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ON THE NONLINEAR MATRIX EQUATION $X^s + A^*F(X)A = Q$ with $s \geq 1$*

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Abstract

This work is concerned with the nonlinear matrix equation $X^s + A^*F(X)A = Q$ with $s \geq 1$. Several sufficient and necessary conditions for the existence and uniqueness of the Hermitian positive semidefinite solution are derived, and perturbation bounds are presented.

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1. Introduction

Let $M(n)$ be the set of all $n \times n$ matrices and $P(n)$ be the set of all $n \times n$ Hermitian positive semidefinite matrices. We consider nonlinear matrix equation

$$X^s + A^*F(X)A = Q \ (s \geq 1), \tag{1.1}$$

where $A \in M(n)$, Q is an $n \times n$ Hermitian positive definite matrix, F is a map from $P(n)$ onto $P(n)$ or $-P(n)$, and the Hermitian positive semidefinite solution X is sought. Here A^* denotes the conjugate transpose of the matrix A. Note that X is a solution of (1.1) if and only if it is a fixed point of

$$G(X) = (Q - A^*F(X)A)^{\frac{1}{s}}.$$

The interest to study (1.1) arose, in particular, in connection with algebraic Riccati equations [2,6,18,21], interpolation [27,30] and the analysis of ladder networks, dynamic programming, control theory, stochastic filtering and statistics [2,16,17]. If $s = 1$, $F(X) = -X^{-1}$, the equation can be written in the form $X = Q + A^*X^{-1}A$. X is a solution of $X = Q + A^*X^{-1}A$ if and only if it is a solution of $X = Q + A^*(Q + A^*X^{-1}A)^{-1}A$. Assuming that A is invertible, this equation can be written as

$$X - F^*XF + F^*X(R + X)^{-1}XF - Q = 0,$$

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where $F = A^{-*}A$, $R = AQ^{-1}A^*$. This is a special case of the discrete algebraic Riccati equation

$$X - S^*XS + S^*XB(R + B^*XB)^{-1}B^*XS - Q = 0,$$

where $Q = Q^*$ and $R = R^*$ is invertible. For detail, see [21]. Several authors have considered such a nonlinear matrix equation, see [2,9-17,19-21,24-29,31-33,35,36] and [23]. It can be categorized as a general system of nonlinear equations in $\mathbb{C}^{n^2}$ space (see [3-7]), which includes the linear and nonlinear matrix equations recently discussed in [3-7,18,24] as special cases.

In [15], El-Sayed and Ran discussed a set of equations of the form $X + A^*F(X)A = Q$, where F maps positive definite matrices either onto positive definite matrices or onto negative definite matrices, and satisfies some monotonicity property. Ran and Reurings [26] also considered the equation $X + A^*F(X)A = Q$. They derived the solutions and perturbation theory. In [29], a perturbation analysis for nonlinear self-adjoint operator equations $X = Q \pm A^*F(X)A$ was provided. Based on the elegant properties of the Thompson metric, Liao, Yao and Duan [24] discussed the equation $X^s - A^*F(X)A = Q$ $(s > 1)$, where $F : P(n) \to P(n)$ is a self-adjoint and nonexpansive map.

The paper is organized as follows. In Section 2, we consider the case $F : P(n) \to P(n)$. The necessary and sufficient conditions for the existence of Hermitian positive semidefinite solution of the matrix equation are derived. A sufficient condition for the existence of a unique Hermitian positive semidefinite solution of the matrix equation is given. Finally, perturbation bounds between (1.1) and the perturbed equation

$$X^s + \widetilde{A}^*F(X)\widetilde{A} = \widetilde{Q} \ (s \geq 1) \tag{1.2}$$

are presented, where $\widetilde{A}$ and $\widetilde{Q}$ are small perturbations of A and Q, respectively. In Section 3, we discuss the case $F : P(n) \to -P(n)$ in a similar way as Section 2. Finally, in Section 4, we give some numerical examples.

Throughout this paper, we write $A \geq B (A > B)$ if both A and B are Hermitian and $A - B$ is positive semidefinite (definite). In particular, $A \geq 0 (A > 0)$ means that A is a Hermitian positive semidefinite (definite) matrix. $\varphi(n)$ denotes the closed set $\{X \in P(n) | X \geq Q^{\frac{1}{s}}\}$. Further, the sets $[A, B]$ and (A, B) are defined by $[A, B] = \{C | A \leq C \leq B\}$, $(A, B) = \{C | A < C < B\}$, whereas $L_{A,B}$ denotes the line segment joining A and B, i.e., $L_{A,B} = \{tA + (1-t)B | t \in [0, 1]\}$. See [28] for more details about these matrix orderings. We use $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ to denote the maximal and the minimal eigenvalues of an $n \times n$ Hermitian positive definite matrix X. $\| \cdot \|$, $\| \cdot \|_2$ and $\| \cdot \|_F$ denote the unitary invariant norm, the spectral norm and the Frobenius norm, respectively.

2. The Case $F : P(n) \to P(n)$

In this section, we derive some necessary and sufficient conditions for the existence and the uniqueness of a solution of (1.1) in the case that $F : P(n) \to P(n)$. The perturbation bound is presented.

Lemma 2.1. ([34]) *If $A \geq B \geq 0$ and $0 \leq r \leq 1$, then*

$$A^r \geq B^r. \tag{2.1}$$

Theorem 2.1. *Let $F : P(n) \to P(n)$ be continuous on $[0, Q^{\frac{1}{s}}]$.*

(i) If Eq. (1.1) *has a Hermitian positive semidefinite solution* $\overline{X}$*, then* $\overline{X} \leq Q^{\frac{1}{s}}$ *and* $A^*F(\overline{X})A \leq Q$.

(ii) If $A^*F(X)A \leq Q$ *for all* $X \in [0, Q^{\frac{1}{s}}]$*, then* (1.1) *has a solution in* $[0, Q^{\frac{1}{s}}]$.

Proof. From $\overline{X} \geq 0$, it follows that $\overline{X}^s \geq 0$. Because F maps $P(n)$ onto $P(n)$, we know that $F(\overline{X}) \geq 0$. This implies that $A^*F(\overline{X})A \geq 0$. According to Lemma 2.1 and $0 \leq \frac{1}{s} \leq 1$, we have

$$\overline{X} = (Q - A^*F(\overline{X})A)^{\frac{1}{s}} \leq Q^{\frac{1}{s}}. \tag{2.2}$$

Because F maps $P(n)$ onto $P(n)$, we know that $F(X) \geq 0$. Then $A^*F(X)A \geq 0$. Assume that $A^*F(X)A \leq Q$ for all $X \in [0, Q^{\frac{1}{s}}]$. Combining this with Lemma 2.1 and $0 \leq \frac{1}{s} \leq 1$, we obtain

$$0 \leq (Q - A^*F(X)A)^{\frac{1}{s}} = G(X) \leq Q^{\frac{1}{s}}, \tag{2.3}$$

for all $X \in [0, Q^{\frac{1}{s}}]$. So G maps $[0, Q^{\frac{1}{s}}]$ onto itself. Since F is continuous, so is G. Hence we can apply Schauder's fixed point theorem (see, e.g., [22], section 106), seeing that a fixed point of G must exist. This fixed point is a solution of (1.1), which proves the second part of the theorem. □

In order to obtain the uniqueness of the solution and the perturbation bound, we restrict the map F to be monotone, i.e., if $X \leq Y$ implies $F(X) \leq F(Y)$.

Theorem 2.2. *Let* $F : P(n) \to P(n)$ *be continuous and monotone, and assume* $A^*F(Q^{\frac{1}{s}})A < Q$*. Then* (1.1) *has a solution* X *and*

$$(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}} \leq X \leq Q^{\frac{1}{s}}. \tag{2.4}$$

In particular, X *is Hermitian positive definite.*

Proof. If $A^*F(Q^{\frac{1}{s}})A < Q$, then for all $X \in [(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$, it follows that

$$0 \leq A^*F(X)A \leq A^*F(Q^{\frac{1}{s}})A < Q. \tag{2.5}$$

Consequently,

$$0 < Q - A^*F(Q^{\frac{1}{s}})A \leq Q - A^*F(X)A \leq Q.$$

Combining this with Lemma 2.1 and $0 \leq \frac{1}{s} \leq 1$, we obtain

$$0 < (Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}} \leq (Q - A^*F(X)A)^{\frac{1}{s}} = G(X) \leq Q^{\frac{1}{s}}.$$

So G maps $[(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$ onto itself and it is continuous on this set. Hence it has a fixed point in this set. This fixed point is a solution X of (1.1) and satisfies (2.4). □

Next we will apply Banach's fixed point theorem to obtain the unique solution of (1.1). We shall prove that the operator G is a strict contraction on the set $[(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$. For this purpose, we will introduce the next lemma.

Lemma 2.2. ([8, Theorem X.3.8]) *Let* f *be a monotone function on* $(0, \infty)$ *and let* A*,* B *be two Hermitian positive operators bounded below by* a*, i.e.,* $A \geq aI$ *and* $B \geq aI$ *for the positive number* a*. If there exists* $f'(a)$*, then for every unitarily invariant norm* $\|\cdot\|$*, we have*

$$\|f(A) - f(B)\| \leq f'(a)\|A - B\|. \tag{2.6}$$

Theorem 2.3. *Assume that the conditions of Theorem 2.2 hold. Let α be the smallest eigenvalue of $Q-A^*F(Q^{\frac{1}{s}})A$. If $b := \frac{1}{s}\alpha^{\frac{1}{s}-1}F'(\alpha^{\frac{1}{s}})\|A\|_2^2 < 1$, then the solution of (1.1) is the unique solution in $[(Q-A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$.*

Proof. Assume that X_1 and X_2 are solutions of (1.1) in $[(Q-A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$ with $X_1 \neq X_2$. Since $F : P(n) \to P(n)$ is continuous and monotone,

$$Q - A^*F(X_j)A \geq Q - A^*F(Q^{\frac{1}{s}})A \geq \alpha I > 0, \quad j = 1, 2. \tag{2.7}$$

So, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\|X_1 - X_2\| &= \|(Q - A^*F(X_1)A)^{\frac{1}{s}} - (Q - A^*F(X_2)A)^{\frac{1}{s}}\| \\ &\leq \frac{1}{s}\alpha^{\frac{1}{s}-1}\|Q - A^*F(X_1)A - Q + A^*F(X_2)A\| \\ &= \frac{1}{s}\alpha^{\frac{1}{s}-1}\|A^*F(X_2)A - A^*F(X_1)A\| \\ &\leq \frac{1}{s}\alpha^{\frac{1}{s}-1}\|A\|_2^2\|F(X_2) - F(X_1)\|.\end{aligned} \tag{2.8}$$

By the spectral mapping theorem, we have

$$0 < \alpha^{\frac{1}{s}} I \leq (Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}} \leq X_1, X_2.$$

Combining this with Lemma 2.2, we get

$$\|X_1 - X_2\| \leq \frac{1}{s}\alpha^{\frac{1}{s}-1}\|A\|_2^2 F'(\alpha^{\frac{1}{s}})\|X_2 - X_1\| = b\|X_1 - X_2\|. \tag{2.9}$$

This contradicts the assumption $b < 1$. Hence (1.1) has a unique Hermitian positive solution X in $[(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$. □

Theorem 2.4. *Assume that the conditions of Theorem 2.3 hold. Then*

$$\lim_{k\to\infty} G^k(X_0) = \overline{X}, \tag{2.10}$$

for all $X_0 \in [(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$. The rate of convergence is given by

$$\|G^k(X_0) - \overline{X}\| \leq \frac{b^k}{1-b}\|G(X_0) - X_0\|. \tag{2.11}$$

Proof. The first statement can be similarly deduced from Lemma 3.2 in [26]. Using Theorem 2.3 we see that

$$\begin{aligned}&\|G^{k+1}(X_0) - G^k(X_0)\| \\ &= \|(Q - A^*F(G^k(X_0))A)^{\frac{1}{s}} - (Q - A^*F(G^{k-1}(X_0))A)^{\frac{1}{s}}\| \\ &\leq b\|G^k(X_0) - G^{k-1}(X_0)\| \leq b^k\|G(X_0) - X_0\|,\end{aligned}$$

which yield (2.11). □

Remark 2.1. According to Theorem 2.4 and $b := \frac{1}{s}\alpha^{\frac{1}{s}-1}F'(\alpha^{\frac{1}{s}})\|A\|_2^2$, we have

$$\|G^k(X_0) - \overline{X}\| \leq \frac{\left(\frac{1}{s}\alpha^{\frac{1}{s}-1}F'(\alpha^{\frac{1}{s}})\|A\|_2^2\right)^k}{1 - \left(\frac{1}{s}\alpha^{\frac{1}{s}-1}F'(\alpha^{\frac{1}{s}})\|A\|_2^2\right)}\|G(X_0) - X_0\|. \tag{2.12}$$

It is easy to see that the convergence rate becomes larger as s increases.

Next we will give an upper bound for $\|\overline{X} - \widetilde{X}\|$, where $\overline{X}$ denotes the unique solution of (1.1) in $[(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$ and $\widetilde{X}$ denotes a solution of (1.2) in $[(\widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A})^{\frac{1}{s}}, \widetilde{Q}^{\frac{1}{s}}]$.

Theorem 2.5. *Let $F : P(n) \to P(n)$ be continuous and monotone. Assume that* (1.1) *has a Hermitian positive solution $\overline{X}$ on $[(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$. Let $\alpha = \min\{\lambda_{\min}(Q - A^*F(Q^{\frac{1}{s}})A), \lambda_{\min}(\widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A})\} > 0$. If $c := \frac{1}{s}\alpha^{\frac{1}{s}-1}$, $b := \frac{1}{s}\alpha^{\frac{1}{s}-1}F'(\alpha^{\frac{1}{s}})\|A\|_2^2 < 1$, then*

$$\|\overline{X} - \widetilde{X}\| \leq \frac{c}{1-b}\Big(\|Q - \widetilde{Q}\| + (\|A\|_2 + \|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A} - A\|\Big), \tag{2.13}$$

for all solutions $\widetilde{X}$ of (1.2).

Proof. According to Theorem 2.3, we know that $\overline{X}$ is the unique solution of (1.1). Since $F : P(n) \to P(n)$ is continuous and monotone and $\alpha = \min\{\lambda_{\min}(Q - A^*F(Q^{\frac{1}{s}})A), \lambda_{\min}(\widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A})\} > 0$, we have

$$Q - A^*F(\overline{X})A \geq Q - A^*F(Q^{\frac{1}{s}})A \geq \alpha I > 0. \tag{2.14}$$

Because $\widetilde{X}$ is a solution of (1.2), we have

$$\widetilde{A}^*F(\widetilde{X})\widetilde{A} \leq \widetilde{Q}, \quad \widetilde{X} \leq \widetilde{Q}^{\frac{1}{s}}.$$

Since $F : P(n) \to P(n)$ is continuous and monotone,

$$\widetilde{Q} - \widetilde{A}^*F(\widetilde{X})\widetilde{A} \geq \widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A}.$$

By the condition $\alpha = \min\{\lambda_{\min}(Q - A^*F(Q^{\frac{1}{s}})A), \lambda_{\min}(\widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A})\} > 0$, we have

$$\widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A} \geq \alpha I > 0, \tag{2.15a}$$

$$\widetilde{Q} - \widetilde{A}^*F(\widetilde{X})\widetilde{A} \geq \widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A} \geq \alpha I > 0. \tag{2.15b}$$

Then $\widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A} \leq \widetilde{Q}$. Applying Theorems 2.2 and 2.3, we know that (1.2) has a unique solution $\widetilde{X}$. We have by Lemma 2.2 that

$$\begin{aligned}
\|\overline{X} - \widetilde{X}\| &= \|(Q - A^*F(\overline{X})A)^{\frac{1}{s}} - (\widetilde{Q} - \widetilde{A}^*F(\widetilde{X})\widetilde{A})^{\frac{1}{s}}\| \\
&\leq \frac{1}{s}\alpha^{\frac{1}{s}-1}\|Q - A^*F(\overline{X})A - \widetilde{Q} + \widetilde{A}^*F(\widetilde{X})\widetilde{A}\| \\
&= c\|Q - \widetilde{Q} + A^*F(\widetilde{X})A - A^*F(\overline{X})A - A^*F(\widetilde{X})A \\
&\qquad + A^*F(\widetilde{X})\widetilde{A} - A^*F(\widetilde{X})\widetilde{A} + \widetilde{A}^*F(\widetilde{X})\widetilde{A}\| \\
&\leq c\Big(\|Q - \widetilde{Q}\| + \|A\|_2^2\|F(\widetilde{X}) - F(\overline{X})\| + (\|A\|_2 + \|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A} - A\|\Big).
\end{aligned}$$

By the spectral mapping theorem, we have

$$0 < \alpha^{\frac{1}{s}}I \leq (Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}} \leq \overline{X}, \tag{2.16a}$$

$$0 < \alpha^{\frac{1}{s}}I \leq (\widetilde{Q} - \widetilde{A}^*F(\widetilde{Q}^{\frac{1}{s}})\widetilde{A})^{\frac{1}{s}} \leq \widetilde{X}. \tag{2.16b}$$

So, by Lemma 2.2, we get

$$\|F(\widetilde{X}) - F(\overline{X})\| \leq F'(\alpha^{\frac{1}{s}})\|\widetilde{X} - \overline{X}\|.$$

Then

$$\begin{aligned}&\|\overline{X}-\widetilde{X}\|\\ &\le c\Big(\|Q-\widetilde{Q}\|+\|A\|_2^2F^{'}(\alpha^{\frac{1}{s}})\|\widetilde{X}-\overline{X}\|+(\|A\|_2+\|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A}-A\|\Big).\end{aligned}$$

Since $b<1$, we have (2.13). □

Remark 2.2. It is easy to see that the the perturbation bound becomes sharper as s increases. Assume that $s=1$ and consider the spectral norm. Then

$$\|\overline{X}-\widetilde{X}\|_2\le\frac{1}{1-F'(\alpha)\|A\|_2^2}\Big(\|Q-\widetilde{Q}\|_2+(\|A\|_2+\|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A}-A\|_2\Big).$$

According to the definition of $\varphi(n)$ in [26], $\varphi(n)=[B,C]$. $M_{\varphi(n)}$ is the smallest possible value such that

$$\sup_{X\in\varphi(n)}\|F'(X)\|_2\le M_{\varphi(n)}.$$

Since $F:P(n)\to P(n)$ is continuous and monotone, we have

$$F'(\alpha)\in\{\|F'(X)\|_2|X\in\varphi(n)\},\quad F'(\alpha)\le\sup_{X\in\varphi(n)}\|F'(X)\|_2\le M_{\varphi(n)}.$$

Consequently,

$$\begin{aligned}\|\overline{X}-\widetilde{X}\|_2\le&\frac{1}{1-M_{\varphi(n)}\|A\|_2^2}\Big(\|Q-\widetilde{Q}\|_2+(\|A\|_2+\|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A}-A\|_2\Big)\\ \le&\frac{\|F(\widetilde{X})\|_2(2\|A\|_2+\|A-\widetilde{A}\|_2)\|A-\widetilde{A}\|_2}{1-M_{\varphi(n)}\|A\|_2^2}+\frac{\|Q-\widetilde{Q}\|_2}{1-M_{\varphi(n)}\|A\|_2^2}.\end{aligned}$$

This is the perturbation bound (16) in [26]. So the perturbation bound we get is smaller than this bound.

The perturbation bound in Theorem 2.5 can be derived by using the approach in [26,29], where perturbations of A, Q, and F are allowed.

3. The Case $F:P(n)\to -P(n)$

In this section, we will consider the case that $F:P(n)\to -P(n)$ is continuous. Consider the map

$$H:H(X)=-F(X).$$

Then

$$\begin{aligned}&F:P(n)\to -P(n) &&\text{iff}\quad H:P(n)\to P(n),\\ &X^s+A^*F(X)A=Q &&\text{iff}\quad X^s-A^*H(X)A=Q.\end{aligned}$$

So the equation we considered in this section is the same as that in [24]. But they assume that F is a self-adjoint and nonexpansive map. The technique they used in the analysis is the elegant properties of the Thompson metric.

Theorem 3.1. *Let $F:P(n)\to -P(n)$ be continuous on $\varphi(n)$.*

(i) *If* (1.1) *has a Hermitian positive semidefinite solution* $\overline{X}$*, then* $\overline{X} \geq Q^{\frac{1}{s}}$*.*

(ii) *If there exists a* $B \geq Q$ *such that*

$$Q - B \leq A^*F(X)A \leq 0, \tag{3.1}$$

for all $X \in [Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$*, then (1.1) has a solution in* $[Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$*. Moreover, if* $Q - B \leq A^*F(X)A \leq 0$ *is satisfied for every* $X \geq Q^{\frac{1}{s}}$*, then all solutions of* (1.1) *are in* $[Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$*.*

Proof. First assume that (1.1) has a solution $\overline{X} \geq 0$. Then $F(\overline{X}) \leq 0$ and

$$\overline{X}^s = Q - A^*F(\overline{X})A \geq Q. \tag{3.2}$$

Combining this inequality, the fact that $0 < \frac{1}{s} \leq 1$ and Lemma 2.1, we have $\overline{X} \geq Q^{\frac{1}{s}}$. Assume that there is a $B \geq Q$ such that $Q - B \leq A^*F(X)A \leq 0$ holds for all $X \in [Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$. Then

$$0 \leq -A^*F(X)A \leq B - Q, \tag{3.3a}$$

$$Q \leq Q - A^*F(X)A \leq B. \tag{3.3b}$$

So, by Lemma 2.1 and $0 < \frac{1}{s} \leq 1$, we have

$$Q^{\frac{1}{s}} \leq (Q - A^*F(X)A)^{\frac{1}{s}} = G(X) \leq B^{\frac{1}{s}}. \tag{3.4}$$

So G maps $[Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$ onto itself and is continuous on this set, by Schauder's fixed point theorem [22], we know that G has a fixed point in $[Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$. This fixed point is a solution of (1.1). Further, assume that $Q - B \leq A^*F(X)A \leq 0$ holds for all $X \geq Q^{\frac{1}{s}}$ and let $\overline{X}$ be a solution of (1.1). Then

$$\overline{X}^s = Q - A^*F(\overline{X})A \leq Q - (Q - B) = B. \tag{3.5}$$

From $0 < \frac{1}{s} \leq 1$ and Lemma 2.1, it follows that $\overline{X} \leq B^{\frac{1}{s}}$. This completes the proof of (ii). □

Next let F be anti-monotone, i.e., $X \leq Y$ implies that $F(X) \geq F(Y)$.

Corollary 3.1. *Let* $F : P(n) \to -P(n)$ *be continuous and anti-monotone and assume that there exists a* B *such that* $Q - A^*F(B^{\frac{1}{s}})A \leq B$*. Then* (1.1) *has a solution in* $X \in [Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$*.*

Proof. Assume that there exists a B such that $Q - A^*F(B^{\frac{1}{s}})A \leq B$. Then for all $X \in [Q^{\frac{1}{s}}, B^{\frac{1}{s}}]$ we have

$$A^*F(X)A \geq A^*F(B^{\frac{1}{s}})A \geq Q - B. \tag{3.6}$$

So condition (ii) of Theorem 3.1 is satisfied and the result follows. □

Next we will introduce the mean-value theorem to get the unique solution of (1.1).

Lemma 3.1. ([1 Theorem I.1.8]) *Let* $F : U \to M(n)$ *(*$U \subset M(n)$ *open) be differentiable at any point of* U*. Then*

$$\|F(X) - F(Y)\| \leq \sup_{Z \in L_{X,Y}} \|DF(Z)\| \|X - Y\|, \tag{3.7}$$

for all $X, Y \in U$*.*

Let $M_{\varphi(n)}$ be the smallest possible value such that

$$\sup_{Z \in \varphi(n)} \|DF(Z)\| \leq M_{\varphi(n)}. \tag{3.8}$$

Theorem 3.2. *Let $F : P(n) \to -P(n)$ be continuous on $\varphi(n)$. Assume that* (1.1) *has a solution in $\varphi(n)$. If $b := \frac{1}{s}\lambda_{\min}^{\frac{1}{s}-1}(Q)M_{\varphi(n)}\|A\|_2^2 < 1$, then this solution is the unique solution in $\varphi(n)$.*

Proof. Assume that X_1 and X_2 are solutions of (1.1) in $\varphi(n)$ with $X_1 \neq X_2$. Since $F : P(n) \to -P(n)$ is continuous on $\varphi(n)$, we have

$$Q - A^*F(X_1)A \geq Q \geq \lambda_{\min}(Q)I > 0, \tag{3.9a}$$

$$Q - A^*F(X_2)A \geq Q \geq \lambda_{\min}(Q)I > 0. \tag{3.9b}$$

Using Lemmas 2.1 and 2.2 we have

$$\begin{aligned}\|X_1 - X_2\| &= \|(Q - A^*F(X_1)A)^{\frac{1}{s}} - (Q - A^*F(X_2)A)^{\frac{1}{s}}\| \\ &\leq \frac{1}{s}\lambda_{\min}^{\frac{1}{s}-1}(Q)\|Q - A^*F(X_1)A - Q + A^*F(X_2)A\| \\ &= \frac{1}{s}\lambda_{\min}^{\frac{1}{s}-1}(Q)\|A^*F(X_2)A - A^*F(X_1)A\| \\ &\leq \frac{1}{s}\lambda_{\min}^{\frac{1}{s}-1}(Q)\|A\|_2^2\|F(X_2) - F(X_1)\|. \end{aligned} \tag{3.10}$$

With the mean-value theorem we obtain

$$\|F(X_2) - F(X_1)\| \leq \sup_{Z \in L_{X_1,X_2}} \|DF(Z)\|\|X_2 - X_1\|. \tag{3.11}$$

Because $X_1, X_2 \in \varphi(n)$, it holds that $L_{X_1,X_2} \subset \varphi(n)$. So

$$\sup_{Z \in L_{X_1,X_2}} \|DF(Z)\| \leq M_{\varphi(n)}.$$

This implies that

$$\|X_1 - X_2\| \leq M_{\varphi(n)}\frac{1}{s}\lambda_{\min}^{\frac{1}{s}-1}(Q)\|A\|_2^2\|X_2 - X_1\| = b\|X_1 - X_2\| < \|X_1 - X_2\|, \tag{3.12}$$

which is a contradiction. So X_1 and X_2 must be equal. □

Theorem 3.3. *Assume that the conditions of Theorem* 3.2 *hold. Then*

$$\lim_{k\to\infty} G^k(X_0) = \overline{X}, \tag{3.13}$$

for all $X_0 \in \varphi(n)$. The rate of convergence is given by

$$\|G^k(X_0) - \overline{X}\| \leq \frac{b^k}{1-b}\|G(X_0) - X_0\|. \tag{3.14}$$

Proof. The proof is similar to that of Theorems 2.4 and is omitted here. □

It is easy to see that the convergence rate becomes larger as s increases.

Theorem 3.4. *Assume that* (1.1) *has a solution $\overline{X}$ in $\varphi(n)$. If $\alpha = \min\{\lambda_{\min}^{\frac{1}{s}}(Q), \lambda_{\min}^{\frac{1}{s}}(\widetilde{Q})\}$, $c := \frac{1}{s}\alpha^{\frac{1}{s}-1}$, and $b := \frac{1}{s}\alpha^{\frac{1}{s}-1}M_{\varphi(n)}\|A\|_2^2 < 1$, then*

$$\|\overline{X} - \widetilde{X}\| \leq \frac{c}{1-b}\Big(\|Q - \widetilde{Q}\| + (\|A\|_2 + \|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A} - A\|\Big), \tag{3.15}$$

for all solutions $\widetilde{X}$ of (1.2).

Proof. According to Theorem 3.2, we know that $\overline{X}$ is the unique solution of (1.1). Since $F : P(n) \to -P(n)$ is continuous on $\varphi(n)$, we have

$$Q - A^*F(\overline{X})A \geq Q \geq \lambda_{\min}(Q)I \geq \alpha I > 0, \tag{3.16a}$$

$$\widetilde{Q} - \widetilde{A}^*F(\widetilde{X})\widetilde{A} \geq \widetilde{Q} \geq \lambda_{\min}(\widetilde{Q})I \geq \alpha I > 0. \tag{3.16b}$$

So, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\|\overline{X} - \widetilde{X}\| \\
=& \|(Q - A^*F(\overline{X})A)^{\frac{1}{s}} - (\widetilde{Q} - \widetilde{A}^*F(\widetilde{X})\widetilde{A})^{\frac{1}{s}}\| \\
\leq& \frac{1}{s}\alpha^{\frac{1}{s}-1}\|Q - A^*F(\overline{X})A - \widetilde{Q} + \widetilde{A}^*F(\widetilde{X})\widetilde{A}\| \\
=& c\|Q - \widetilde{Q} + A^*F(\widetilde{X})A - A^*F(\overline{X})A - A^*F(\widetilde{X})A + A^*F(\widetilde{X})\widetilde{A} - A^*F(\widetilde{X})\widetilde{A} + \widetilde{A}^*F(\widetilde{X})\widetilde{A}\| \\
\leq& c\Big(\|Q - \widetilde{Q}\| + \|A\|_2^2\|F(\widetilde{X}) - F(\overline{X})\| + (\|A\|_2 + \|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A} - A\|\Big).
\end{aligned}$$

According to Lemma 3.1, we have

$$\|\overline{X} - \widetilde{X}\| \leq c\Big(\|Q - \widetilde{Q}\| + \|A\|_2^2 M_{\varphi(n)}\|\widetilde{X} - \overline{X}\| + (\|A\|_2 + \|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A} - A\|\Big).$$

Since $b < 1$, it leads to (3.15). □

Remark 3.1. It is easy to see that the perturbation bound becomes sharper as s increases. Assume $s = 1$ and consider the spectral norm. Then

$$\begin{aligned}
\|\overline{X} - \widetilde{X}\|_2 \leq& \frac{1}{1 - M_{\varphi(n)}\|A\|_2^2}\Big(\|Q - \widetilde{Q}\|_2 + (\|A\|_2 + \|\widetilde{A}\|_2)\|F(\widetilde{X})\|_2\|\widetilde{A} - A\|_2\Big) \\
\leq& \frac{\|F(\widetilde{X})\|_2(2\|A\|_2 + \|A - \widetilde{A}\|_2)\|A - \widetilde{A}\|_2}{1 - M_{\varphi(n)}\|A\|_2^2} + \frac{\|Q - \widetilde{Q}\|_2}{1 - M_{\varphi(n)}\|A\|_2^2}.
\end{aligned}$$

This is the perturbation bound (16) in [26]. So the perturbation bound we get is smaller than this bound.

The perturbation bound in Theorem 3.4 can be derived by using the approach in [26,29], where perturbations of A, Q, and F are allowed.

4. Numerical Experiments

In this section, we use the methods given in Theorems 2.4 and 3.3 to compute the unique Hermitian positive definite solution of (1.1). All numerical experiments are run in MATLAB version 7.9. We denote the residual error by $\epsilon(X) = \|X^s + A^*F(X)A - Q\|_F < 10^{-10}$.

Experiment 4.1. We consider (1.1) when $A \in \mathbb{R}^{n\times n}$ is given as in Example 6.2 from [5]:

$$A = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix},$$

and $Q = 256I_n$. Assume that $s = 2$ and $F(X) = X^{\frac{1}{2}}$. Then $F : P(n) \to P(n)$ is continuous and monotone. It is easy to verify that $A^*F(Q^{\frac{1}{s}})A < Q$. If we take $n = 256$, then $b = 0.2614 < 1$. (1.1) has a unique solution $\overline{X}$ on $[(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}]$ and

$$\lim_{k\to\infty} G^k(X_0) = \overline{X}, \quad \forall X_0 \in [(Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}, Q^{\frac{1}{s}}].$$

Indeed, when we take $X_0 = (Q - A^*F(Q^{\frac{1}{s}})A)^{\frac{1}{s}}$, after 19 iterations we obtain the unique Hermitian positive definite solution $\overline{X}$ and its residual error: $\epsilon(X) = 8.0029 \times 10^{-11}$. When we take $X_0 = Q^{\frac{1}{s}}$, after 20 iterations we obtain the same solution $\overline{X}$ and its residual error: $\epsilon(X) = 8.0029 \times 10^{-11}$.

Experiment 4.2. We consider (1.1) when A is the same as in [6,18]:

$$A = \begin{pmatrix} 3 & -1 & & \\ & 3 & \ddots & \\ & & \ddots & -1 \\ -1 & & & 3 \end{pmatrix} \in \mathbb{R}^{n\times n},$$

and $Q = 256I_n$. Assume that $s = 2$ and $F(X) = -X$. Then $F : P(n) \to -P(n)$ is continuous. If we take $n = 256$, then $b = 0.5 < 1$. (1.1) has a unique solution $\overline{X}$ in $\varphi(n)$ and

$$\lim_{k\to\infty} G^k(X_0) = \overline{X}, \quad \forall X_0 \in \varphi(n).$$

Indeed, when we take $X_0 = Q^{\frac{1}{s}}$, after 27 iterations we obtain the unique Hermitian positive definite solution $\overline{X}$ and its residual error: $\epsilon(X) = 6.6241 \times 10^{-11}$. When we take $X_0 = 5Q^{\frac{1}{s}}$, after 28 iterations we obtain the same solution $\overline{X}$ and its residual error: $\epsilon(X) = 6.9649 \times 10^{-11}$.

5. Conclusions

We have considered the more general nonlinear matrix equation (1.1). In Section 2, the case $F : P(n) \to P(n)$ was considered. The necessary and sufficient conditions for the existence of Hermitian positive semidefinite solutions of the matrix equation are derived. Based on fixed point theorem of contraction map, we prove that (1.1) always has a unique positive definite solution. An iterative method is proposed to compute the unique positive definite solution. We also show that the iterative method becomes more effective as s increases. Finally, perturbation bound for the unique positive definite solution is presented. The bound improves some recent results. In Section 3, we discuss the case $F : P(n) \to -P(n)$ in a similar way to Section 2.

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References

[1] A. Ambrosetti and G. Prodi, A Primer of Nonlinear Analysis, Cambridge Studies in Advanced Mathematics, Vol. 34, Cambridge University Press, Cambridge, 1993.

[2] W.N. Anderson, T.D. Morley and G.E. Trapp, Positive solutions to $X = A - BX^{-1}B^*$, *Linear Algebra Appl.*, **134** (1990), 53-62.

[3] Z.-Z. Bai, A class of iteration methods based on the Moser formula for nonlinear equations in Markov chains, *Linear Algebra Appl.*, **266** (1997), 219-241.

[4] Z.-Z. Bai, On Hermitian and skew-Hermitian splitting iteration methods for continuous Sylvester equations, *J. Comput. Math.*, **29** (2011), 185-198.

[5] Z.-Z. Bai and Y.-H. Gao, Modified Bernoulli iteration methods for quadratic matrix equation, *J. Comput. Math.*, **25** (2007), 498-511.

[6] Z.-Z. Bai, X.-X. Guo and S.-F. Xu, Alternately linearized implicit iteration methods for the minimal nonnegative solutions of nonsymmetric algebraic Riccati equations, *Numer. Linear Algebra Appl.*, **13** (2006), 655-674.

[7] Z.-Z. Bai, X.-X. Guo and J.-F. Yin, On two iteration methods for the quadratic matrix equations, *Intern. J. Numer. Anal. Modeling*, **2** (2005), 114-122.

[8] R. Bhatia, Matrix Analysis, Grad. Texts in Math. 169, Springer-Verlag, New York, 1997.

[9] J. Cai and G.-L. Chen, Some investigation on Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$, *Linear Algebra Appl.*, **430** (2009), 2448-2456.

[10] J. Cai and G.-L. Chen, On the Hermitian positive definite solutions of nonlinear matrix equation $X^s + A^*X^{-t}A = Q$, *Appl. Math. Comput.*, **217** (2010), 117-123.

[11] S.-H. Du and J.-C. Hou, Positive definite solutions of operator equations $X^m + A^*X^{-n}A = I$, *Linear and Multilinear Algebra*, **51** (2003), 163-173.

[12] X.-F. Duan and A.-P. Liao, On the existence of Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$, *Linear Algebra Appl.*, **429** (2008), 673-687.

[13] S.M. El-Sayed, An algorithm for computing positive definite solutions of the nonlinear matrix equation $X + A^*X^{-1}A = I$, *Int. J. Comput. Math.*, **80**:12 (2003), 1527-1534.

[14] S.M. El-Sayed and A.M. Al-Dbiban, On positive definite solutions of the nonlinear matrix equations $X + A^*X^{-n}A = I$, *Appl. Math. Comput.*, **151** (2004), 533-541.

[15] S.M. El-Sayed and A.C.M. Ran, On an iterative method for solving a class of nonlinear matrix equations, *SIAM J. Matrix Anal. Appl.*, **23** (2001), 632-645.

[16] J.C. Engwerda, On the existence of a positive definite solution of the matrix equation $X + A^TX^{-1}A = I$, *Linear Algebra Appl.*, **194** (1993), 91-108.

[17] J.C. Engwerda, A.C.M. Ran and A.L. Rijkeboer, Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^*X^{-1}A = I$, *Linear Algebra Appl.*, **186** (1993), 255-275.

[18] Y.-H. Gao and Z.-Z. Bai,On inexact Newton methods based on doubling iteration scheme for nonsymmetric algebraic Riccati equations, *Numer. Linear Algebra Appl.*, **18** (2011), 325-341.

[19] X.-X. Guo, On Hermitian positive definite solution of nonlinear matrix equation $X+A^*X^{-2}A = Q$, *J. Comput. Math.*, **23** (2005), 513-526.

[20] V.I. Hasanov and S.M. El-Sayed, On the positive definite solutions of nonlinear matrix equation $X + A^*X^{-\delta}A = Q$, *Linear Algebra Appl.*, **412** (2006), 154-160.

[21] V.I. Hasanov, I.G. Ivanov and F. Uhlig, Improved perturbation estimates for the matrix equations $X \pm A^*X^{-1}A = Q$, *Linear Algebra Appl.*, **379** (2004), 113-135.

[22] V.I. Istratescu, Fixed Point Theorem, Mathematics and Its Applications, Vol. 7, Reidel, Dordrecht, 1981.

[23] M. M. Konstantinov, D. Gu, V. Mehrmann and P. Petkov, Perturbation Theory of Matrix Equations, Elsevier, New York, 2003.

[24] A.-P. Liao, G.-Z. Yao and X. Duan, Thompson metric method for solving a class of nonlinear

matrix equation, *Appl. Math. Comput.*, **216** (2010), 1831-1836.

[25] X.-G. Liu and H. Gao, On the positive definite solutions of the matrix equations $X^s \pm A^T X^{-t} A = I$, *Linear Algebra Appl.*, **368** (2003), 83-97.

[26] A.C.M. Ran and M.C.B. Reurings, On the nonlinear matrix equation $X + A^* F(X) A = Q$: Solutions and perturbation theory, *Linear Algebra Appl.*, **346** (2002), 15-26.

[27] A.C.M. Ran and M.C.B.Reurings, A nonlinear matrix equation connected to interpolation theory, Linear Algebra Appl., **379** (2004), 289-302.

[28] A.C.M. Ran and M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, *Proc. Amer. Math. Soc.*, **132** (2004), 1435-1443.

[29] A.C.M. Ran, M.C.B. Reurings and A.L. Rodman, A perturbation analysis for nonlinear selfadjoint operators, *SIAM J. Matrix Anal. Appl.*, **28** (2006), 89-104.

[30] L.A. Sakhnovich, Interpolation Theory and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[31] Y.-T. Yang, The iterative method for solving nonlinear matrix equation $X^s + A^* X^{-t} A = Q$, *Appl. Math. Comput.*, **188** (2007), 46-53.

[32] G.-Z. Yao, A.-P. Liao and X.-F. Duan, Positive definite solution of the matrix equation $X = Q + A^*(I \otimes X - C)^\delta A$, *Numer Algor.*, **56** (2011), 349-361.

[33] X.-Z. Zhan and J.-J. Xie, On the matrix equation $X + A^T X^{-1} A = I$, *Linear Algebra Appl.*, **247** (1996), 337-345.

[34] X.-Z. Zhan, Matrix Inequalities, Lecture Notes in Mathematics, Vol. 1790, Springer-Verlag, Berlin, Heidelberg, 2002.

[35] Y.-H. Zhang, On Hermitian positive definite solutions of matrix equation $X + A^* X^{-2} A = I$, *Linear Algebra Appl.*, **372** (2003), 295-304.

[36] D.-M. Zhou, G.-L. Chen, G.-X. Wu and X.-Y. Zhang, Some properties of the nonlinear matrix equation $X^s + A^* X^{-t} A = Q$, *J. Math. Anal. Appl.*, **392** (2012), 75-82.