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CONVERGENCE ANALYSIS FOR SPECTRAL APPROXIMATION TO A SCALAR TRANSPORT EQUATION WITH A RANDOM WAVE SPEED*

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Abstract

This paper is concerned with the initial-boundary value problems of scalar transport equations with uncertain transport velocities. It was demonstrated in our earlier works that regularity of the exact solutions in the random spaces (or the parametric spaces) can be determined by the given data. In turn, these regularity results are crucial to convergence analysis for high order numerical methods. In this work, we will prove the spectral convergence of the stochastic Galerkin and collocation methods under some regularity results or assumptions. As our primary goal is to investigate the errors introduced by discretizations in the random space, the errors for solving the corresponding deterministic problems will be neglected.

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1. Introduction

In numerical simulation, accounting for uncertainties in input quantities (such as model parameters, initial and boundary conditions, and geometry) becomes an important issue in recent years, especially in risk analysis, safety, and optimal design, see, e.g., [1, 7, 9, 20, 23, 27]. Many works have been recently devoted to the analysis and the implementation of the Stochastic Galerkin (SG) methods and Stochastic Collocation (SC) techniques for such problems. These methods are promising since they can exploit the possible regularity of the solution with respect to the stochastic parameters to achieve faster convergence. SG methods and SC methods can be classified as parametric techniques, since both approximate u, the solution of the underlying problems as a linear combination of suitable deterministic basis functions in probability space. The Stochastic Galerkin is a projection technique over a set of orthogonal polynomials with respect to the probability measure at hand [25, 26] and this methods is also called the general Polynomial Chaos (gPC) methods which is first introduced in [24], while Stochastic Collocation is a sum of Lagrangian interpolants over the probability space (see e.g., [10, 12, 17, 18]).

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Many numerical analysis results for the linear stochastic elliptic equation have been given by researchers. Babuška et al. [3,4] analyze the convergence properties for both the SG methods and SC methods for the stochastic elliptic equation, they show that both two methods achieves exponential convergence provided that the input random data are infinitely differentiable with respect to the random variables, under very mild assumptions on the growth of such derivatives, as is the case for standard expansions of random fields. Schwab and co-workers [14,15] provided similar results for the stochastic parabolic problems and the second order wave equations with random coefficients. They also discussed the convergence properties of the *Best N-term* approximation. The application of stochastic spectral methods to hyperbolic problems of conservation laws poses additional challenges. Very few works have been investigated for uncertain hyperbolic problems, especially for theoretical part. The scalar wave equation with a random wave number has been treated with gPC methods by Gottlieb and Xiu [13]. After that, Tang and Zhou [22,28] give some rigorous regularity results for the similar model problem, and show that the regularity results are important for the analysis of convergence rate of the SG methods and SC methods. In this paper, for the initial-boundary value problems of linear transport equation, we will show the analytic regularity of the solution with respect to the random parameter. Such results are crucial for analyzing the convergence properties of high order numerical methods. By using the analytic regularity results together with complex analysis, the spectral convergence of the Stochastic Galerkin and Collocation methods are shown. We note that related works on the second order wave equations with random data has been done by Nobile et.al. [6]. We also remark that numerical treatment for nonlinear hyperbolic problems are also discussed by many researchers, see, e.g., [19].

The paper is organized as follows. In Section 2, we set up the problems and discuss some analytic regularity results of the solutions in the parametric spaces. Spectral convergence of the Stochastic Gelerkin and collocation methods will be investigated in Section 3. Then, we provided with an numerical example in Section 4. Some conclusion remarks will be provided in the final section.

2. Problem Set Up and Solution Regularity

2.1. Problem set up

Let $x \in D \equiv [-1,1]$ be the spatial coordinate, and t be the time variable in $T \equiv [0,T]$, and $(\Omega, \mathcal{A}, \mathcal{P})$ be a complete probability space, whose event(ω) space is Ω and is equipped with σ-algebra $\mathcal{A}$, and $P : \mathcal{A} \to [0,1]$ is a probability measure. We consider the following class of linear scalar transport equations with random velocity: Find a random function, $u : T \times D \times \Omega \to \mathbb{R}$ such that P-almost everywhere in Ω, or in other words, almost surely the following equation holds:

$$\frac{\partial u(x,t,y(\omega))}{\partial t} = c(y(\omega))\frac{\partial u(x,t,y(\omega))}{\partial x}, \tag{2.1a}$$

$$u(x,t=0,y(\omega)) = u_0(x,y(\omega)). \tag{2.1b}$$

A well-posed boundary conditions can be given by

$$u(-1,t;y(\omega)) = u_L(t;y(\omega)) \qquad c(y(\omega)) < 0, \tag{2.2a}$$

$$u(+1,t;y(\omega)) = u_R(t;y(\omega)) \qquad c(y(\omega)) > 0. \tag{2.2b}$$

Eqs. (2.1)-(2.2) complete the set up of the problem.

In what follows, we will denote with $\Gamma \equiv y(\Omega)$ the image of $y(\omega)$. We will also assume for simplicity that the random variable admits a *probability density function* $\rho(y(\omega))$. For notation simplicity, we will omit the symbol ω. Then, problem (2.1)-(2.2) appears the following formula:

$$\frac{\partial u(x,t,y)}{\partial t} = c(y)\frac{\partial u(x,t,y)}{\partial x}, \tag{2.3a}$$

$$u(x,t=0,y) = u_0(x,y), \tag{2.3b}$$

$$u(-1,t;y) = u_L(t;y) \qquad c(y) < 0, \tag{2.3c}$$

$$u(+1,t;y) = u_R(t;y) \qquad c(y) > 0. \tag{2.3d}$$

Namely, one can view y as a parameter in the parametric space Γ. We will focus on the finite support case and assume that $\Gamma \equiv [-1,1]$ without loss of generalization. In the following, we also denote with $\Gamma^- \equiv [-1,0)$ and $\Gamma^+ \equiv [0,1]$.

2.2. Analytic regularity in the parametric space

For many stochastic PDEs, the exact solutions exhibit analytic regularity in the random spaces even if the regularity in the physic space is low, see, e.g., [3, 4] for elliptic problems and [6, 15, 15] for other problems, to name a few. In [22], some basic regularity results for problem (2.3) in the parametric space are provided. For example, the following results were obtained in [22].

Lemma 2.1. *Consider the problem* (2.3)*. If $c'(y)$ is bounded in the distribution sense and if the following conditions are satisfied:*

$$\int_\Gamma\int_D \rho(y)\Big(\partial_x u_0(x,y)\Big)^2 dxdy < \infty, \qquad \int_\Gamma\int_D \rho(y)\Big(\partial_y u_0(x,y)\Big)^2 dxdy < \infty, \tag{2.4a}$$

$$\int_0^T\int_{\Gamma^+} \frac{\rho(y)}{c(y)}\Big(\partial_t u_R(t;y)\Big)^2 dydt < \infty, \qquad \int_0^T\int_{\Gamma^+} \frac{\rho(y)}{c(y)}\Big(\partial_y u_R(t;y)\Big)^2 dydt < \infty, \tag{2.4b}$$

$$\int_0^T\int_{\Gamma^-} \frac{\rho(y)}{|c(y)|}\Big(\partial_t u_L(t;y)\Big)^2 dydt < \infty, \qquad \int_0^T\int_{\Gamma^-} \frac{\rho(y)}{|c(y)|}\Big(\partial_y u_L(t;y)\Big)^2 dydt < \infty, \tag{2.4c}$$

then

$$\int_\Gamma\int_D \rho(y)\Big(u_x^2 + u_y^2\Big) dxdy < C(T), \quad 0 < t \le T, \tag{2.5}$$

where $\rho(y) > 0$ is the probability distribution function and $C(T)$ is a positive constant depending on T.

We can view the regularity property in the above lemma as the regularity in the *Stochastic Hilbert Spaces*. Using the similar technique as in [22], we can provide with the following regularity properties with respect to the high order derivatives of the solution in the random spaces.

$$\max_{y\in\Gamma} \|\partial_y u(\cdot,t;y)\|_{L^2(D)} < C(T), \quad 0 < t \le T, \tag{2.6}$$

where $C(T)$ is a finite number depending on T and the given data. Based on these basic results, we want to give further regularity properties. It is nature that additional assumptions for the given data should be used. More precisely, we will use the following assumptions.

Assumption 2.1. *The given initial and boundary conditions satisfy the following properties:*

$$\max_{y\in\Gamma} |\partial_y^k c(y)| \le \gamma^k, \tag{2.7a}$$

$$\max_{\Gamma\otimes T} |\partial_y^k d_R| \le \delta_R^k, \quad \max_{\Gamma\otimes T} |\partial_y^k u_R| \le \delta_R^k, \tag{2.7b}$$

$$\max_{\Gamma\otimes T} |\partial_y^k d_L| \le \delta_L^k, \quad \max_{\Gamma\otimes T} |\partial_y^k u_L| \le \delta_L^k, \tag{2.7c}$$

$$\max_{y\in\Gamma} \|\partial_y^k u_0'\|_V^2 \le \eta^k, \;\; \max_{y\in\Gamma} \|\partial_y^k u_0\|_V^2 \le \eta^k, \tag{2.7d}$$

where $V = L^2(D)$, $k \in \mathbb{N}$, $\gamma, \delta_R, \delta_L, \eta$ *are positive constants,* u' *stands for* $\partial_x u$, *and* $d_R(y,t), d_L(y,t)$ *stand for* $\partial_t u_R/c(y), \partial_t u_L/c(y)$ *respectively. Without loss of generality, we also assume that* $\gamma \ge \max\{\delta_R, \delta_R, \eta\}$.

Note that Assumption 2.1 covers assumptions used in Lemma 2.1, which also implies the result of (2.6). We are now ready to state and prove the following result.

Theorem 2.1. *Under Assumption* 2.1*, we have*

$$\max_{\Gamma} \|\partial_y^k u(\cdot,t,\cdot)\|_V^2 \le C_k(T)\left(\delta_R^k + \delta_L^k + \eta^k\right) < +\infty, \tag{2.8}$$

for every $k \in \mathbb{N}$, *where* $C_k(T)$ *is a constant related to* k, γ *and* T.

Proof. Using the basic results of (2.6), we can conduct on the index k. Assume for the index $i \le k$, the solution u of problem (2.1) satisfies

$$\max_{\Gamma} \|\partial_y^i u(\cdot;y)\|_V^2 \le C_i(T)\left(\delta_R^i + \delta_L^i + \eta^i\right), \tag{2.9}$$

provided that the given initial and boundary condition satisfy (2.7). Then, for index of $k+1$, we have

$$\begin{aligned}\frac{d}{dt}(\partial_y^{k+1}u) &= \sum_{l=0}^{k+1} \binom{l}{k} \left(\partial_y^{k-l+1}c\right)\left(\partial_y^l u_x\right)\\ &= \sum_{l=0}^{k} \binom{l}{k} \left(\partial_y^{k-l+1}c\right)\left(\partial_y^l u_x\right) + c(y)\left(\partial_y^{k+1}u_x\right).\end{aligned} \tag{2.10}$$

Multiply $\partial_y^{k+1}u$ in both sides and integral in D one gets

$$\begin{aligned}\frac{d}{dt}\|\partial_y^{k+1}u\|_V^2 &= \int_D (\partial_y^{k+1}u)\sum_{l=0}^{k}\binom{l}{k}\left(\partial_y^{k-l+1}c\right)\left(\partial_y^l u_x\right)dx + c(y)\int_{-1}^{1}\left(\partial_y^{k+1}u\right)_x^2 dx\\ &\le \sum_{l=0}^{k}\binom{l}{k}\left(\partial_y^{k-l+1}c\right)\|(\partial_y^{k+1}u)\|_V^2 + \sum_{l=0}^{k}\binom{l}{k}\left(\partial_y^{k-l+1}c\right)\|(\partial_y^l u_x)\|_V^2\\ &\quad + C_0\left[(\partial_y^{k+1}u_R)^2 + (\partial_y^{k+1}u_L)^2\right],\end{aligned}$$

which yields

$$\begin{aligned}&\max_{y\in\Gamma}\|\partial_y^{k+1}u\|_V^2\\ &\le \left(\sum_{l=0}^{k}\binom{l}{k}\left(\partial_y^{k-l+1}c\right)\right)\int_T \max_{y\in\Gamma}\|(\partial_y^{k+1}u)\|_V^2 dt + \sum_{l=0}^{k}\binom{l}{k}\gamma^{k-l+1}\int_T \max_{y\in\Gamma}\|(\partial_y^l u_x)\|_V^2 dt\\ &\quad + C_0\int_T \max_{y\in\Gamma}\left(|\partial_y^{k+1}u_R|^2 + |\partial_y^{k+1}u_L|^2\right)dt + \max_{y\in\Gamma}\|\partial_y^{k+1}u_0\|_V^2.\end{aligned} \tag{2.11}$$

Note that $v = u_x$ satisfies the following problem:

$$\begin{aligned}
&\frac{\partial v(x,t,y)}{\partial t} = c(y)\frac{\partial v(x,t,y)}{\partial x}, \\
&v(x,t=0,y) = u_0'(x,y), \\
&v(-1,t;y) = u_x(-1,t,y) = d_R(t,y) \qquad c(y) < 0, \\
&v(+1,t;y) = u_x(+1,t,y) = d_L(t,y) \qquad c(y) > 0.
\end{aligned}$$

That is, u_x satisfies a problem similar with (eqn2.1). Note that the initial and boundary conditions are given in a general setting, and u_0', d_R, d_L are supposed to have the same properties as u_0, u_R, u_L (see Assumption 2.1). Consequently, by the conduction assumption (2.9), we have

$$\max_{\Gamma} \|\partial_y^i u_x(\cdot,t;y)\|_V^2 \le C_i(T)\left(\delta_R^i + \delta_L^i + \eta^i\right), \tag{2.13}$$

which leads to

$$\begin{aligned}
&\sum_{l=0}^{k} \binom{l}{k} \gamma^{k-l+1} \int_T \|(\partial_y^l u_x)\|_V^2 dt \\
&\le C_M(T)\Big((\gamma+\delta_R)^{k+1} + (\gamma+\delta_L)^{k+1} + (\gamma+\eta)^{k+1}\Big),
\end{aligned} \tag{2.14}$$

where $C_M(T) = \max_{i\le k} C_i(T)$. Substitute (2.14) into (2.11) one gets

$$\begin{aligned}
\max_{y\in\Gamma} \|\partial_y^{k+1} u\|_V^2 &\le (1+\gamma)^{k+1}\int_T \max_{y\in\Gamma}\|\partial_y^{k+1}u\|_V^2 dt + C_0(T)\left(\delta_R^{k+1} + \delta_L^{k+1} + \eta^{k+1}\right) \\
&\quad + C_M(T)\Big((\gamma+\delta_R)^{k+1} + (\gamma+\delta_R)^{k+1} + (\gamma+\eta)^{k+1}\Big) \\
&\le (1+\gamma)^{k+1}\int_T \max_{y\in\Gamma}\|\partial_y^{k+1}u\|_V^2 dt + \widetilde{C}(T)\left(\delta_R^{k+1} + \delta_L^{k+1} + \eta^{k+1}\right),
\end{aligned} \tag{2.15}$$

where $\widetilde{C}(T) = C_0(T) + 2^{k+1}C_M(T)$. Then, the desired result (2.8) follows by using the Gronwell inequality with $C_{k+1}(T) = \widetilde{C}(T)e^{T(1+\gamma)^{k+1}}$. $\square$

Some simple remarks are listed below:

- Regularity results of weaker norms can also be derived, for example, the above results imply the following:

$$\begin{aligned}
&\|\partial_y^k u(\cdot,t,\cdot)\|_{Mean-Square}^2 \\
&= \int_\Gamma\int_D (\partial_y^k u)^2 dxdy \le C_k(T)\left(\delta_R^k + \delta_L^k + \eta^k\right) < +\infty.
\end{aligned} \tag{2.16}$$

  This norm was used in [22] to analyze the convergence property of the stochastic collocation methods.

- We remark that the estimation in the above theorem is not optimal in the sense that $C_k(T)$ is related to k, γ and T in general case (even not uniform bounded). This is different from the cases of problems involving symmetric Laplace operator. However, for some special cases, for instance, the initial conditions and the boundary conditions are compatible, then, the solution will has similar properties as the initial function, and we can expect that $C_k(T)$ can be bounded by a positive constant uniformly. An spacial case is "c(y)=y" which was used as an example in [13,22]. We also remark that one usually use the Karhunen-Lòeve expansion to deal with the stochastic functions when space variable is involved:

$$\kappa(x,\omega) = \kappa_0(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\kappa_i(x)y_i(\omega),$$

where $\{\lambda_i\}_{i=1}^{\infty}$ and $\{\kappa_i\}_{i=1}^{\infty}$ are the eigenvalues and orthogonal eigenfunctions of $\mathrm{Cov}_\kappa(x,x')$, i.e.,

$$\int_\Omega \mathrm{Cov}_\kappa(x,z)\kappa_i(z)dz = \lambda_i\kappa_i(x).$$

Note the expansion above is a linear combination of random variables, and thus, for such cases, we may also expect that $C_k(T)$ can also be bounded by a uniform constant with compatible assumption.

For ease of notations, we denote with $\zeta^k = \max\{\delta_R^k, \delta_L^k, \eta^k\}$, and we will also assume that the coefficients $\{C_k(T)\}_{k=0}^{\infty}$ can be bounded by a constant $C(T)$ which is independent of k. We will use the following functional spaces

$$C^0(\Gamma,V) = \left\{u:\Gamma\to V,\ u\ \textit{continuous in } y,\ \max_{y\in\Gamma}\|u(y)\|_V < +\infty\right\}, \tag{2.17a}$$

$$L^\infty[T,C^0(\Gamma,V)] = \left\{u: T\to C^0(\Gamma,V),\ \max_{t\in T}\|u(t)\|_{C^0(\Gamma,V)} < +\infty\right\}, \tag{2.17b}$$

$$L^1[T,C^0(\Gamma,V)] = \left\{u: T\to C^0(\Gamma,V),\ \int_T \|u(t)\|_{C^0(\Gamma,V)}dt < +\infty\right\}. \tag{2.17c}$$

It is clear that Theorem 2.1 implies that $u \in L^\infty[T,C^0(\Gamma,V)]$. We are ready to give the following result whose proof follows very closely with [3,4].

Theorem 2.2. *The solution $u(x,t;y)$ as a function of y, $u:\Gamma\to L^\infty(T,V)$ admits an analytic extension $u(x,t;z), z\in\mathbb{C}$ in the region of the complex plane*

$$\Sigma(\Gamma,\tau) \equiv \{z\in\mathbb{C}, \mathtt{dist}(z,\Gamma)\le\tau\} \tag{2.18}$$

with $0<\tau<1/\sqrt{\zeta}$.

Proof. We define for every $y\in\Gamma$, the power series $u:\mathbb{C}\to L^\infty(T,V)$ as

$$u(z,x,t) = \sum_{k=0}^{\infty}(z-y)^k\partial_y^k u(y,x,t),$$

which yields

$$\|u(z)\|_{L^\infty[T,V]} \le \sum_{k=0}^{\infty}|z-y|^k\|\partial_y^k u(y)\|_{L^\infty(T,V)} \le C(T)\sum_{k=0}^{\infty}(|z-y|\sqrt{\zeta})^k. \tag{2.19}$$

The series converges for all $z\in\mathbb{C}$ satisfying $|z-y|\le\tau<1/\sqrt{\zeta}$. Moreover, in the ball $|z-y|\le\tau$, we have

$$\|u(z)\|_{L^\infty(T,V)} \le \frac{C(T)}{1-\tau\sqrt{\zeta}}. \tag{2.20}$$

The power series converges for every $y\in\Gamma$; hence, the function u can be extended analytically on the whole region $\Sigma(\Gamma,\tau)$ by a continuation argument. $\square$

3. Spectral Convergence Analysis

For the convergence analysis of stochastic numerical methods such as SG methods and SC methods, the key point is to analyze the regularity properties of the corresponding exact solution in the random spaces. Once we have the regularity results, we can use the framework in [3, 4] (for elliptic model problem) to get the convergence results for SG and SC methods. The framework is also used in [16] to analyze the convergence property for stochastic parabolic equations and in [6] for second order wave equations with a random speed. In the following, we will give the convergence analysis for model problem (2.1)-(2.2). We remark that the following results are also based on framework in [3, 4] although different models are considered. Some of the following results are slightly extension of results in [3, 4] and some are direct consequence of results in [3, 4].

3.1. Stochastic collocation methods

In stochastic collocation methods, we use the roots of the corresponding orthogonal polynomials as the collocation points, and the type of the polynomials is decided according to the distribution information of the random parameter. The commonly seen correspondences between the polynomials $\{P_k(y)\}$ and the distribution of the random variable y include Hermite-Gaussian, Legendre-uniform, etc., cf. [2, 25]. Let $\Theta = \{y_k\}_{k=0}^{p} \in \Gamma$ (the parameter space) be such a set of nodes. A Lagrange interpolation of the solution $u(x, y)$ can be written as

$$\mathcal{I}_p u(x,y) = \sum_{k=0}^{p} \tilde{u}(y_k) l_k(y), \tag{3.1}$$

where

$$l_k \in \mathcal{P}_p, \quad l_i(y_k) = \delta_{ik}, \quad 1 \le i, k \le N, \tag{3.2}$$

are the Lagrange interpolation polynomials. In this section, we prove the spectral convergence of such methods. As the primary goal of this paper is to investigate the error introduced by the random space, therefore we will neglect the error introduced when solving the corresponding deterministic problems. We will need the following lemmas which slightly generalize those in [3].

Lemma 3.1. *The operator $\mathcal{I}_p : L^\infty[T, C^0(\Gamma, V)] \to L^\infty(T, L^2_\rho(\Gamma, V))$ is continuous.*

Proof. Observe that

$$\begin{aligned} \|\mathcal{I}_p u\|_{L^\infty(T, L^2_\rho(\Gamma,V))} &= \left\| \int_\Gamma \rho(y) \| \sum_{k=0}^{p} u(\cdot,\cdot,y_k) l_k(y) \|_V^2 dy \right\|_{L^\infty(T)} \\ &\le \left\| \int_\Gamma \rho(y) \left(\sum_{k=0}^{p} \|u(\cdot,\cdot,y_k)\|_V l_k(y) \right)^2 dy \right\|_{L^\infty(T)}. \end{aligned} \tag{3.3}$$

It follows from the orthogonality property of Lagrange interpolation functions that

$$\begin{aligned} \|\mathcal{I}_p u\|_{L^\infty(T, L^2_\rho(\Gamma,V))} &\le \left\| \int_\Gamma \rho(y) \left(\sum_{k=0}^{p} \|u(\cdot,\cdot,y_k)\|_V^2 l_k^2(y) \right) dy \right\|_{L^\infty(T)} \\ &\le \left(\sum_{k=0}^{p} \omega_k \right) \left\| \max_{k=0,1,\cdots,p} \|u(\cdot,\cdot,y_k)\|_V^2 \right\|_{L^\infty(T)} \\ &\le C \|u\|_{L^\infty[T, C^0(\Gamma,V)]}. \end{aligned} \tag{3.4}$$

This completes the proof. $\square$

Lemma 3.2. *For every function $u \in L^\infty[T, C^0(\Gamma, V)]$, the interpolation error satisfies*

$$\|u - \mathcal{I}_p u\|_{L^\infty[T, L^2_\rho(\Gamma, V)]} \le \tilde{C} \inf_{v \in L^\infty[T, \mathcal{P}_p(\Gamma) \otimes V]} \|u - v\|_{L^\infty[T, C^0(\Gamma, V)]}. \tag{3.5}$$

Proof. Let us note that for all $v \in L^\infty[T, \mathcal{P}_p(\Gamma) \otimes V]$, it holds that $\mathcal{I}_p v = v$. Hence,

$$\|u(\cdot,\cdot,t) - \mathcal{I}_p u(\cdot,\cdot,t)\|_{L^2_\rho(\Gamma,V)} \le \|u(\cdot,\cdot,t) - v(\cdot,\cdot,t)\|_{L^2_\rho(\Gamma,V)} + \|\mathcal{I}_p(u-v)(\cdot,\cdot,t)\|_{L^2_\rho(\Gamma,V)}, \tag{3.6}$$

and the desired result follows by using Lemma 3.1. $\square$

Now, we are ready to give the spectral convergence of the Stochastic Collocation methods for which the proof is omitted, and one can find the proof in [3].

Theorem 3.1. *Given a function $u \in L^\infty[T, C^0(\Gamma, V)]$ which admits an analytic extension in the region of complex plane*

$$\Sigma(\Gamma, \tau) \equiv \{z \in \mathbb{C}, \mathtt{dist}(z, \Gamma) \le \tau\}$$

for some $\tau > 0$, it holds that

$$\begin{aligned} &\min_{v \in L^1[T, \mathcal{P}_p(\Gamma) \otimes V]} \|u - v\|_{L^\infty[T, C^0(\Gamma, V)]} \\ \le\ & \frac{2}{\varrho - 1} e^{-p log(\varrho)} \max_{z \in \Sigma(\Gamma, \tau)} \|u(z)\|_{L^\infty(T, V)}, \end{aligned} \tag{3.7}$$

where $1 < \varrho = \tau + \sqrt{1 + \tau^2}$.

3.2. Stochastic Galerkin methods

Following the standard gPC expansion, we assume that $u(x,t,y)$ has a converging expansion of the form

$$u(x,t,y) = \sum_{k=0}^{\infty} \tilde{u}_k(x,t) P_k(y). \tag{3.8}$$

Similar to the Stochastic Collocation methods, the orthogonal polynomials $\{P_k(y)\}_{k=0}^\infty$ are chosen according to the distribution of the random variable y. For simplicity we will discuss in this paper the case of random variable y with uniformly distributed random variable y, for which the corresponding polynomials are Legendre polynomials. By truncating the expansion (3.8) with the first $P+1$ terms and employing a Galerkin projection, it is straightforward to verify that the first $p+1$ coefficients $\{\tilde{u}_k\}_{k=0}^p$ satisfy the following system of equations

$$\frac{\partial \tilde{u}_k(x,t)}{\partial t} = \sum_{j=0}^{p} a_{j,k} \frac{\partial \tilde{u}_k(x,t)}{\partial x}, \quad j = 0, \cdots, p, \tag{3.9}$$

with the vector form

$$\frac{\partial \vec{v}(x,t)}{\partial t} = \mathrm{A} \frac{\partial \vec{v}(x,t)}{\partial x}, \tag{3.10}$$

where $\vec{v} = (\tilde{u}_0, \cdots, \tilde{u}_p)^T$ and

$$\begin{aligned} &\mathrm{A} = (a_{jk}), \quad 0 \le j,k \le p, \\ &a_{jk} = \int_\Gamma \rho(y) c(y) P_j(y) P_k(y) dy. \end{aligned}$$

Basic properties of the coefficient matrix are given in [13]. For details of this numerical methods, see [13]. As to the convergence property for SG methods, we can use the error estimate results for polynomial approximation. We give the lemma in the following:

Lemma 3.3. *Suppose that $u \in H^m(I)$ and $m \geq 1$, then, for sufficient large N, the projection error satisfy* [5, 21]

$$||u - \Pi_N u||_{L^2(I)} \leq C_I N^{-m} |u|_{H^m(I)},$$

where $|u|_{H^m(I)} = ||\partial^m u||_{L^2(I)}$ and C_I is a constant independent of N.

Using together Theorem 2.1 and Lemma 3.3, it is easy to get the following convergence result

Theorem 3.2. *The error estimate of the N-term SG methods satisfy*

$$|| \left(u - u_{SG}^N\right)(\cdot, t, \cdot)||_{L^2(\Gamma, V)} \leq C_\Gamma \sqrt{C(T)} (\sqrt{\zeta} N)^{-m},$$

where m is an integer index related to the regularity of the solution in the random spaces.

Remark 3.1. The error estimate in Theorem 3.2 is rather rough. A better estimation can be done using complex analysis with the usage of Theorem 2.2. But, unlike the random elliptic/parabolic problems, the solutions of the random hyperbolic equations are not analytic in general with respect to the random parameters, and the solution can not be extend to a complex region like $\Sigma_\tau(z) = \{z \in \mathbb{C}, |z| \leq 1 + \tau\}$, where τ has the same definition as theorem 2.2. The extension can be done only for very spacial cases, for example, problem (2.1) with periodic boundary conditions and analytic given data. We state the complex analysis for this spacial case in the following. One can also find similar results in [4] for random elliptic problem and [16] for random parabolic problem.

Note that for the spacial case of initial problems with periodic solutions, the regularity results in Section 2 still hold. The following convergence result is provided in [13]:

Lemma 3.4. *For any finite time t, the error of the gPC methods behaves like*

$$\mathbb{E}\|u - u^p\|_2^2 \leq C(T) \sum_{k=p+1}^{\infty} \|\tilde{u}_k\|_1^2, \tag{3.11}$$

where $u^p = \sum_{k=0}^{p} \tilde{u}_k P_k(y)$, and the norm $\|u\|_1$ is defined by

$$\|u\|_1^2 = \int_D (u^2 + u_x^2) dx.$$

To prove the spectral convergence, we only need to link the decay rate of $\tilde{u}_k$ as $k \to \infty$ to the analytic results in Section 3. Recall that $u : \Gamma \to L^\infty(T, V)$ admits an analytic extension $u(x, t, z), z \in \mathbb{C}$ in the region of the complex plane $\Sigma(\Gamma, \tau)$. We estimate the Fourier coefficients in the following:

Lemma 3.5. *Consider the corresponding initial problem with periodic solution of* (2.3)*, and assume $c(z), u_0$ are analytic in the complex plat. Then, the Fourier coefficients of u behave as*

$$\|\widetilde{u}_n\|_1 \leq \frac{C_\Sigma}{2^n} \sqrt{\frac{2n+1}{2}} \int_{-1}^{1} \left(\frac{1 - y^2}{1 - |y| + \tau}\right)^n dy, \tag{3.12}$$

where C_Σ will be defined during the derivation

Proof. We recall the Legendre polynomials

$$P_n(y) = \frac{1}{2^n n!}\frac{d^n}{dy}\left((y^2-1)^n\right), \quad n = 0, 1, \cdots,$$

which satisfy

$$\int_\Gamma P_n(y)P_m(y)dy = \frac{2}{2n+1}\delta_{nm}.$$

As Lemma 3.4 was derived using normalized polynomials, we need to work with

$$\tilde{P}_n(y) = \sqrt{\frac{2n+1}{2}}P_n(y).$$

We have the error representation

$$\mathbb{E}\|u - u^p\|_2^2 \leq C(T)\sum_{n=p+1}^{\infty}\|\tilde{u}_n\|_1^2,$$

with the corresponding Fourier coefficients

$$\tilde{u}_n(x,t) \equiv \int_\Gamma u(x,t,y)\tilde{P}_n(y)dy = \sqrt{\frac{2n+1}{2}}\frac{(-1)^n}{n!2^n}\int_\Gamma (1-y^2)^n\frac{d^n}{dy^n}udy. \tag{3.13}$$

Using the analytic continuation of the real function u to the complex domain, an application of Cauchy's formula gives

$$\frac{d^n}{dy^n}u(x,t) = \frac{n!(-1)^n}{2\pi i}\int_{\sigma_y}\frac{u(\theta,x,t)}{(\theta-y)^{n+1}}d\theta, \tag{3.14}$$

where σ_y is a positively oriented closed circumference with the center at the real point $y \in \Gamma$, with radius r_y and such that all singularities from u are exterior to σ_y.

Consider the nature extension of parameter y in (2.1) from Γ to Σ_τ(Σ_τ is defined in Remark 3.1, and one should notice the difference between Σ_τ and $\Sigma(\Gamma,\tau)$),

$$\begin{aligned} &u_t = c(z)\nabla u,\\ &u(x,t=0,z) = u_0(x,z). \end{aligned}$$

We will show in the following the solution u is analytic in Σ_τ. We clear this by showing that the solution satisfies the Cauchy-Riemann conditions (see also in [16]).

Let $u = u^R + iu^I, u_0 = u_0^R + iu_0^I$ and $c = c^R + ic^I$. Then the problem becomes

$$\begin{pmatrix} u^R \\ u^I \end{pmatrix}_t = \begin{pmatrix} c^R & -c^I \\ c^I & c^R \end{pmatrix}\begin{pmatrix} u^R \\ u^I \end{pmatrix}_x.$$

In fact, the problem is a scaler conservation law [11], it is easy to show that

$$\max_{z\in\Sigma_\tau}||u(\cdot,t,\cdot)||_V = \max_{z\in\Sigma_\tau}||u_0||_V = C_\Sigma.$$

Differentiating the equation with respect to $\mathbf{Re}z = s$ and $\mathbf{Im}z = w$, we obtain

$$\begin{aligned}
\partial_t\partial_s u^R &= \partial_s c^R\nabla u^R - \partial_s c^I\nabla u^I + c^R\nabla\partial_s u^R - c^I\nabla\partial_s u^I,\\
\partial_t\partial_s u^I &= \partial_s c^I\nabla u^R + \partial_s c^R\nabla u^I + c^R\nabla\partial_s u^I + c^I\nabla\partial_s u^R,\\
\partial_t\partial_w u^R &= \partial_w c^R\nabla u^R - \partial_w c^I\nabla u^I + c^R\nabla\partial_w u^R - c^I\nabla\partial_w u^I,\\
\partial_t\partial_w u^I &= \partial_w c^I\nabla u^R + \partial_w c^R\nabla u^I + c^R\nabla\partial_w u^I + c^I\nabla\partial_w u^R.
\end{aligned}$$

Then consider the functions $\Theta(z) = \partial_s u^R - \partial_w u^I$ and $\Xi(z) = \partial_w u^R + \partial_s u^I$, they satisfy

$$\begin{aligned}
&\partial_t\Theta(z) - c^R\nabla\Theta(z) + c^I\nabla\Xi(z) = (\partial_s c^R - \partial_w c^I)\nabla u^R - (\partial_w c^R + \partial_s c^I)\nabla u^I,\\
&\partial_t\Theta(z) - c^I\nabla\Theta(z) - c^R\nabla\Xi(z) = (\partial_w c^R + \partial_s c^I)\nabla u^R + (\partial_s c^R - \partial_w c^I)\nabla u^I.
\end{aligned}$$

Note that c is analytic and thus satisfies the Cauchy-Riemann conditions, then the right hand sides of the above equations vanish. Also, the above system has zero initial conditions as u_0 is analytic, therefore, the system admits a unique solution $\Theta(z) = \Xi(z) = 0$, and this proves the analytic of u.

Hence, we have

$$\left\|\frac{d^n}{dy^n}u\right\|_1 \le \frac{n!C_\Sigma}{(r_y)^n}. \tag{3.16}$$

To make sure that $\sigma_y \in \Sigma_z$ in (3.16), we can set $r_y = 1 - |y| + \tau$. Thus, the desired result can be arrived by substitute (3.16) into (3.13). $\square$

We are now ready to obtain the following convergence result which is direct consequence of Lemma 3.4-3.5, Lemma 6.2-6.3 in [4].

Theorem 3.3. *The error of gPC methods for problem* (2.1)-(2.2) *behaves like*

$$\left(\mathbb{E}\|u - u^p\|_2^2\right)^{\frac{1}{2}} \le \sqrt{C_\Sigma C(T)}\sqrt{\pi}\left(\sqrt{1-r^2} + \mathcal{O}(\frac{1}{p^{1/3}})\right)\frac{r^{p+1}}{\sqrt{1-r^2}},$$

where

$$r \equiv \frac{1}{|\xi| + \sqrt{\xi^2 - 1}}, \quad 0 < r < 1, \tag{3.17}$$

with $\xi = -1 - \tau < -1$.

4. An Illustrated Example

We consider the problem which is used in [13]:

$$\begin{cases} u_t(x,t;y) = y u_x(x,t;y) & 0 < x < 2\pi \;\; t > 0,\\ u(x,0;y) = \cos(y) & 0 < x < 2\pi. \end{cases}$$

The boundary conditions of the type (2.2) are given so that the exact solution is of the form

$$u(x,t;y) = \cos(x - yt).$$

It can be verified that the exact solution belongs to $H_y^{(m)}(-1,1)$ for any given positive integer m. Consequently, it is expected that exponential rate of convergence can be obtained. In Figs. 4.1 and 4.2, we plot the mean-square error against the order N for several time levels. For both SG methods(Top) and SC methods(Bottom), the exponential rate of convergence is observed. Also the errors are proportional to the increase of t, and this phenomenon is a severe problem for the polynomial chaos method and stochastic collocation method.

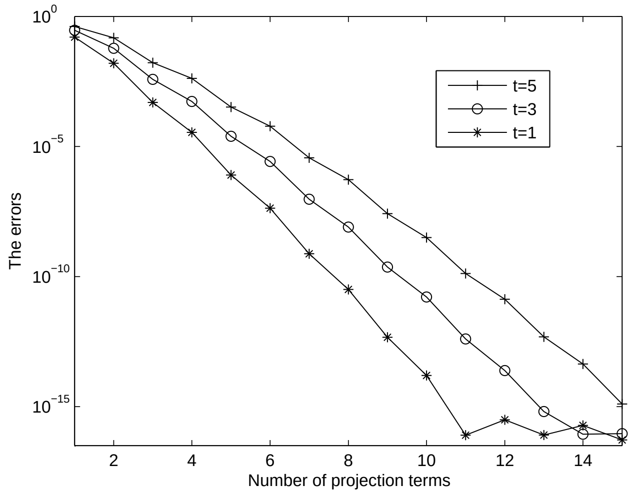

Fig. 4.1. Mean-square errors for SG methods against the number of projection terms.

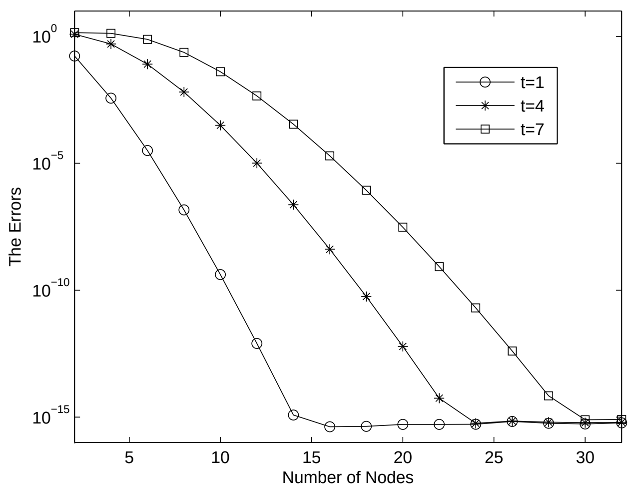

Fig. 4.2. Mean-square errors for SC methods against number of collocation points.

5. Conclusions

In this paper, scalar transport equations with a random wave variable is considered. We discussed the analytic regularity results of the solutions in the random space. Using these regularity results together with complex analysis, the spectral convergence properties of both Stochastic Galerkin methods and Stochastic Collocation methods are obtained. These theoretical results confirmed the numerical observations provided in [13, 22]. In this sense, this paper can be regarded as a theoretical complementary of the numerical papers [13, 22, 28]. We close this paper by pointing out that although the results of this paper are established for one dimension of the random space, the results given in Sections 3 can be easily extended to higher dimensions in random space.

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