be found in [1, 10] while numerical solutions of integro-differential equation are studied in [2, 12, 18] etc. Many numerical analysts used graded meshes to develop numerical schemes with an optimal order of convergence, see, e.g., [1,9,11,12]. A hybrid collocation method was also proposed to solve Volterra equations with weakly singular kernels [5] and Fredholm singular equations [3,4]. Recently, in [6,7] a spectral Jacobi-collocation method was proposed and analyzed to solve Volterra equations. The basic idea is to collocate equations at some Jacobi points and use a highly accurate quadrature to approximate the integration in (1.1). In this article, however, instead of numerical integration, we apply exact integration to the composition of the Legendre polynomials and the weakly singular kernel. The exact integration leads to a more accurate solution and reduces the computation cost. It will be shown that a geometric (supergeometric) rate of convergence can be achieved by using our method if $y(t)$ satisfies condition (R): $\|y^{(k)}\|_{L^\infty[0,T]} \leq Ck!R^{-k}$ (condition (M): $\|y^{(k)}\|_{L^\infty[0,T]} \leq CM^k, M > 0$) not only for Volterra equations but also for Fredholm equations as well as their corresponding integro-differential equations. Here, R is sufficiently large. If R is small, an *hp*-version of our method is necessary. For a Volterra equation, if the solution is not smooth enough we may take some function transformations as in [7] to obtain a new equation which possesses better regularity.

In the traditional analysis of spectral methods, the error bound is given in the form $\mathcal{O}(p^{-k})$ for any positive integer k, where p is the polynomial degree. This is to say that the convergence is superior to any polynomial rate as long as the exact solution is in C^∞. Nevertheless, it is still not geometric convergence in its precise sense. The fundamental difference of our analysis here is that we establish a convergence rate in the form $\mathcal{O}(e^{-\sigma p}) = \mathcal{O}(R^{-p})$ (geometric convergence) or $\mathcal{O}(e^{-\sigma p(\ln p - \ln \gamma)}) = \mathcal{O}((\gamma/p)^{\sigma p})$ (super-geometric convergence). For the former, we need condition (R) on the exact solution, and for the later we need condition (M), a more restricted assumption.

Let us elaborate some more on those two conditions. Actually, condition (R) is the analytic assumption. Based on the Cauchy integral formulae, an analytic function f, and its kth derivative $f^{(k)}$, can be expressed as,

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw, \quad f^{(k)}(z) = \frac{k!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{k+1}}dw,$$

in its analytic region D with boundary Γ. Condition (M) characterizes a class of entire functions. Typical candidates are $\sin Mz, \cos Mz, e^{Mz}$ and their combinations. For more details, see the discussion in [20].

We see that both conditions are very restricted and satisfied only in special situations, such as when the singular kernel is replaced by a smooth kernel $k(t,s)$ (or $\mu = 0$) and input data a, b are analytic functions. Nevertheless, our numerical schemes is still valid for problems with the singular kernel discussed in this paper, even though the associated analysis for solutions with singularity is lacking. We would emphasis that the spectral method is particularly efficient for equations with sufficiently smooth solutions. Regarding supergeometric convergence of spectral collocation method for differential equations, readers are referred to [19, 20].

This paper is organized as follows: In Sections 2, some preliminary knowledge as well as algorithms for both types of integral equations are given. In Section 3, convergence analysis for integral equations is discussed. We prove the error estimates in the L^∞ norm. Section 4 is devoted to the algorithm and convergence analysis of integro-differential equations. We also present in Section 3 and Section 4 some numerical results, which demonstrate the accuracy of our algorithms.

Throughout the paper, C stands for a generic constant that is independent of p but may depend on T, the given function $b(t)$ and the index μ.

2. Algorithms

Let $h(t)$ be a sufficiently smooth function on $[0,T]$ and write $f(x) = h(\frac{T}{2}(1+x)), x \in [-1,1]$. Let T_p be the Chebyshev polynomial of the first kind with degree p and we define an interpolation operator I_p, such that $I_p f \in P_p[-1,1]$ interpolates f at $p+1$ Chebyshev points, i.e., zeros of T_{p+1}: $x_i = \cos\frac{2i+1}{2p+2}\pi$. Then the remainder of the interpolation is

$$f(x) - I_p f(x) = f[x_0, x_1, \cdots, x_p, x]\nu(x), \tag{2.1}$$

where $\nu(x) = (x-x_0)(x-x_1)\cdots(x-x_p)$. Hence,

$$f(x) - I_p f(x) = \frac{f[x_0, x_1, \cdots, x_p, x]}{2^{p+1}} T_{p+1}(x), \tag{2.2}$$

since the leading coefficient of $T_{p+1}(x)$ is 2^{p+1}. Moreover, if $f \in C^{p+1}[-1,1]$, the divided difference

$$f[x_0, x_1, \cdots, x_p, x] = \frac{f^{(p+1)}(\xi_x)}{(p+1)!}, \qquad \xi_x \in (-1,1).$$

Therefore, by the Stirling formula,

$$\|f - I_p f\|_{L^\infty[-1,1]} \le C \begin{cases} \left(\frac{T}{4R}\right)^{p+1}, & \text{if } f \text{ satisfies condition (R);} \\ \sqrt{p+1}\left(\frac{eMT}{4(p+1)}\right)^{p+1}, & \text{if } f \text{ satisfies condition (M).} \end{cases} \tag{2.3}$$

Define a weighted L^2 norm by

$$\|v\|_{w^{\alpha,\beta}} = \left(\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta |v(x)|^2 dx \right)^{\frac{1}{2}}.$$

From (2.2), the rates of interpolation error under condition (R) and condition (M) in the weighted L^2 norm are the same as those in the L^∞ norm, respectively.

Next, we introduce two identities, which will be essential in this paper.

Lemma 2.1. ([13]) *Let a, b be positive constants and $L_n(x)$ be the Legendre polynomial with degree n on $[-1,1]$, then*

$$\int_a^b (s-a)^{\alpha-1} L_n(\frac{s}{b}) ds = \frac{n!}{(\alpha)_{n+1}} (b-a)^\alpha P_n^{(\alpha,-\alpha)}(\frac{a}{b}), \quad -b < a < b;\ \alpha > 0, \tag{2.4}$$

$$\int_{-a}^b (b-s)^{\beta-1} L_n(\frac{s}{a}) ds = \frac{n!}{(\beta)_{n+1}} (b+a)^\beta P_n^{(-\beta,\beta)}(\frac{b}{a}), \quad -a < b < a;\ \beta > 0, \tag{2.5}$$

where $(k)_{n+1} = k(k+1)\cdots(k+n)$ and $P_n^{(-k,k)}(x)$ is the Jacobi polynomail of degree n with index $(-k,k)$ and $k = \alpha$ or β.

Specifically, if we choose $a = 1, b = x, \beta = 1 - \mu$ in (2.5), then we obtain

$$\int_{-1}^{x} (x-t)^{-\mu} L_n(t)dt = \frac{n!}{(1-\mu)_{n+1}}(1+x)^{1-\mu} P_n^{(\mu-1,1-\mu)}(x), \tag{2.6}$$

and $a = x, b = 1, \alpha = 1 - \mu$ in (2.4), we achieve

$$\int_{x}^{1} (t-x)^{-\mu} L_n(t)dt = \frac{n!}{(1-\mu)_{n+1}}(1-x)^{1-\mu} P_n^{(1-\mu,\mu-1)}(x). \tag{2.7}$$

2.1. Algorithm for (1.1)

After a change of variable $t = \frac{T}{2}(1+x)$, (1.1) can be written as

$$u(x) - \int_0^{T(1+x)/2} \left(\frac{T}{2}(1+x) - s\right)^{-\mu} y(s)ds = g(x), \tag{2.8}$$

where $x \in [-1, 1]$ and

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \quad g(x) = b\left(\frac{T}{2}(1+x)\right). \tag{2.9}$$

Consequently, making another change of variable $s = \frac{T}{2}(1+\tau),\ \tau \in [-1, x]$ gives

$$u(x) - \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x} (x-\tau)^{-\mu} u(\tau)d\tau = g(x). \tag{2.10}$$

We look for numerical approximation of $u(x)$ in the form $u_p(x) = \sum_{j=0}^{p} c_j L_j(x)$, where c_j's are determined by the following equations at the collocation points,

$$\sum_{j=0}^{p} c_j L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{-1}^{x_i} (x_i-\tau)^{-\mu} L_j(\tau)d\tau = g(x_i), \quad i = 0, \cdots, p. \tag{2.11}$$

By virtue of (2.6),

$$\sum_{j=0}^{p} c_j \left(L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \right) = g(x_i), \qquad i = 0, \cdots, p. \tag{2.12}$$

2.2. Algorithm for (1.2)

After the same change of variable as before, we reach

$$u(x) - \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x} \frac{u(\tau)}{(x-\tau)^{\mu}} d\tau - \left(\frac{T}{2}\right)^{1-\mu} \int_{x}^{1} \frac{u(\tau)}{(\tau-x)^{\mu}} d\tau = g(x),\ x \in [-1, 1], \tag{2.13}$$

where $u(x)$ and $g(x)$ are defined the same as those in (2.9).

Let $u_p(x) = \sum_{j=0}^{p} c_j L_j(x)$ be the approximation of $u(x)$. Then, for $i = 0, \cdots, p$, c_j's satisfy the equations

$$\begin{aligned}\sum_{j=0}^{p} c_j L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{-1}^{x_i} (x_i - \tau)^{-\mu} L_j(\tau) d\tau \\ - \left(\frac{T}{2}\right)^{1-\mu} \sum_{j=0}^{p} c_j \int_{x_i}^{1} (\tau - x_i)^{-\mu} L_j(\tau) d\tau = g(x_i).\end{aligned} \tag{2.14}$$

Applying (2.6) and (2.7), we obtain

$$\begin{aligned}\sum_{j=0}^{p} c_j \Bigg(L_j(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \\ - \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) \Bigg) = g(x_i), \quad i = 0, \cdots, p.\end{aligned}$$

2.3. Algorithm for (1.3)

We take the same notations and variable transformation as in Algorithm for (1.1), then we obtain

$$\frac{2}{T} u'(x) = f(x)u(x) + \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x} (x-\tau)^{-\mu} u(s) ds + g(x), \tag{2.15}$$

where $f(x) = a(\frac{T}{2}(1+x))$. Let $u_p(x) = y_0 + \sum_{j=1}^{p} c_j (L_j(x) + L_{j-1}(x))$, and note that

$$\frac{d}{dx} L_n(x) = \frac{n+1}{2} P_{n-1}^{(1,1)}(x). \tag{2.16}$$

We have

$$\begin{aligned}\frac{1}{T} \sum_{j=1}^{p} c_j \left[(j+1) P_{j-1}^{(1,1)}(x_i) + j P_{j-2}^{(1,1)}(x_i) \right] = f(x_i) \left(y_0 + \sum_{j=1}^{p} c_j (L_j(x_i) + L_{j-1}(x_i)) \right) \\ + \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x_i} \frac{y_0 + \sum\limits_{j=1}^{p} c_j (L_j(\tau) + L_{j-1}(\tau))}{(x_i - \tau)^{\mu}} d\tau + g(x_i), \quad i = 0, \cdots, p.\end{aligned} \tag{2.17}$$

By virtue of (2.6) again,

$$\begin{aligned}&\sum_{j=1}^{p} c_j \Bigg[\frac{j+1}{T} P_{j-1}^{(1,1)}(x_i) + \frac{j}{T} P_{j-2}^{(1,1)}(x_i) - (L_j(x_i) + L_{j-1}(x_i)) f(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \\ &\frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) - \left(\frac{T}{2}\right)^{1-\mu} \frac{(j-1)!}{(1-\mu)_j} (1+x_i)^{1-\mu} P_{j-1}^{(\mu-1,1-\mu)}(x_i) \Bigg] \\ &= g(x_i) + f(x_i) y_0 + \left(\frac{T}{2}\right)^{1-\mu} \frac{y_0}{1-\mu} (1+x_i)^{1-\mu}, \quad i = 0, \cdots, p.\end{aligned} \tag{2.18}$$

2.4. Algorithm for (1.4)

We take the same notation and variable transformation as in Algorithm for (1.2) and we obtain

$$\begin{aligned}\frac{2}{T}u'(x) = f(x)u(x) + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x}\frac{u(\tau)}{(x-\tau)^{\mu}}d\tau\\ + \left(\frac{T}{2}\right)^{1-\mu}\int_{x}^{1}\frac{u(\tau)}{(\tau-x)^{\mu}}d\tau + g(x), \quad x\in[-1,1].\end{aligned} \tag{2.19}$$

Again, $f(x)$ has the same definition as in Algorithm for (1.3). Let $u_p(x) = y_0 + \sum_{j=1}^{p} c_j(L_j(x)+L_{j-1}(x))$ be the approximation of $u(x)$. Then c_j must satisfy

$$\begin{aligned}g(x_i) = \sum_{j=1}^{p}\frac{2c_j}{T}\Big((L_j'(x_i)+L_{j-1}'(x_i)\Big) - f(x_i)\Big(y_0 + \sum_{j=1}^{p}c_j(L_j(x_i)+L_{j-1}(x_i))\Big)\\ -\left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}\frac{y_0+\sum_{j=1}^{p}c_j(L_j(\tau)+L_{j-1}(\tau))}{(x_i-\tau)^{\mu}}d\tau\\ -\left(\frac{T}{2}\right)^{1-\mu}\int_{x_i}^{1}(\tau-x_i)^{-\mu}\left[y_0+\sum_{j=1}^{p}c_j(L_j(\tau)+L_{j-1}(\tau))\right]d\tau.\end{aligned} \tag{2.20}$$

Applying (2.6), (2.7) and (2.16) gives, for $i = 0,\cdots,p$,

$$\begin{aligned}&\sum_{j=1}^{p}c_j\Bigg[\frac{j+1}{T}P_{j-1}^{(1,1)}(x_i)+\frac{j}{T}P_{j-2}^{(1,1)}(x_i)-(L_j(x_i)+L_{j-1}(x_i))f(x_i)\\ &-\left(\frac{T}{2}\right)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1+x_i)^{1-\mu}P_j^{(\mu-1,1-\mu)}(x_i)-\left(\frac{T}{2}\right)^{1-\mu}\frac{(j-1)!}{(1-\mu)_j}(1+x_i)^{1-\mu}P_{j-1}^{(\mu-1,1-\mu)}(x_i)\\ &-\left(\frac{T}{2}\right)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1-x_i)^{1-\mu}P_j^{(1-\mu,\mu-1)}(x_i)-\left(\frac{T}{2}\right)^{1-\mu}\frac{(j-1)!}{(1-\mu)_j}(1-x_i)^{1-\mu}P_{j-1}^{(\mu-1,1-\mu)}(x_i)\Bigg]\\ &= g(x_i)+f(x_i)y_0+\left(\frac{T}{2}\right)^{1-\mu}\frac{y_0}{1-\mu}(1+x_i)^{1-\mu}+\left(\frac{T}{2}\right)^{1-\mu}\frac{y_0}{1-\mu}(1-x_i)^{1-\mu}.\end{aligned} \tag{2.21}$$

We need to delete one equation from the last two algorithms because the initial condition is given.

For the sake of analysis, we define the Lagrange interpolation at $(p+1)$ Chebyshev points, i.e., $I_p u(x_i) = u(x_i)$, $0 \le i \le p$. It is clear that $I_p u(x) = \sum_{i=0}^{p} u(x_i)l_i(x)$, where $l_i(x)$ is the Lagrange interpolation basis function at Chebyshev points.

It is obvious that

$$y(t) = y\left(\frac{T}{2}(1+x)\right) = u(x), \quad t\in[0,T] \text{ and } x\in[-1,1]. \tag{2.22}$$

Consequently,

$$y_p(t) = y_p\left(\frac{T}{2}(1+x)\right) = u_p(x), \quad t\in[0,T] \text{ and } x\in[-1,1], \tag{2.23}$$

which gives that $(y-y_p)(t)=(u-u_p)(x):=e(x)$.

3. Convergence Analysis of Integral Equations

This section is devoted to the convergence analysis of our algorithms to the Volterra integral equation(1.1) or the Fredholm equation (1.2).

3.1. Volterra integral equations

We first present some useful lemmas.

Lemma 3.1. ([1]) *Let $\{l_j(x)\}_{j=0}^{p}$ be the Lagrange interpolation polynomials at the Chebyshev points. Then*

$$\|I_p\|_{L^\infty} := \max_{x\in[-1,1]} \sum_{j=0}^{p} |l_j(x)| = \mathcal{O}(\log p). \tag{3.1}$$

Let $C^{r,\kappa}([0,T])$ denote the space of function whose r-th derivatives are Hólder continuous with exponent κ, endowed with the usual norm $\|\cdot\|_{r,\kappa}$. Then from the result of [14, 15], we have the following lemma.

Lemma 3.2. *Let r be a non-negative integer and $\kappa\in(0,1)$. Then for any $v\in C^{r,\kappa}([-1,1])$, there exists a polynomial function $\mathcal{T}_N v\in P_N$ such that*

$$\|v-\mathcal{T}_N v\|_{L^\infty[-1,1]} \le CN^{-(r+\kappa)}\|v\|_{r,\kappa}. \tag{3.2}$$

Here, $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}$ to P_N.

3.1.1. Error estimate in L^∞

Theorem 3.1. *Let $y(t)$ and $u_p(x)$ be the exact solution and spectral approximation of* (1.1), *respectively.*

1) *If y satisfies the condition (R), then, for sufficiently large p,*

$$\|u-u_p\|_{L^\infty} \le C\left(\frac{T}{4R}\right)^{p+1}; \tag{3.3}$$

2) *If y satisfies the condition (M), then, for sufficiently large p,*

$$\|u-u_p\|_{L^\infty} \le \frac{C}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{3.4}$$

Proof. The structure of the proof is similar to the proof of Theorem 4.1 in [6]. However, our emphasis here is the geometric or supergeometric rate of convergence. Using (2.11), we have

$$u_i=\left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}u_p(\tau)d\tau+g(x_i), \tag{3.5}$$

where $u_i=\sum_{j=0}^{p}c_jL_j(x_i)$. Note that the true solution at the Chebyshev points satisfies

$$u(x_i)=\left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}u(\tau)d\tau+g(x_i). \tag{3.6}$$

Denoting, let $e(x) = u(x) - u_p(x)$ yields

$$u(x_i) - u_i = \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x_i} (x_i - \tau)^{-\mu} e(\tau) d\tau. \tag{3.7}$$

Multiplying both sides by $l_i(x)$ and summing up from 0 to p and applying the fact that $u_p(x) = \sum_{j=0}^{p} c_j L_j(x) = \sum_{j=0}^{p} u_j l_j(x)$ give

$$(I_p u - u_p)(x) = \left(\frac{T}{2}\right)^{1-\mu} I_p \left[\int_{-1}^{x} (x - \tau)^{-\mu} e(\tau) d\tau \right]. \tag{3.8}$$

We write $e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x)$. Obviously,

$$\begin{aligned} e(x) = &\left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{x} (x - \tau)^{-\mu} e(\tau) d\tau + \underbrace{u(x) - I_p u(x)}_{I_1} \\ &+ \underbrace{\left(\frac{T}{2}\right)^{1-\mu} \left\{ (I_p - I) \left[\int_{-1}^{x} (x - \tau)^{-\mu} e(\tau) d\tau \right] \right\}}_{I_2}, \end{aligned} \tag{3.9}$$

where I is the identity operator. By the generalized Gronwall's inequality [8], we have

$$e(x) \le |I_1 + I_2| + C \int_{-1}^{x} (x - \tau)^{-\mu} |I_1 + I_2| d\tau, \tag{3.10}$$

which yields $\|e\|_{L^\infty} \le C(\|I_1\|_{L^\infty} + \|I_2\|_{L^\infty})$. It follows from (2.3) that if $y(t)$ satisfies condition (R), then

$$\|I_1\|_{L^\infty} \le C\Big(\frac{T}{4R}\Big)^{p+1},$$

and if $y(t)$ satisfies condition (M), then

$$\|I_1\|_{L^\infty} \le \frac{C}{\sqrt{p+1}}\Big(\frac{eMT}{4(p+1)}\Big)^{p+1}.$$

To estimate I_2, we define a linear integral operators $\mathcal{M}_1$ by

$$\mathcal{M}_1 v(x) = \int_{-1}^{x} (x - \tau)^{-\mu} v(\tau) d\tau. \tag{3.11}$$

From [16], for any function $v \in C([-1, 1])$, there exists a positive constant C such that

$$\|\mathcal{M}_1 v\|_{0,\kappa} \le C\|v\|_{L^\infty[-1,1]}. \tag{3.12}$$

Therefore, by virtue of Lemma 3.2, we have

$$\|\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e\|_{L^\infty} \le C p^{-\kappa} \|\mathcal{M}_1 e\|_{0,\kappa}, \quad \kappa \in (0, 1 - \mu). \tag{3.13}$$

Thus,

$$\begin{aligned} \|I_2\|_{L^\infty} &= \left(\frac{T}{2}\right)^{1-\mu} \|(I_p - I)\mathcal{M}_1 e\|_{L^\infty} = \left(\frac{T}{2}\right)^{1-\mu} \|(I_p - I)(\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e)\|_{L^\infty} \\ &\le \left(\frac{T}{2}\right)^{1-\mu} (1 + \|I_p\|_{L^\infty}) \|\mathcal{M}_1 e - \mathcal{T}_p \mathcal{M}_1 e\|_{L^\infty} \le C p^{-\kappa} \log p \|e\|_{L^\infty}. \end{aligned} \tag{3.14}$$

Consequently, for sufficiently large p, we obtain $\|I_2\|_{L^\infty} \le \frac{1}{2C}\|e\|_{L^\infty}$. This completes the proof. □

Table 3.1: Example 3.1: L^∞ and weighted L^2 errors with respect to N for $t \in [0, 6]$

N		4	8	12	14
$\|\cdot\|_{L^\infty}$	Our Method	4.8408e-03	6.5600e-04	2.1539e-04	1.8947e-04
	Spectral Jacobi Mehtod	8.4195e-03	5.4146e-04	1.6376e-04	1.4467e-04
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	7.0517e-03	7.0605e-04	2.1543e-04	1.8630e-04
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	1.3429e-02	6.8552e-04	1.9362e-04	1.6830e-04
N		18	20	24	26
$\|\cdot\|_{L^\infty}$	Our method	8.7019e-05	8.1105e-06	2.0336e-07	2.9167e-09
	Spectral Jacobi Method	7.0872e-05	3.8413e-06	8.7916e-07	1.7642e-07
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	8.3915e-05	7.7757e-06	1.9346e-07	2.7674e-09
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	8.0931e-05	4.3619e-06	9.9085e-07	1.9832e-07

Table 3.2: Example 3.2 L^∞ and weighted L^2 errors with respect to N for $t \in [0, 4]$

N		4	6	8	10
$\|\cdot\|_{L^\infty}$	Our Method	4.8974e-01	2.1410e-02	5.7430e-04	9.7091e-06
	Spectral Jacobi Mehtod	4.0381e-01	1.6403e-02	4.1529e-04	7.3809e-06
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	5.8315e-01	2.1210e-02	4.9030e-04	7.3712e-06
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	7.5536e-01	2.7139e-02	6.2278e-04	1.0286e-05
N		12	14	16	18
$\|\cdot\|_{L^\infty}$	Our method	1.3511e-07	1.4702e-09	1.0267e-11	3.4817e-13
	Spectral Jacobi Method	9.9530e-08	1.0766e-09	1.2221e-11	2.2382e-12
$\|\cdot\|_{w^{-1/2,-1/2}}$	Our method	9.4324e-08	9.6632e-10	6.4585e-12	2.3305e-13
$\|\cdot\|_{w^{-\mu,0}}$	Spectral Jacobi Method	1.3159e-07	1.3714e-09	1.5158e-11	2.7214e-12

3.1.2. Numerical examples

In this subsection, we find numerical approximations to solutions of two examples to demonstrate the theoretical results of Theorem 3.1. Unlike the numerical scheme in [6, 7] which is called spectral Jacobi method, our scheme is of the form

$$LC_p - AC_p = G_p, \tag{3.15}$$

where $C_p = [c_0, \cdots, c_p]^T$ and $G_p = [g(x_0), \cdots, g(x_p)]^T$ and the elements of the matrix $A = (a_{ij})$ and $L = (l_{ij})$ are given by

$$a_{ij} = \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1 + x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i), \qquad l_{ij} = L_j(x_i),$$

which are derived from (2.12).

In the figures below, reference curve 1 is the graph of function $f(p) = \frac{1}{2^{p+1}}$ associating with condition (R) and reference curve 2 is the graph of $f(p) = \frac{1}{\sqrt{p+1}}(\frac{eMT}{4(p+1)})^{p+1}$ corresponding to condition (M), where $M = \sqrt{e}$.

Example 3.1. Consider a Volterra integral equation of the form (1.1) on [0, 6] with $\mu = \frac{1}{2}$ and

$$b(t) = (t+2)^{2/3} - \frac{3\pi}{8}(t+2)^2 + \frac{3}{4}(t+2)^2 \arctan\left(\sqrt{\frac{2}{t}}\right) - \frac{\sqrt{2t}}{4}(3t+10).$$

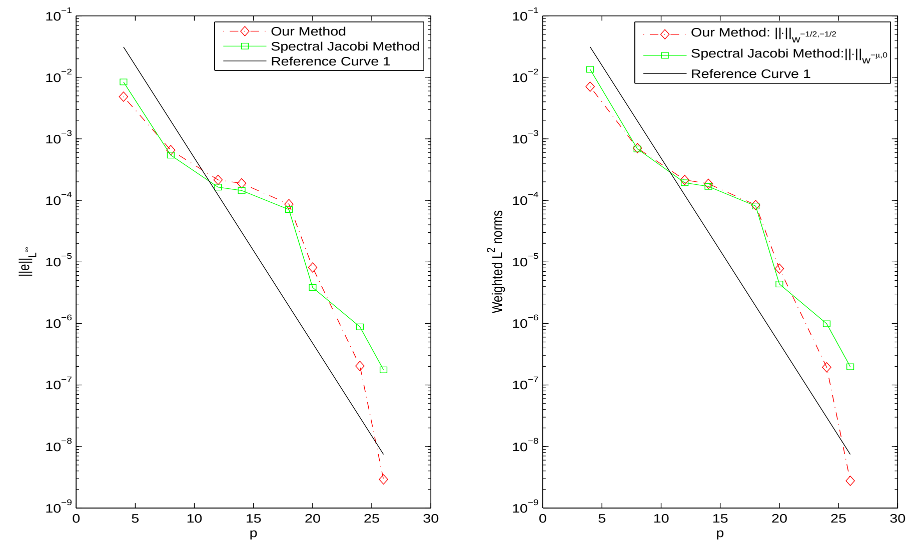

Fig. 3.1. Example 3.1: L^∞ error (left) and weighted L^2 error with variation of p.

The exact solution for this example is $y(t) = (2+t)^{3/2}$. Obviously, this solution satisfies condition (R). Hence, we expect a geometric rate of convergence. Numerical errors for our method and the spectral Jacobi method are presented in Table 3.1 and Fig. 3.1. We see that our method outperforms the spectral Jacobi method for larger $p > 0$.

Example 3.2. Consider a Volterra equation of the form (1.1) on $[0,\ 4]$ with $\mu = \frac{2}{3}$ and $b(t) = e^t(1 - \gamma(\frac{1}{3}, t))$, where $\gamma(a, x)$ is the lower incomplete Gamma function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt.$$

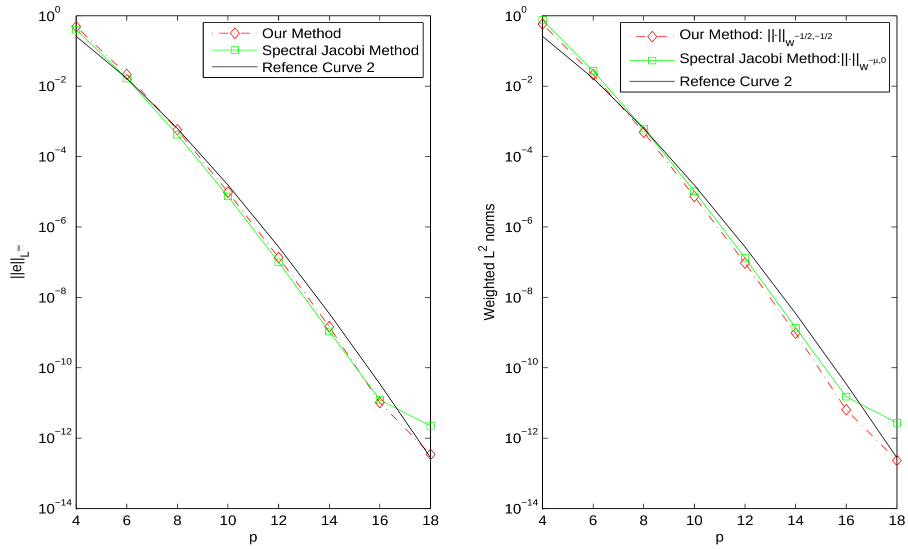

Fig. 3.2. Example 3.2: L^∞ error (left) and weighted L^2 error with variation of p.

The exact solution of this equation is $y(t) = e^t$, which satisfies condition (M). We expect a supergeometric rate of convergence for numerical approximations, see Table 3.2 and Fig. 3.2.

3.2. Fredholm integral equations

For our method, the error analysis of the L^∞ norm for Fredholm equations is similar to that of Volterra equations. However, the analysis for Fredholm type equations is not included in [6, 7]. Let us start with a property of weakly singular integral operators.

Lemma 3.3. ([17] Corollary 2.1) *Weakly singular integral operators are compact from L^∞ to $C[0,T]$ (and hence from L^∞ to L^∞) and from $C[0,T]$ to $C[0,T]$.*

3.2.1. Error estimate in L^∞

Theorem 3.2. *Let $y(t)$ and $u_p(x)$ be the exact solution and spectral approximation of (1.2), respectively.*

1) *If y satisfies the condition (R), then, for sufficiently large p,*

$$\|u - u_p\|_{L^\infty} \leq C\left(\frac{T}{4R}\right)^{p+1}; \tag{3.16}$$

2) *If y satisfies the condition (M), then for sufficiently large p,*

$$\|u - u_p\|_{L^\infty} \leq \frac{C}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{3.17}$$

Proof. Letting $e(x) = u(x) - u_p(x)$ and following the same routine as the proof in Theorem 3.1, we have

$$u(x_i) - u_i = \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}e(\tau)d\tau + \left(\frac{T}{2}\right)^{1-\mu}\int_{x_i}^{1}(\tau-x_i)^{-\mu}e(\tau)d\tau. \tag{3.18}$$

Multiply both sides by $l_i(x)$ and sum up from 0 to p. Applying the fact that $u_p(x) = \sum_{j=0}^{p} c_j L_j(x) = \sum_{j=0}^{p} u_j l_j(x)$ give

$$I_p u - u_p = \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{-1}^{x}(x-\tau)^{-\mu}e(\tau)d\tau\right] + \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{x}^{1}(\tau-x)^{-\mu}e(\tau)d\tau\right]. \tag{3.19}$$

We write $e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x)$. Then,

$$\begin{aligned} e(x) = {} & \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{1}|x-\tau|^{-\mu}e(\tau)d\tau + \underbrace{u(x) - I_p u(x)}_{I_1} \\ & + \underbrace{\left(\frac{T}{2}\right)^{1-\mu}\left\{(I_p - I)\left[\int_{-1}^{1}|x-\tau|^{-\mu}e(\tau)d\tau\right]\right\}}_{I_2}, \end{aligned} \tag{3.20}$$

By Lemma 3.4 and the Fredholm Alternative, we have

$$\|e\|_{L^\infty} \leq C(\|I_1\|_{L^\infty} + \|I_2\|_{L^\infty}), \tag{3.21}$$

where I_1 has exactly the same estimation as that in Theorem 3.1.

We define an operator $\mathcal{M}_1$ as we did in Theorem 3.1. Consequently, for any function $v \in C([-1,1])$, there exists a positive constant C such that [16]

$$\|\mathcal{M}_1 v\|_{0,\kappa} \leq C\|v\|_{L^\infty[-1,1]}. \tag{3.22}$$

We further define another linear integral operators $\mathcal{M}_2$ by

$$\mathcal{M}_2 v(x) = \int_x^1 (\tau - x)^{-\mu} v(\tau) d\tau. \tag{3.23}$$

We can obtain the same estimate for $\mathcal{M}_2$ as (3.22) following [16]. Therefore, we derive from Lemma 3.2 that

$$\|\mathcal{M}_i e - \mathcal{T}_p \mathcal{M}_i e\|_{L^\infty} \leq C p^{-\kappa} \|\mathcal{M}_i e\|_{0,\kappa}, \quad \kappa \in (0, 1-\mu),\ i = 1, 2. \tag{3.24}$$

These results lead to

$$\begin{aligned}
\|I_2\|_{L^\infty} &= \left(\frac{T}{2}\right)^{1-\mu} \left\|(I_p - I)\mathcal{M}_1 e + (I_p - I)\mathcal{M}_2 e\right\|_{L^\infty} \\
&= \left(\frac{T}{2}\right)^{1-\mu} \left(\left\|(I_p - I)(\mathcal{M}_1 e - \mathcal{T}_p\mathcal{M}_1 e)\right\|_{L^\infty} + \left\|(I_p - I)(\mathcal{M}_2 e - \mathcal{T}_p\mathcal{M}_2 e)\right\|_{L^\infty}\right) \\
&\leq \left(\frac{T}{2}\right)^{1-\mu} (1 + \|I_p\|_{L^\infty}) \left(\left\|\mathcal{M}_1 e - \mathcal{T}_p\mathcal{M}_1 e\right\|_{L^\infty} + \left\|\mathcal{M}_1 e - \mathcal{T}_p\mathcal{M}_2 e\right\|_{L^\infty}\right) \\
&\leq C p^{-\kappa} \log p \|e\|_{L^\infty}.
\end{aligned} \tag{3.25}$$

The desired results (3.16) and (3.17) follow. □

3.2.2. Numerical examples

From Algorithm for (1.2), we can obtain the scheme

$$LC_p - AC_p = G_p,$$

where $C_p = [c_0, \cdots, c_p]^T$, $G_p = [g(x_0), \cdots, g(x_p)]^T$, the elements of the matrix $A = (a_{ij})$ are given by

$$\begin{aligned}
a_{ij} =& \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \\
&+ \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i),
\end{aligned}$$

and the elements of the matrix $L = (l_{ij})$ are given by

$$l_{ij} = L_j(x_i).$$

In Fig. 3.3 and Fig. 3.4 below, reference curve 1 is the graph of function $f(p) = \frac{1}{4^{p+1}}$ associating with condition (R) and reference curve 2 is the graph of $f(p) = \frac{1}{\sqrt{p+1}}(\frac{eT}{4(p+1)})^{p+1}$ corresponding to condition (M).

Table 3.3: Example 3.3: errors against N for $t \in [0, 6]$

N	4	6	8	10
$\|\cdot\|_{L^\infty}$	4.1805e-04	1.2503e-04	4.7910e-06	1.5799e-07
$\|\cdot\|_{w^{-1/2,-1/2}}$	6.5631e-04	1.6064e-04	5.7565e-06	1.8530e-07
N	12	14	18	22
$\|\cdot\|_{L^\infty}$	2.4987e-08	6.1532e-10	5.0531e-11	6.8390e-14
$\|\cdot\|_{w^{-1/2,-1/2}}$	3.4284e-08	8.4890e-10	1.2349e-10	1.3079e-13

Table 3.4: Example 3.4: errors against N for $t \in [0, 10]$

N	4	8	10	12
$\|\cdot\|_{L^\infty}$	2.5522e-01	2.2144e-02	1.0467e-03	1.5876e-05
$\|\cdot\|_{w^{-1/2,-1/2}}$	5.3307e-01	3.0980e-02	1.2953e-03	1.8215e-05
N	16	18	20	22
$\|\cdot\|_{L^\infty}$	1.8332e-08	9.5780e-11	1.8910e-12	2.0761e-14
$\|\cdot\|_{w^{-1/2,-1/2}}$	2.8859e-08	1.2175e-10	2.7897e-12	3.3669e-14

Example 3.3 Consider a Fredholm equation of the form

$$y(t) = \int_0^6 |t-s|^{-\mu} y(s)ds + b(t), \quad t \in [0, 6]. \tag{3.26}$$

Specifically, we choose $\mu = 1/2$ and

$$\begin{aligned} b(t) =& \sqrt{t+2} - \frac{\pi}{2}(t+2) + (t+2)\arctan(\sqrt{2/t}) - \sqrt{2t} - 2\sqrt{12-2t} \\ &- (t+2)\log(2\sqrt{6-t} + 4\sqrt{2}) + (t+2)\log(2\sqrt{t+2}), \end{aligned}$$

so that the true solution is $y(t) = \sqrt{t+2}$. Clearly, the solution satisfies condition (R). Numerical errors are reported in Table 3.3 and Fig. 3.3.

Example 3.4 Consider a Fredholm equation of the form

$$y(t) = \int_0^{10} |t-s|^{-\mu} y(s)ds + b(t), \quad t \in [0, 10]. \tag{3.27}$$

We choose $\mu = 1/2$ and

$$\begin{aligned} b(t) =& \sin(t) + \sqrt{2\pi}\left[\cos(t)S\sqrt{2t/\pi} - \sin(t)C\sqrt{2t/\pi}\right] \\ &- \sqrt{2\pi}\left[\sin(t)C\sqrt{2(10-t)/\pi} + \cos(t)S\sqrt{2(10-t)/\pi}\right] \end{aligned}$$

Here, $C(u)$ and $S(u)$ are Fresnel integrals defined by

$$C(u) = \int_0^u \cos(\pi x^2/2)dx, \quad S(u) = \int_0^u \sin(\pi x^2/2)dx.$$

Then, the true solution is $y(t) = \sin(t)$. It is obvious that the solution satisfies condition (M). Numerical results are given in Table 3.4 and Fig. 3.4.

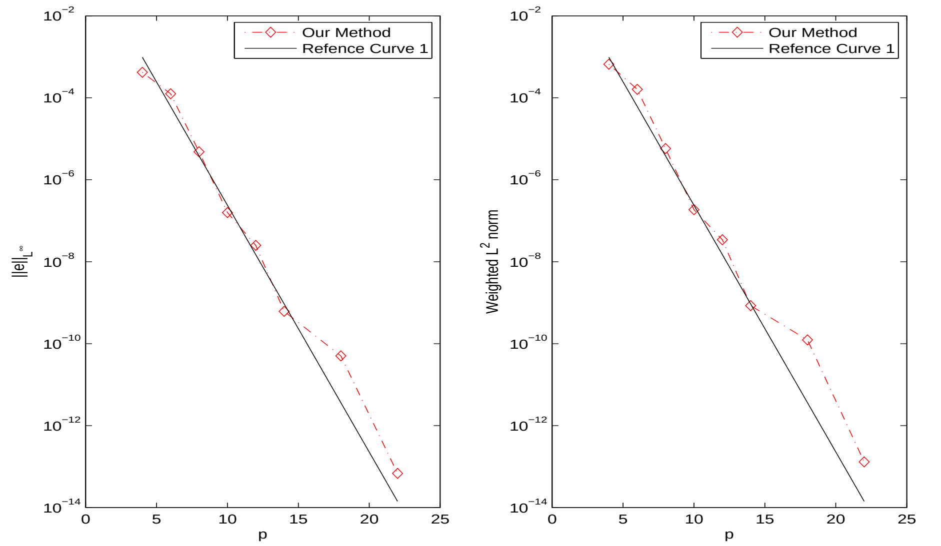

Fig. 3.3. Example 3.3 : L^∞ error (left) and weighted L^2 error with variation of p.

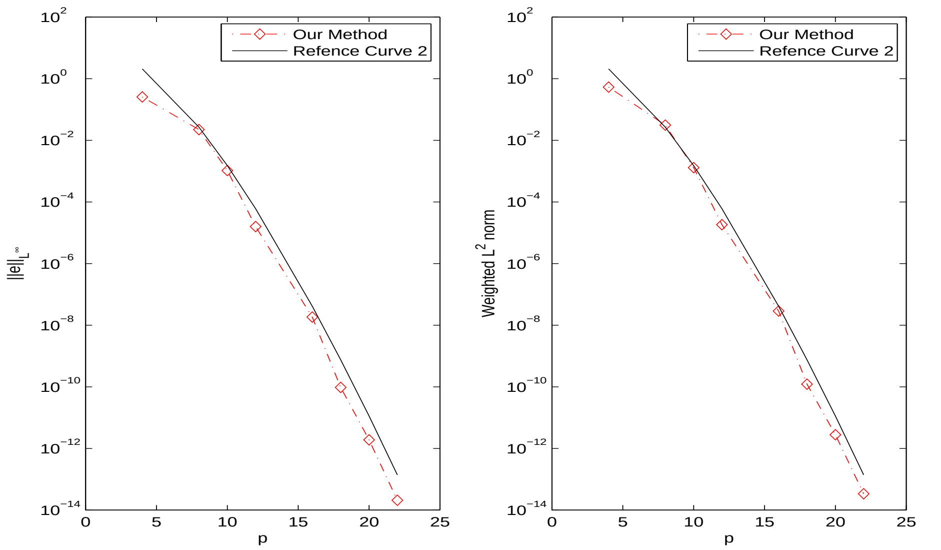

Fig. 3.4. Example 3.4 : L^∞ error (left) and weighted L^2 error with variation of p.

4. Convergence analysis of integro-differential equations

4.1. Volterra integro-differential equations

Theorem 4.1. *Let y and y_p be the exact solution and spectral approximation of* (1.3), *respectively.*

1) *If $y(t)$ satisfies the condition (R), then, for sufficiently large p,*

$$\|y - y_p\|_{L^\infty} \le \frac{C(p+2)}{R}\left(\frac{T}{4R}\right)^{p+1}; \tag{4.1}$$

2) *If y satisfies the condition (M), then, for sufficiently large p,*

$$\|y - y_p\|_{L^\infty} \le \frac{C\log p}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{4.2}$$

Proof. Using (2.17), we have

$$\left.\frac{du_p}{dx}\right|_{x_i} = f(x_i)u_i + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}u_p(\tau)d\tau + g(x_i), \tag{4.3}$$

where $u_i = y_0 + \sum_{j=1}^{p} c_j(L_j(x_i) + L_{j-1}(x_i))$. Note that the true solution at the Chebyshev points satisfies

$$\left.\frac{du}{dx}\right|_{x_i} = f(x_i)u(x_i) + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}u(\tau)d\tau + g(x_i). \tag{4.4}$$

Letting $e(x) = u(x) - u_p(x)$, gives

$$\left.\left(\frac{du}{dx} - \frac{du_p}{dx}\right)\right|_{x_i} = f(x_i)e(x_i) + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x_i}(x_i-\tau)^{-\mu}e(\tau)d\tau. \tag{4.5}$$

Multiply both sides by $l_i(x)$ and sum up from 0 to p. Applying the fact that $u_p(x) = y_0 + \sum_{j=1}^{p} c_j(L_j(x) + L_{j-1}(x)) = y_0 l_0(x) + \sum_{j=1}^{p} u_j l_j(x)$, we obtain

$$\begin{aligned} &I_p u'(x) - I_p u_p'(x) - I_p\Big[f(x)(I_p u(x) - u_p(x))\Big] \\ =&\left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{-1}^{x}(x-\tau)^{-\mu}e(\tau)d\tau\right] + I_p\Big[f(x)(u(x) - I_p u(x))\Big]. \end{aligned} \tag{4.6}$$

We write

$$e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x),\ \ e'(x) = u'(x) - I_p u'(x) + I_p u'(x) - u_p'(x). \tag{4.7}$$

Thus, by (4.6) and simple calculations,

$$\begin{aligned} &e'(x) - f(x)e(x) \\ =&u'(x) - I_p u'(x) - f(x)(u(x) - I_p u(x)) - (I - I_p)[f(x)(I_p u(x) - u_p(x))] \\ &\quad + I_p u'(x) - u_p'(x) - I_p[f(x)(I_p u(x) - u_p(x))] \\ =&\underbrace{u'(x) - I_p u'(x)}_{I_1} - \underbrace{f(x)(u(x) - I_p u(x))}_{I_2} - \underbrace{(I - I_p)[f(x)(I_p u(x) - u_p(x))]}_{I_3} \\ &\quad + \underbrace{\left(\frac{T}{2}\right)^{1-\mu}\left\{(I_p - I)\left[\int_{-1}^{x}(x-\tau)^{-\mu}e(\tau)d\tau\right]\right\}}_{I_4} + \underbrace{I_p\Big[f(x)\Big(u(x) - I_p u(x)\Big)\Big]}_{I_5} \\ &\quad + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{x}(x-\tau)^{-\mu}e(\tau)d\tau. \end{aligned} \tag{4.8}$$

Denoting $I(x) = \sum_{k=1}^{5} I_k(x)$ and integrating both sides from -1 to x, we derive

$$\begin{aligned} e(x) &= \int_{-1}^{x} I(s)ds + \int_{-1}^{x} \left(f(s) + (\frac{T}{2})^{1-\mu} \int_{s}^{x} (v-s)^{-\mu} dv \right) e(s)ds \\ &= \int_{-1}^{x} I(s)ds + \int_{-1}^{x} \left(f(s) + (\frac{T}{2})^{1-\mu} \frac{(x-s)^{1-\mu}}{1-\mu} \right) e(s)ds \\ &\leq \tilde{I}(x) + L \int_{-1}^{x} e(s)ds, \end{aligned} \tag{4.9}$$

where

$$\tilde{I}(x) = \int_{-1}^{x} I(s)ds, \qquad L = \frac{T^{1-\mu}}{2^{2-2\mu}} \frac{1}{1-\mu} + \max_{x\in[-1,1]} |f(x)|.$$

By the Gronwall inequality, we have

$$e(x) \leq |\tilde{I}(x)| + C \int_{-1}^{x} |\tilde{I}(\tau)| d\tau. \tag{4.10}$$

Therefore, $\|e\|_{L^\infty} \leq C\|\tilde{I}\|_{L^\infty} \leq 2C\|I\|_{L^\infty}$. Now, let us estimate terms from I_1 to I_5. If y satisfies condition (R), then

$$\|I_1\|_{L^\infty} \leq \frac{C(p+2)}{R} \Big(\frac{T}{4R}\Big)^{p+1}, \tag{4.11a}$$

and if y satisfies condition (M), then

$$\|I_1\|_{L^\infty} \leq \frac{C}{\sqrt{p+1}} \Big(\frac{eMT}{4(p+1)}\Big)^{p+1}. \tag{4.11b}$$

Using (2.3) and the fact that $|f(x)| \leq M$ on $[0, T]$, we know that if y satisfies condition (R), then

$$\|I_2\|_{L^\infty} \leq C\Big(\frac{T}{4R}\Big)^{p+1}, \tag{4.12a}$$

and if y satisfies condition (M), then

$$\|I_2\|_{L^\infty} \leq \frac{C}{\sqrt{p+1}} \Big(\frac{eMT}{4(p+1)}\Big)^{p+1}. \tag{4.12b}$$

As $f(x)$ is analytic and $I_p u(x) - u_p(x)$ are polynomials, their product is also analytic. Consequently,

$$\|I_3\|_{L^\infty} \leq \frac{C}{\sqrt{p+1}} \left(\frac{eT}{4(p+1)}\right)^{p+1}. \tag{4.13}$$

The estimate of I_4 is exactly the same as I_2 in the proof of Theorem 3.1. Therefore, if p is sufficiently large, from the estimate of I_2 in Theorem 3.1, we obtain

$$\|I_4\|_{L^\infty} \leq \frac{1}{4C}\|e\|_{L^\infty}. \tag{4.14}$$

Finally, noting $\|I_5\|_{L^\infty} \leq C \log p \|u(x) - I_p u(x)\|_{L^\infty}$, we have

$$\|I_5\|_{L^\infty} \leq C \log p \left(\frac{T}{4R}\right)^{p+1}, \qquad (y \text{ satisfies condition (R)}), \tag{4.15a}$$

$$\|I_5\|_{L^\infty} \le \frac{C\log p}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}, \qquad (y \text{ satisfies condition (M)}). \tag{4.15b}$$

The decided results follow by combining all the estimates above.

4.2. Fredholm integro-differential equation

Theorem 4.2. *Let y and y_p be the exact solution and spectral approximation of* (1.4)*, respectively. Then*

1) *If y satisfies the condition (R), then, for sufficiently large p,*

$$\|y-y_p\|_{L^\infty} \le \frac{C(p+2)}{R}\left(\frac{T}{4R}\right)^{p+1}; \tag{4.16}$$

2) *If y satisfies the condition (M), then, for sufficiently large p,*

$$\|y-y_p\|_{L^\infty} \le \frac{C\log p}{\sqrt{p+1}}\left(\frac{eMT}{4(p+1)}\right)^{p+1}. \tag{4.17}$$

Proof. Following exactly the same technique as in the proof of Theorem 4.1, we obtain

$$\begin{aligned} &I_p u'(x) - I_p u_p'(x) - I_p\Big[f(x)(I_p u(x) - u_p(x))\Big] \\ =& \left(\frac{T}{2}\right)^{1-\mu} I_p\left[\int_{-1}^{1} |x-\tau|^{-\mu} e(\tau)d\tau\right] + I_p\Big[f(x)(u(x) - I_p u(x))\Big]. \end{aligned} \tag{4.18}$$

Again, we write

$$e(x) = u(x) - I_p u(x) + I_p u(x) - u_p(x), \;\; e'(x) = u'(x) - I_p u'(x) + I_p u'(x) - u_p'(x). \tag{4.19}$$

Thus, by (4.18) and simple calculations,

$$\begin{aligned} &e'(x) - f(x)e(x) \\ =& u'(x) - I_p u'(x) - f(x)(u(x) - I_p u(x)) - (I - I_p)[f(x)(I_p u(x) - u_p(x))] \\ &+ I_p u'(x) - u_p'(x) - I_p[f(x)(I_p u(x) - u_p(x))] \\ =& \underbrace{u'(x) - I_p u'(x)}_{I_1} - \underbrace{f(x)(u(x) - I_p u(x))}_{I_2} - \underbrace{(I - I_p)[f(x)(I_p u(x) - u_p(x))]}_{I_3} \\ &+ \underbrace{\left(\frac{T}{2}\right)^{1-\mu}\left\{(I_p - I)\left[\int_{-1}^{1} |x-\tau|^{-\mu} e(\tau)d\tau\right]\right\}}_{I_4} + \underbrace{I_p[f(x)(u(x) - I_p u(x))]}_{I_5} \\ &+ \left(\frac{T}{2}\right)^{1-\mu} \int_{-1}^{1} |x-\tau|^{-\mu} e(\tau)d\tau. \end{aligned} \tag{4.20}$$

Denoting $\tilde{I}(x) = \sum_{k=1}^{5} I_k(x)$ and adopting the idea from Theorem 4.2 in [12], we let $z(x) = e'(x)$. From our algorithm, it is clear that $z(-1) = 0$. Then, by change the order of integration, we have

$$
\begin{aligned}
z(x) &= \tilde{I}(x) + f(x)\int_{-1}^{x} z(s)ds + \left(\frac{T}{2}\right)^{1-\mu}\int_{-1}^{1}\int_{-1}^{s}|x-s|^{-\mu}dsz(u)du \\
&= \tilde{I}(x) + f(x)\int_{-1}^{x} z(s)ds + \left(\frac{T}{2}\right)^{1-\mu}\frac{1}{1-\mu}\int_{-1}^{x}(x-u)^{1-\mu}z(u)du \\
&\quad + \left(\frac{T}{2}\right)^{1-\mu}\frac{1}{1-\mu}\int_{x}^{1}(1-x)^{1-\mu}z(u)du. \\
&=: \tilde{I}(x) + Az
\end{aligned}
\tag{4.21}
$$

Clearly, the operator A defined above is compact by the Arzelá-Ascoli theory. Hence, (4.21) can be written as $z = Az + \tilde{I}$. From our assumption and the Fredholm Alternative, $(I - A)$ has a bounded inverse. Thus, $\|z\|_{L^\infty} \leq C\|\tilde{I}\|_{L^\infty}$. Note $e(x) = \int_{-1}^{x} z(s)ds, x \in [-1, 1]$. We have

$$\|e\|_{L^\infty} \leq 2\|z\|_{L^\infty} \leq C\|\tilde{I}\|_{L^\infty}.$$

Clearly, I_1, I_2, I_3 and I_5 are the same as those of Theorem 4.1 and I_4 is the same as I_2 in Theorem 3.2; hence, we obtain the same estimates for these terms. Thus our result holds. □

Remark 4.1. In Theorems 3.1, 3.2, 4.1 and 4.2, if $\mu = 0$, the kernels in (1.1)-(1.4) are smooth and thus compact, we can obtain geometric or supergeometric convergence for these equations in exactly the same fashion.

4.3. Numerical examples

Our scheme for integro-differential equations has the form

$$DC_p + LC_p - AC_p = G_p,$$

where $C_p = [c_0, \cdots, c_p]^T$,

$$(G_p)_i = g(x_i) + f(x_i)y_0 + \big(\frac{T}{2}\big)^{1-\mu}\frac{y_0}{1-\mu}(1+x_i)^{1-\mu},$$

and the elements of the matrix $A = (a_{ij})$, $L = (l_{ij})$ and $D = (d_{ij})$ are given by

Scheme for (1.3)*:*

$$
\begin{aligned}
a_{ij} &= \left(\frac{T}{2}\right)^{1-\mu}\frac{j!}{(1-\mu)_{j+1}}(1+x_i)^{1-\mu}P_j^{(\mu-1,1-\mu)}(x_i) \\
&\quad + \left(\frac{T}{2}\right)^{1-\mu}\frac{(j-1)!}{(1-\mu)_j}(1+x_i)^{1-\mu}P_{j-1}^{(\mu-1,1-\mu)}(x_i), \\
l_{ij} &= -\Big(L_j(x_i) + L_{j-1}(x_i)\Big)f(x_i), \\
d_{ij} &= \frac{j+1}{T}P_{j-1}^{(1,1)}(x_i) + \frac{j}{T}P_{j-2}^{(1,1)}(x_i).
\end{aligned}
$$

Table 4.1: Example 4.1: errors against N for $t \in [0, 6]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	2.5788e-04	4.2887e-06	3.3610e-08	4.6499e-09
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	7.8877e-11	1.0431e-11	1.4531e-12	5.1514e-13

Table 4.2: Example 4.2: errors against N for $t \in [0, 6]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	4.9325e+01	7.9231e-02	2.9523e-05	3.0818e-07
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	1.7735e-11	2.6148e-12	4.4338e-12	3.4106e-13

Scheme for (1.4)*:*

$$\begin{aligned}
a_{ij} =& \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1+x_i)^{1-\mu} P_j^{(\mu-1,1-\mu)}(x_i) \\
&+ \left(\frac{T}{2}\right)^{1-\mu} \frac{(j-1)!}{(1-\mu)_j} (1+x_i)^{1-\mu} P_{j-1}^{(\mu-1,1-\mu)}(x_i) \\
&+ \left(\frac{T}{2}\right)^{1-\mu} \frac{j!}{(1-\mu)_{j+1}} (1-x_i)^{1-\mu} P_j^{(1-\mu,\mu-1)}(x_i) \\
&+ \left(\frac{T}{2}\right)^{1-\mu} \frac{(j-1)!}{(1-\mu)_j} (1-x_i)^{1-\mu} P_{j-1}^{(1-\mu,\mu-1)}(x_i), \\
l_{ij} =& -\Big(L_j(x_i) + L_{j-1}(x_i)\Big) f(x_i), \\
d_{ij} =& \frac{j+1}{T} P_{j-1}^{(1,1)}(x_i) + \frac{j}{T} P_{j-2}^{(1,1)}(x_i).
\end{aligned}$$

In this whole section, reference curve 1 associate with Examples 4.1 and 4.2 is the graph of $f(p) = \frac{p+2}{3^{p+1}}$ and reference curve 1 associate with Examples 4.3 and 4.4 is the graph of $f(p) = \frac{p+2}{(17/5)^{p+1}}$. Reference curve 2 is the graph of $f(p) = \frac{\log p}{\sqrt{p+1}}(\frac{eMT}{4(p+1)})^{p+1}$ for all four examples.

Example 4.1. Now let us consider integro-differential equations. First, we consider an equation of the form (1.3) on [0, 6] with $\mu = \frac{1}{2}$, $a(t) = e^t$ and

$$b(t) = \frac{3}{2}\sqrt{t+2} - e^t(t+2)^{2/3} - \frac{3\pi}{8}(t+2)^2 + \frac{3}{4}(t+2)^2 \arctan(\sqrt{2/t}) - (3t+10)\sqrt{t/8}.$$

The exact solution for this example is $y(t) = (2+t)^{3/2}$. Numerical errors are reported in Table 4.1 and left part of Fig. 4.1.

Example 4.2. Consider an integro-differential equation of the form (1.3) on [0, 6] with $\mu = \frac{1}{4}$, $a(t) = \sin(t)$ and $b(t) = e^t - e^t(\sin(t) - \gamma(\frac{1}{3}, t))$, where $\gamma(a, x)$ is the lower incomplete Gamma function defined in Section 3. The exact solution of this equation is $y(t) = e^t$. Numerical results are listed in Table 4.2 and the right part of Fig. 4.1.

Table 4.3: Example 4.3: errors against N for $t \in [0, 6]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	4.1506e-04	4.4205e-06	2.9091e-08	2.8076e-09
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	2.4562e-11	2.3024e-12	2.2715e-13	2.3093e-14

Table 4.4: Example 4.4: errors against N for $t \in [0, 10]$

N	4	8	12	14
$\|\cdot\|_{L^\infty}$	3.0272e-01	2.9460e-02	9.5393e-05	2.7053e-06
N	18	20	22	24
$\|\cdot\|_{L^\infty}$	1.1763e-09	1.8301e-11	2.1344e-13	2.0872e-14

Example 4.3. Consider a Fredholm integro-differential equation of the form (1.4) on [0, 6]. We choose $\mu = 1/2$, $a(t) = e^t$ and

$$b(t) = \frac{1}{2\sqrt{t+2}} - e^t\sqrt{t+2} - \frac{\pi}{2}(t+2) + (t+2)\arctan(\sqrt{2/t}) - \sqrt{2t} - 2\sqrt{12-2t}$$
$$-(t+2)\log(2\sqrt{6-t} + 4\sqrt{2}) + (t+2)\log(2\sqrt{t+2}),$$

so that the true solution is $y(t) = \sqrt{t+2}$. Readers are referred to Table 4.3 and left part of Fig. 4.2 for numerical report.

Example 4.4. Consider an integro-differential equation of the form (1.4) on [0, 10]. We choose $\mu = 1/2$ and

$$b(t) = \cos(t) - a(t)\sin(t) + \sqrt{2\pi}\left[\cos(t)S(\sqrt{2t/\pi}) - \sin(t)C(\sqrt{2t/\pi})\right]$$
$$-\sqrt{2\pi}\left[\sin(t)C(\sqrt{2(10-t)/\pi}) + \cos(t)S(\sqrt{2(10-t)/\pi})\right],$$

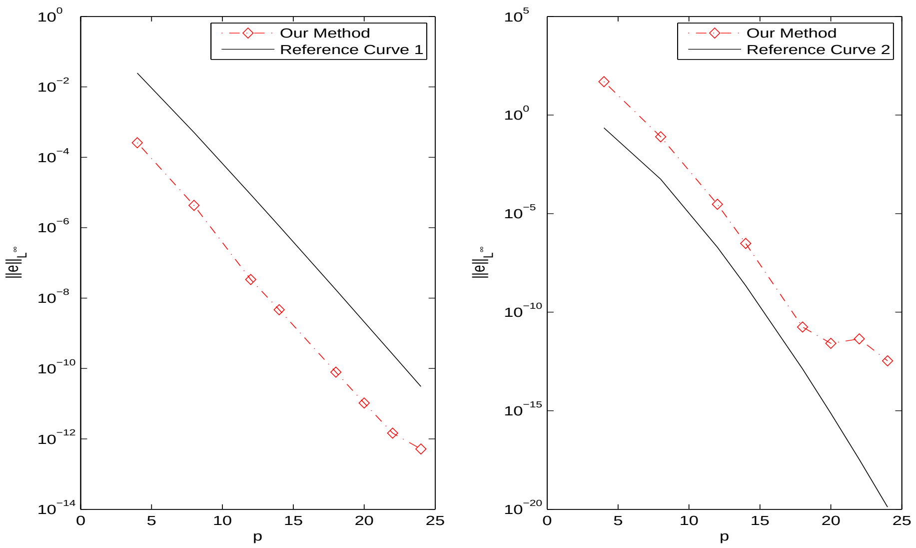

Fig. 4.1. $\|\cdot\|_{L^\infty}$ errors for Example 4.1 (left) and for Example 4.2 (right).

so that the true solution is $y(t) = \sin(t)$. Here, $C(u)$ and $S(u)$ are Fresnel integrals defined as in Example 3.4 and

$$a(t) = \begin{cases} 0, & t \le 5; \\ 1, & t > 5. \end{cases} \tag{4.22}$$

Numerical results are given in Table 4.4 and right part of Fig. 4.2.

From our numerical experiments, we see that the condition on $a(t)$ can be relaxed as long as the solution $y(t)$ is sufficiently smooth.

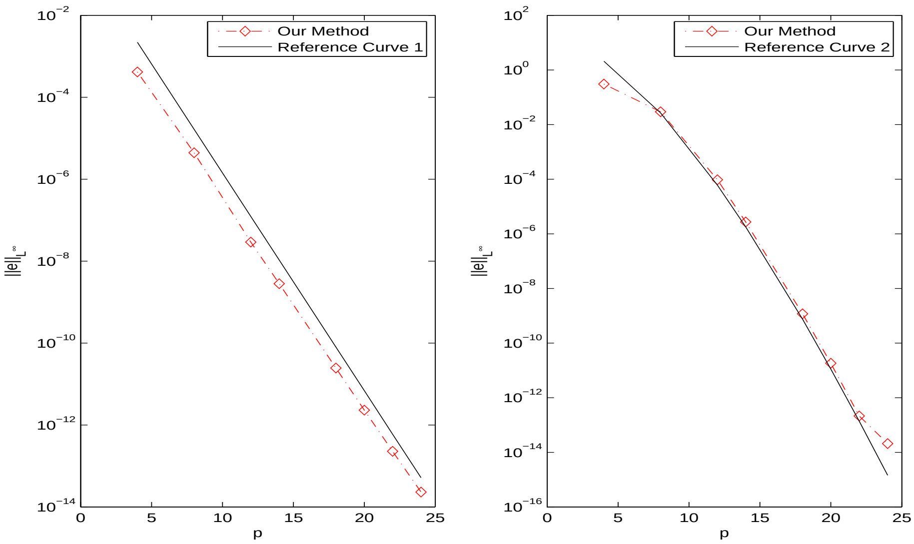

Fig. 4.2. $\|\cdot\|_{L^\infty}$ errors for Example 4.3 (left) and for Example 4.4 (right).

5. Concluding Remarks

In this work, we propose a spectral collocation method for various types of integral equations by utilizing some identities in the development of numerical algorithms. Geometric/supergeometric convergence results are established under certain regularity conditions. Our method has the following two merits:

• The exact integration in dealing with the composition of the Legendre polynomials and weakly singular kernel avoids numerical integration error and reduces the computational cost.

• Our method works not only for Volterra integral equations, but also for Fredholm integral equations and their related integro-differential equations.

Finally, we would like to emphasize the benefit of using different orthogonal polynomials in this study. Theoretically, the Chebyshev polynomial serves as a bridge to simplify our error estimate. In practice, we are able to reduce the computational cost by utilizing those mathematical formula relating different orthogonal polynomials, especially, the Legendre polynomial.

Acknowledgments. This research is partially supported by the GRF grants of Hong Kong Research Grant Council; the FRG grants of Hong Kong Baptist University; the US National Science Foundation through grant DMS-0612908; the Ministry of Education of China through the Changjiang Scholars program; and Guangdong Provincial Government of China through the "Computational Science Innovative Research Team" program.

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