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APPROXIMATION OF NONCONFORMING QUASI-WILSON ELEMENT FOR SINE-GORDON EQUATIONS*

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Abstract

In this paper, nonconforming quasi-Wilson finite element approximation to a class of nonlinear sine-Gordan equations is discussed. Based on the known higher accuracy results of bilinear element and different techniques from the existing literature, it is proved that the inner product $(\nabla(u - I_h^1 u), \nabla v_h)$ and the consistency error can be estimated as order $O(h^2)$ in broken $H^1 - norm/L^2 - norm$ when $u \in H^3(\Omega)/H^4(\Omega)$, where $I_h^1 u$ is the bilinear interpolation of u, v_h belongs to the quasi-Wilson finite element space. At the same time, the superclose result with order $O(h^2)$ for semi-discrete scheme under generalized rectangular meshes is derived. Furthermore, a fully-discrete scheme is proposed and the corresponding error estimate of order $O(h^2 + \tau^2)$ is obtained for the rectangular partition when $u \in H^4(\Omega)$, which is as same as that of the bilinear element with ADI scheme and one order higher than that of the usual analysis on nonconforming finite elements.

Mathematics subject classification: 65N15, 65N30.
Key words: Sine-Gordon equations, Quasi-Wilson element, Semi-discrete and fully-discrete schemes, Error estimate and superclose result.

1. Introduction

Consider the following nonlinear sine-Gordon equations [1]:

$$\begin{cases} u_{tt} + \alpha u_t - \gamma \Delta u + \beta \sin u = f, & (X,t) \in \Omega \times (0,T], \\ u \mid_{\partial\Omega} = 0, & t \in (0,T], \\ u(X,0) = u_0(X), \quad u_t(X,0) = u_1(X), & X \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ is a convex and bounded region with Lipschitz boundary $\partial\Omega$, $X = (x,y)$, $u = u(X,t)$, α, β, γ are positive constants, $u_0, u_1, f = f(X,t)$ are known smooth functions.

There have been a lot of studies devoted to (1.1). For example, [1] proved the existence and uniqueness of the solution; [2] presented an explicit finite difference method for the numerical solution; [3] obtained analytical solutions to the unperturbed sine-Gordon equation with zero damping, i.e., $\alpha \equiv 0$; [4] established the Fourier quasi-spectrum explicit scheme and gave the convergence and error estimation; [5] discussed two implicit difference schemes and provided the numerical results; [6] studied an ADI scheme of bilinear element and deduced $O(\tau^2 + h^2)$ order estimates; [7] considered the general approximation scheme for a class of low order

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nonconforming finite elements satisfying some assumptions and derived the optimal order error estimates.

As we know, Wilson element [8,9] has been widely used in engineering computation, but it is only convergent for rectangular and parallelogram meshes. In order to extend this element to arbitrary quadrilateral meshes, various improved Wilson elements were developed. For instance, [10] proposed a quasi-Wilson element by simply adding a high order term, which is independent of the element geometry, to the nonconforming part of the shape function; [11] generalized the result of [10] to a class of quasi-Wilson arbitrary quadrilateral elements; In [12], a special property is discovered, i.e., the consistency error is of order $O(h^2)$ in broken H^1-norm, one order higher than that of its interpolation error $O(h)$, which is similar to the famous nonconforming rectangular EQ_1^{rot} element[13,14], the Q_1^{rot} square element[15,16] and the constrained Q_1^{rot} element[17,18]; Especially, [19] applied this quasi-Wilson element to second-order problems on narrow quadrilateral meshes. However, all of the above studies on quasi-Wilson element are only limited to the linear problems.

In this paper, as a continuous work of [7], we will apply this quasi-Wilson finite element to problem (1.1). Based on the known higher accuracy results of bilinear element and different approaches from the existing literature, we prove that the estimations of the inner product of gradients of the difference between u and its bilinear interpolation $I_h^1 u$ with any polynomial of the finite element space and the consistency error are of order $O(h^2)$ in broken H^1-norm or L^2-norm when $u\in H^3(\Omega)$ or $H^4(\Omega)$ (see Lemmas 2.2–2.3 below). At the same time, the superclose result with order $O(h^2)$ is obtained although the mean values of quasi-Wilson element across the edges between elements are not continuous which does not satisfy the requirement (III) of [7]. Furthermore, a kind of fully-discrete scheme is proposed, and the $O(h^2+\tau^2)$ order error estimate is derived, which improves the result $O(h+\tau^2)$ of [7] by one order with respect to h, and as same as [6] with ADI scheme.

The rest of the paper is organized as follows. In the next section, we introduce the nonconforming quasi-Wilson element, and prove the important characters of the element, and derive the superclose result. In Section 3, a kind of fully-discrete scheme is proposed and the optimal order error estimate is gained.

Throughout this paper, c denotes a general positive constant which is independent of h, where $h=max_K h_K$, h_K is the diameter of the element K, τ is the time step for the partition of the time interval $[0,T]$.

2. Quasi-Wilson Element and Superclose Result

Let $\hat{K}=[0,1]\times[0,1]$ be the reference element with vertices $\hat{M}_1(0,0)$, $\hat{M}_2(1,0)$, $\hat{M}_3(1,1)$, $\hat{M}_4(0,1)$. We define on $\hat{K}$ the finite element $(\hat{K},\hat{P},\hat{\Sigma})$ as follows:

$$\hat{P}=span\Big\{N_i(\xi,\eta),(i=1,2,3,4),\hat{\Psi}(\xi),\hat{\Psi}(\eta)\Big\},$$

where $N_1(\xi,\eta)=(1-\xi)(1-\eta),\ N_2(\xi,\eta)=\xi(1-\eta),\ N_3(\xi,\eta)=\xi\eta,\ N_4(\xi,\eta)=(1-\xi)\eta,\ \hat{\Psi}(s)=-\frac{3}{4}s(s-1)+\frac{5}{32}[(2s-1)^4-1]$. The degrees of freedom are taken as $\hat{\Sigma}=\{\hat{v}_1,\cdots,\hat{v}_4,\hat{\beta}_1,\hat{\beta}_2\}$, where $\hat{v}_i=\hat{v}(\hat{M}_i),\ i=1,2,3,4.$ $\hat{\beta}_1=\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi^2}d\xi d\eta,\ \hat{\beta}_2=\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\eta^2}d\xi d\eta$. We have

$$\hat{v}(\xi,\eta)=\sum_{i=1}^{4}\hat{v}_iN_i(\xi,\eta)+\hat{\beta}_1\hat{\Psi}(\xi)+\hat{\beta}_2\hat{\Psi}(\eta).$$

Let K be a generalized rectangular(see, e.g., [20]) with vertices $M_i(x_i, y_i),\ 1 \le i \le 4$. Then there exists a unique mapping $F_K : \hat{K} \to K$

$$x^K = \sum_{i=1}^{4} N_i(\xi,\eta)x_i, \quad y^K = \sum_{i=1}^{4} N_i(\xi,\eta)y_i.$$

For any function $v(x, y)$ defined on K, we define $\hat{v}(\xi, \eta)$ by

$$\hat{v}(\xi,\eta) = v(x^K(\xi,\eta), y^K(\xi,\eta)) \text{ or } \hat{v} = v \circ F_K.$$

On the generalized rectangular element K, we define the shape function space P_K by

$$P_K = \{p = \hat{p} \circ F_K^{-1}, \hat{p} \in \hat{P}\}.$$

Let $\bar{\Omega} = \bigcup_{K \in T_h} K$ be a decomposition of Ω with diameters $\le h$, such that T_h satisfies the usual regular assumption ([26]). The finite element space is $V^h = \{v_h : v_h|_K \in P_K,\ \forall K \in T_h, v$ is continuous at the vertices of elements and vanishing at the vertices on the boundary of $\Omega\}$.

Then for every $v_h \in V^h$, v_h can be written as

$$v_h = \bar{v}_h + v_h^1,$$

where $\bar{v}_h$ and v_h^1 are the conforming part and the nonconforming part of v_h, respectively. Let S_h^1 and S_h^2 be the bilinear and biquadratic space, $I_h^1 : H^2(\Omega) \to S_h^1$ and $I_h^2 : H^2(\Omega) \to S_h^2$ be the associated interpolation operators with respect to T_h.

The weak form of (1.1) is: Find $u \in H_0^1(\Omega)$, such that

$$\begin{cases} (u_{tt}, v) + \alpha(u_t, v) + \gamma(\nabla u, \nabla v) + \beta(\sin u, v) = (f, v), & \forall v \in H_0^1(\Omega), \\ u(X,0) = u_0(X), \quad u_t(X,0) = u_1(X). \end{cases} \tag{2.1}$$

The corresponding semi-discrete finite element procedure to (2.1) is: Find $u^h \in V^h$, such that

$$\begin{cases} (u_{tt}^h, v_h) + \alpha(u_t^h, v_h) + \gamma(\nabla_h u^h, \nabla_h v_h)_h + \beta(\sin u^h, v_h) = (f, v_h), & \forall v_h \in V^h, \\ u^h(X,0) = I_h^1 u_0, \quad u_t^h(X,0) = I_h^1 u_1, \end{cases} \tag{2.2}$$

where ∇_h denotes the gradient operator defined on V^h piecewisely with

$$(u, v)_h = \sum_K \int_K u \cdot v dxdy.$$

Lemma 2.1. *Let T_h be the generalized rectangular meshes. Then for any $v_h \in V^h$ and $v_h = \bar{v}_h + v_h^1$, we have*

$$\|v_h\|_h^2 = \|\bar{v}_h\|_h^2 + \|v_h^1\|_h^2, \quad \|v_h^1\|_0 \le ch\|v_h\|_h, \tag{2.3a}$$

$$\int_K v_h^1 dxdy = \int_K q_1 \frac{\partial v_h^1}{\partial x} dxdy = \int_K q_1 \frac{\partial v_h^1}{\partial y} dxdy = 0. \tag{2.3b}$$

Furthermore, if T_h is the rectangular partition,

$$\int_K q_1 v_h^1 dxdy = \int_K q_2 \frac{\partial v_h^1}{\partial x} dxdy = \int_K q_2 \frac{\partial v_h^1}{\partial y} dxdy = 0, \tag{2.4}$$

where $\|\cdot\|_h=(\sum\limits_K|\cdot|_{1,K}^2)^{\frac12}$ *is a norm on* V^h, $q_1\in S_h^1,\ q_2\in S_h^2$.

Proof. The proof of (2.3) can be found in [12]. Here we only need to prove (2.4). In this case, assume that the four vertices of $K\in T_h$ are $M_1(x_K-h_{x,K},y_K-h_{y,K}),\ M_2(x_K+h_{x,K},y-h_{y,K}),\ M_3(x_K+h_{x,K},y_K+h_{y,K}),\ M_4(x_K-h_{x,K},y+h_{y,K})$. Then there exists a unique mapping $F_K^1:\hat K\to K$

$$x=x_K+(2\xi-1)h_{x,K},\ y=y_K+(2\eta-1)h_{y,K}. \tag{2.5}$$

Let $|J|$ be the Jacobian of (2.5), because $q_1\in S_h^1,\ q_2\in S_h^2$. We can check that

$$\begin{aligned}
\int_K xv_h^1dxdy&=\int_{\hat K}(x_K+(2\xi-1))(\beta_1\hat\Psi(\xi)+\beta_2\hat\Psi(\eta))|J|dxdy\\
&=\int_{\hat K}(x_K+(2\xi-1))4h_{x,K}h_{y,K}(\beta_1\hat\Psi(\xi)+\beta_2\hat\Psi(\eta))dxdy,\\
\int_K yv_h^1dxdy&=\int_{\hat K}(y_K+(2\eta-1))(\beta_1\hat\Psi(\xi)+\beta_2\hat\Psi(\eta))|J|dxdy\\
&=\int_{\hat K}(y_K+(2\eta-1))4h_{x,K}h_{y,K}(\beta_1\hat\Psi(\xi)+\beta_2\hat\Psi(\eta))dxdy,\\
\int_K x^2y^2\frac{\partial v_h^1}{\partial x}dxdy&=\int_{\hat K}(x_K+(2\xi-1)h_{x,K})^2(y_K+(2\eta-1)h_{y,K})^2\frac{\partial \hat{v_h^1}}{\partial\xi}|J|d\xi d\eta\\
&=\int_{\hat K}(x_K+(2\xi-1)h_{x,K})^2(y_K+(2\eta-1)h_{y,K})^2 2h_{y,K}\hat\Psi'(\xi)d\xi d\eta.
\end{aligned}$$

On the other hand, by the definition of $\hat\Psi(s)$, it is easy to check

$$\int_0^1\hat\Psi(s)ds=\int_0^1\hat\Psi'(s)ds=\int_0^1 s\hat\Psi(s)ds=\int_0^1 s\hat\Psi'(s)ds=\int_0^1 s^2\hat\Psi'(s)ds=0. \tag{2.6}$$

Then

$$\int_K xv_h^1dxdy=\int_K yv_h^1dxdy=\int_K x^2y^2\frac{\partial v_h^1}{\partial x}dxdy=0.$$

Similarly, we can treat all other terms appeared in (2.4). The proof is completed. □

Note that for all $v_h\in V^h$, we get from (2.3) that

$$\|v_h\|_0^2\le\|\bar v_h\|_0^2+\|v_h^1\|_0^2\le c\|v_h\|_h^2+ch\|v_h\|_h^2\le c\|v_h\|_h^2. \tag{2.7}$$

Now we start to prove the other two important lemmas.

Lemma 2.2. *Let* $\omega=u-I_h^1u$, T_h *be the generalized rectangular meshes. Then if* $u\in H^3(\Omega)$, *there holds*

$$(\nabla_h\omega,\nabla_h v_h)_h\le ch^2|u|_3\|v_h\|_h,\ \forall v_h\in V^h. \tag{2.8}$$

Furthermore, if $u\in H^4(\Omega)$, T_h *is a family of rectangular meshes, we have*

$$(\nabla_h\omega,\nabla_h v_h)_h\le ch^2|u|_4\|v_h\|_0. \tag{2.9}$$

Proof. For all $v_h\in V^h$, it has been shown in [20] that

$$(\nabla_h\omega,\nabla_h\bar v_h)_h\le\begin{cases}ch^2|u|_3\|\bar v_h\|_h, & u\in H^3(\Omega),\\ ch^2|u|_4\|\bar v_h\|_0, & u\in H^4(\Omega).\end{cases} \tag{2.10}$$

On the other hand, by the first term of (2.10) and (2.3) of Lemma 2.1, we obtain

$$\begin{aligned}(\nabla_h\omega,\nabla_h v_h)_h &= (\nabla_h\omega,\nabla_h\bar{v}_h)_h+(\nabla_h\omega,\nabla_h v_h^1)_h\\ &= (\nabla_h\omega,\nabla_h\bar{v}_h)_h+(\nabla_h\omega-I_h^1\nabla_h\omega,\nabla_h v_h^1)_h\\ &\le ch^2|u|_3\|\bar{v}_h\|_h+ch^2\sum_K|\nabla_h\omega|_{2,K}\|\nabla_h v_h^1\|_{0,K}\\ &\le ch^2|u|_3\|\bar{v}_h\|_h+ch^2|u|_3\|v_h^1\|_h\le ch^2|u|_3\|v_h\|_h.\end{aligned}$$

When $u\in H^4(\Omega)$, T_h is rectangular partition, by use of the second estimate of (2.10) and the inverse inequality, we have

$$\begin{aligned}(\nabla_h\omega,\nabla_h v_h)_h &= (\nabla_h\omega,\nabla_h\bar{v}_h)_h+(\nabla_h\omega,\nabla_h v_h^1)_h\\ &= (\nabla_h\omega,\nabla_h\bar{v}_h)_h+(\nabla_h\omega-I_h^2\nabla_h\omega,\nabla_h v_h^1)_h\\ &\le ch^2|u|_4\|\bar{v}_h\|_0+ch^3\sum_K|\nabla_h\omega|_{3,K}\|\nabla_h v_h^1\|_{0,K}\\ &\le ch^2|u|_4\|\bar{v}_h\|_0+ch^3|u|_4\|v_h^1\|_h\le ch^2|u|_4\|v_h\|_0.\end{aligned}$$

The proof is complete. □

Lemma 2.3. *For $u\in H^3(\Omega)$, there holds*

$$|\sum_K\int_{\partial K}\frac{\partial u}{\partial n}v_h ds|\le ch^2|u|_3\|v_h\|_h,\quad \forall v_h\in V^h. \tag{2.11}$$

Furthermore, if $u\in H^4(\Omega)$, T_h is rectangular partition, we have

$$\left|\sum_K\int_{\partial K}\frac{\partial u}{\partial n}v_h ds\right|\le ch^2|u|_4\|v_h\|_0, \tag{2.12}$$

where n denotes the unit out normal vector over ∂K.

Proof. As (2.11) has been proved in [11], here we just need to prove (2.12). In fact, for each $v\in V^h$,

$$\sum_K\int_{\partial K}\frac{\partial u}{\partial n}\bar{v}_h ds=0. \tag{2.13}$$

On the other hand,

$$\sum_K\int_{\partial K}\frac{\partial u}{\partial n}v_h^1 ds=(\Delta_h u,v_h^1)_h+(\nabla_h u,\nabla_h v_h^1)_h, \tag{2.14}$$

where Δ_h is the Laplace operator defined piecewisely. By the interpolation theory and Lemma 2.2, we have

$$(\Delta_h u,v_h^1)_h=(\Delta_h u-I_h^1\Delta_h u,v_h^1)_h\le ch^2\sum_K|\Delta_h u|_{2,K}\|v_h^1\|_0\le ch^2|u|_4\|v_h\|_0, \tag{2.15}$$

and

$$\begin{aligned}|(\nabla_h u,\nabla_h v_h^1)_h| &= |(\nabla_h u-I_h^2\nabla_h u,\nabla_h v_h^1)_h\\ &\le ch^3\sum_K|\nabla u|_3\|\nabla_h v_h^1\|_0\le ch^3|u|_4\|v_h\|_h\le ch^2|u|_4\|v_h\|_0.\end{aligned} \tag{2.16}$$

Substituting (2.15)-(2.16) into (2.14) yields

$$\left|\sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h^1 ds\right| \le ch^2 |u|_4 \|v_h\|_0,$$

which joins together with (2.13) leads to (2.12). The proof is complete. $\square$

Now we are ready to prove one of the main results of this paper.

Theorem 2.1. *Suppose u and u^h are the solutions of* (2.1) *and* (2.2)*, respectively. If $u, u_t \in H^3(\Omega), u_{tt} \in H^2(\Omega)$, then there holds*

$$\|u_t^h(t) - I_h^1 u_t(t)\|_0 + \|u^h(t) - I_h^1 u(t)\|_h \le ch^2 \left(\int_0^t \|u\|_*^2 d\tau + |u|_3^2\right)^{\frac{1}{2}},$$

where $\|u\|_*^2 = |u_{tt}|_2^2 + |u|_2^2 + |u_t|_3^2$.

Proof. Let $\theta = u^h - I_h^1 u$, for all $v_h \in V^h$, there holds the following error equation:

$$\begin{aligned} &(\theta_{tt}, v_h) + \gamma(\nabla_h \theta, \nabla_h v_h)_h + \alpha(\theta_t, v_h) \\ =\ &(\omega_{tt}, v_h) + \gamma(\nabla_h \omega, \nabla_h v_h)_h + \alpha(\omega_t, v_h) - \beta(\sin u^h - \sin u, v_h) \\ &-\gamma \sum_K \int_{\partial K} \frac{\partial u}{\partial n} v_h ds. \end{aligned} \tag{2.17}$$

Noting that

$$\begin{aligned} |\beta(\sin u^h - \sin u, v_h)| &\le \beta\|u^h - u\|_0 \|v_h\|_0 \\ &\le \beta(\|u^h - I_h^1 u\|_0 + \|I_h^1 u - u\|_0)\|v_h\|_0. \end{aligned} \tag{2.18}$$

Choosing $v_h = \theta_t \in V^h$ in (2.17), by (2.8) and (2.18), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\theta_t\|_0^2 + \gamma\|\theta\|_h^2\right) + \alpha\|\theta_t\|_0^2 \\ \le\ &ch^2|u_{tt}|_2\|\theta_t\|_0 + \gamma\frac{d}{dt}(\nabla_h\omega, \nabla_h\theta)_h - \gamma(\nabla_h\omega_t, \nabla_h\theta)_h + ch^2|u_t|_2\|\theta_t\|_0 \\ &+ ch^2|u|_2\|\theta_t\|_0 + \beta\|\theta\|_0\|\theta_t\|_0 - \gamma\frac{d}{dt}\sum_K\int_{\partial K}\frac{\partial u}{\partial n}\theta ds + \gamma\sum_K\int_{\partial K}\frac{\partial u_t}{\partial n}\theta ds \\ \le\ &ch^2|u_{tt}|_2\|\theta_t\|_0 + \gamma\frac{d}{dt}(\nabla_h\omega, \nabla_h\theta)_h + ch^2|u_t|_3\|\theta\|_h + ch^2|u_t|_2\|\theta_t\|_0 \\ &+ ch^2|u|_2\|\theta_t\|_0 + \beta\|\theta\|_0\|\theta_t\|_0 - \gamma\frac{d}{dt}\sum_K\int_{\partial K}\frac{\partial u}{\partial n}\theta ds + \gamma\sum_K\int_{\partial K}\frac{\partial u_t}{\partial n}\theta ds. \end{aligned}$$

Applying (2.7), (2.11), Schwartz's inequality and noting that $\alpha\|\theta_t\|_0^2 \ge 0$, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\theta_t\|_0^2 + \gamma\|\theta\|_h^2\right) \\ \le\ &ch^4\|u\|_*^2 + c(\|\theta\|_0^2 + \|\theta_t\|_0^2) - \gamma\frac{d}{dt}\sum_K\int_{\partial K}\frac{\partial u}{\partial n}\theta ds + \gamma\frac{d}{dt}(\nabla_h\omega, \nabla_h\theta)_h \\ \le\ &ch^4\|u\|_*^2 + c(\|\theta\|_h^2 + \|\theta_t\|_0^2) - \gamma\frac{d}{dt}\sum_K\int_{\partial K}\frac{\partial u}{\partial n}\theta ds + \gamma\frac{d}{dt}(\nabla_h\omega, \nabla_h\theta)_h. \end{aligned} \tag{2.19}$$

Integrating the both sides of (2.19) with respect to time from 0 to t, and noting $\theta(X,0) = \theta_t(X,0) = 0$, we obtain

$$\begin{aligned}&\|\theta_t\|_0^2 + \gamma\|\theta\|_h^2\\ \le\ & ch^4\int_0^t \|u\|_*^2 d\tau + c\int_0^t \Big(\|\theta\|_h^2 + \|\theta_t\|_0^2\Big)d\tau + \gamma\sum_K\int_{\partial K}\frac{\partial u}{\partial n}\theta ds + \gamma\frac{d}{dt}\Big(\nabla_h\omega, \nabla_h\theta\Big)_h.\end{aligned}$$

Taking (2.11) and Schwartz's inequality, we have

$$\|\theta_t\|_0^2 + \|\theta\|_h^2 \le ch^4\int_0^t \|u\|_*^2 d\tau + c\int_0^t \Big(\|\theta\|_h^2 + \|\theta_t\|_0^2\Big)d\tau + ch^4|u|_3^2. \tag{2.20}$$

Applying Gronwall's lemma and (2.20) yields the desired result. □

3. Fully-Discrete Scheme and Error Estimates

In this section, we will propose a fully-discrete scheme for approximating solution u of problem (1.1) and discuss the related error estimate.

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a given partition of the time interval $[0,T]$ with step length $\tau = \frac{T}{M}$ for some positive integer M. For a function φ on $[0,T]$, define $\varphi^n = \varphi(X, t_n)$,

$$\begin{aligned}&\varphi^{n+\frac12} = \frac12(\varphi^{n+1} + \varphi^n),\ \ \bar\partial_t\varphi^{n+\frac12} = \frac1\tau(\varphi^{n+1} - \varphi^n),\\ &\varphi^{n,\frac14} = \frac14(\varphi^{n+1} + 2\varphi^n + \varphi^{n-1}) = \frac12(\varphi^{n+\frac12} + \varphi^{n-\frac12}),\\ &\bar\partial_t\varphi^n = \frac{\varphi^{n+1} - \varphi^{n-1}}{2\tau} = \frac{\varphi^{n+\frac12} - \varphi^{n-\frac12}}{\tau} = \frac12(\bar\partial_t\varphi^{n+\frac12} + \bar\partial_t\varphi^{n-\frac12}),\\ &\bar\partial_{tt}\varphi^n = \frac{\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}}{\tau^2} = \frac{(\bar\partial_t\varphi^{n+\frac12} - \bar\partial_t\varphi^{n-\frac12})}{\tau}.\end{aligned}$$

The problem (1.1) is equivalent to the following formulation

$$\begin{cases}(\bar\partial_{tt}u^n, v) + \alpha(\bar\partial_t u^n, v) + \gamma(\nabla u^{n,\frac14}, \nabla v) + \beta(\sin^{n,\frac14} u, v) = (f^{n,\frac14}, v)\\ \qquad\quad +(r^n, v) + \alpha(p^n, v), \qquad \forall\ v \in H_0^1(\Omega),\\ u^0 = u_0(X),\ \ u_t^0 = u_1(X), \qquad \forall X \in \Omega,\end{cases} \tag{3.1}$$

where

$$r^n = \bar\partial_{tt}u^n - u_{tt}^{n,\frac14} = O(\tau^2), \qquad p^n = \bar\partial_t u^n - u_t^{n,\frac14} = O(\tau^2).$$

We can establish the fully-discrete scheme as: Find $U^n \in V^h$, such that

$$\begin{cases}(\bar\partial_{tt}U^n, v_h) + \alpha(\bar\partial_t U^n, v_h) + \gamma(\nabla_h U^{n,\frac14}, \nabla_h v_h)_h + \beta(\sin^{n,\frac14} U, v_h)\\ \quad = (f^{n,\frac14}, v_h), \qquad \forall\ v_h \in V^h,\\ U^0 = I_h^1 u_0(X),\ \ U^1 = I_h^1(u_0(X) + \tau u_1(X) + \frac{\tau^2}{2}u_{tt}(X,0)).\end{cases} \tag{3.2}$$

In order to get the error estimate, let $U^n - u^n = (U^n - I_h^1 u^n) + (I_h^1 u^n - u^n) = \rho^n + \sigma^n$. then for all $v_h \in V^h$, we get from (3.1) and (3.2) that

$$\begin{aligned}&(\bar\partial_{tt}\rho^n, v_h) + \gamma(\nabla_h\rho^{n,\frac14}, \nabla_h v_h)_h + \alpha(\bar\partial_t\rho^n, v_h)\\ =\ & -(\bar\partial_{tt}\sigma^n, v_h) - \alpha(\bar\partial_t\sigma^n, v_h) - \gamma(\nabla_h\sigma^{n,\frac14}, \nabla_h v_h)_h - (r^n, v_h)\\ &\quad -\alpha(p^n, v_h) - \gamma\sum_K\int_{\partial K}\frac{\partial u^{n,\frac14}}{\partial n}v_h ds - \beta(\sin^{n,\frac14} U - \sin^{n,\frac14} u, v_h).\end{aligned} \tag{3.3}$$

Now we state the other main result of this paper.

Theorem 3.1. *Assume that u and U^n are solutions of* (1.1) *and* (3.2) *respectively, if $u,\ u_t \in H^3(\Omega),\ u_{tt} \in H^2(\Omega),\ T_h$ is a family of generalized rectangular meshes, and τ is sufficiently small, there holds*

$$\|\bar{\partial}_t(U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}})\|_0^2 + \|U^n - u^n\|_h^2 \le c\Big(\tau^4 + h^2\Big),\quad n = 1, 2, \cdots, M. \tag{3.4}$$

Furthermore, if $u \in H^4(\Omega),\ T_h$ is rectangular partition, then we have

$$\|\bar{\partial}_t(U^{n-\frac{1}{2}} - u^{n-\frac{1}{2}})\|_0^2 + \|U^n - u^n\|_h^2 \le c\Big(\tau^4 + h^4\Big). \tag{3.5}$$

Proof. Choosing $v_h = \bar{\partial}_t\rho^n$ in (3.3), there holds

$$\begin{aligned}
&(\bar{\partial}_{tt}\rho^n, \bar{\partial}_t\rho^n) + \gamma(\nabla_h\rho^{n,\frac{1}{4}}, \nabla_h\bar{\partial}_t\rho^n)_h + \alpha(\bar{\partial}_t\rho^n, \bar{\partial}_t\sigma^n)\\
&= -(\bar{\partial}_{tt}\sigma^n, \bar{\partial}_t\rho^n) - \alpha(\bar{\partial}_t\sigma^n, \bar{\partial}_t\rho^n) - (r^n, \bar{\partial}_t\rho^n) - \alpha(p^n, \bar{\partial}_t\rho^n)\\
&\quad -\gamma(\nabla_h\sigma^{n,\frac{1}{4}}, \nabla_h\bar{\partial}_t\rho^n)_h - \gamma\sum_K\int_{\partial K}\frac{\partial u^{n,\frac{1}{4}}}{\partial n}\bar{\partial}_t\rho^n ds - \beta(\sin^{n,\frac{1}{4}}U - \sin^{n,\frac{1}{4}}u, \bar{\partial}_t\rho^n).
\end{aligned} \tag{3.6}$$

The terms on the left hand of (3.6) can be rewritten as:

$$\begin{aligned}
(\bar{\partial}_{tt}\rho^n, \bar{\partial}_t\rho^n) &= \Big(\frac{1}{\tau}(\bar{\partial}_t\rho^{n+\frac{1}{2}} - \bar{\partial}_t\rho^{n-\frac{1}{2}}), \frac{1}{2}(\bar{\partial}_t\rho^{n+\frac{1}{2}} + \bar{\partial}_t\rho^{n-\frac{1}{2}})\Big)\\
&= \frac{1}{2\tau}(\|\bar{\partial}_t\rho^{n+\frac{1}{2}}\|_0^2 - \|\bar{\partial}_t\rho^{n-\frac{1}{2}}\|_0^2), \qquad (3.7)\\
\gamma(\nabla_h\rho^{n,\frac{1}{4}}, \nabla_h\bar{\partial}_t\rho^n)_h &= \gamma\left(\frac{1}{2}\nabla_h(\rho^{n+\frac{1}{2}} + \rho^{n-\frac{1}{2}}), \frac{1}{\tau}\nabla_h(\rho^{n+\frac{1}{2}} - \rho^{n-\frac{1}{2}})\right)_h\\
&= \frac{\gamma}{2\tau}(\|\rho^{n+\frac{1}{2}}\|_h^2 - \|\rho^{n-\frac{1}{2}}\|_h^2), \qquad (3.8)
\end{aligned}$$

and

$$\alpha(\bar{\partial}_t\rho^n, \bar{\partial}_t\rho^n) = \alpha\|\bar{\partial}_t\rho^n\|_0^2. \tag{3.9}$$

Now we estimate each term of the right hand of (3.6). Firstly, by Schwartz's inequality and the definitions of r^n and p^n, we have

$$\begin{aligned}
&-(\bar{\partial}_{tt}\sigma^n, \bar{\partial}_t\rho^n) + \alpha(\bar{\partial}_t\sigma^n, \bar{\partial}_t\rho^n) \le c(\|\bar{\partial}_{tt}\sigma^n\|_0^2 + \|\bar{\partial}_t\sigma^n\|_0^2) + c\|\bar{\partial}_t\rho^n\|_0^2, \qquad (3.10)\\
&(r^n, \bar{\partial}_t\rho^n) + \alpha(p^n, \bar{\partial}_t\rho^n) \le c\tau^4 + c\|\bar{\partial}_t\rho^n\|_0^2. \qquad (3.11)
\end{aligned}$$

Then by (2.8), for $t = t_n$, we get

$$\begin{aligned}
\gamma(\nabla_h\sigma^{n,\frac{1}{4}}, \nabla_h\bar{\partial}_t\rho^n)_h &= ch^2|u^{n,\frac{1}{4}}|_3\|\bar{\partial}_t\rho^n\|_h\\
&\le ch|u^{n,\frac{1}{4}}|_3\|\bar{\partial}_t\rho^n\|_0 \le ch^2|u^{n,\frac{1}{4}}|_3^2 + c\|\bar{\partial}_t\rho^n\|_0^2.
\end{aligned} \tag{3.12}$$

Applying (2.11) yields

$$\begin{aligned}
\gamma\sum_K\int_{\partial K}\frac{\partial u^{n,\frac{1}{4}}}{\partial n}\bar{\partial}_t\rho^n ds &\le ch^2|u^{n,\frac{1}{4}}|_3\|\bar{\partial}_t\rho^n\|_h \le ch|u^{n,\frac{1}{4}}|_3\|\bar{\partial}_t\rho^n\|_0\\
&\le ch^2|u^{n,\frac{1}{4}}|_3^2 + c\|\bar{\partial}_t\rho^n\|_0^2,
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\beta(\sin^{n,\frac{1}{4}} U-\sin^{n,\frac{1}{4}} u,\bar{\partial}_t\rho^n)\\
=&\frac{\beta}{4}\Big((\sin U^{n+1}+2\sin U^n+\sin U^{n-1})-(\sin u^{n+1}+2\sin u^n+\sin u^{n-1}),\bar{\partial}_t\rho^n\Big)\\
\le& c\Big(\|U^{n+1}-u^{n+1}\|_0+\|U^n-u^n\|_0+\|U^{n-1}-u^{n-1}\|_0\Big)\|\bar{\partial}_t\rho^n\|_0\\
\le& c\Big(\|\rho^{n+1}\|_0^2+\|\rho^n\|_0^2+\|\rho^{n-1}\|_0^2\Big)+c\|\bar{\partial}_t\rho^n\|_0^2+c\Big(\|\sigma^{n+1}\|_0^2+\|\sigma^{n+1}\|_0^2+\|\sigma^{n+1}\|_0^2\Big)\\
\le& c\Big(\|\rho^{n+1}\|_0^2+\|\rho^n\|_0^2+\|\rho^{n-1}\|_0^2\Big)+c\|\bar{\partial}_t\rho^n\|_0^2+ch^4\Big(|u^{n+1}|_2^2+|u^n|_2^2+|u^{n-1}|_2^2\Big).
\end{aligned}\tag{3.14}$$

Substituting (3.7)-(3.14) into (3.6), we have

$$\begin{aligned}
&\frac{1}{2\tau}\Big(\|\bar{\partial}_t\rho^{n+\frac{1}{2}}\|_0^2-\|\bar{\partial}_t\rho^{n-\frac{1}{2}}\|_0^2\Big)+\frac{\gamma}{2\tau}\Big(\|\rho^{n+\frac{1}{2}}\|_h^2-\|\rho^{n-\frac{1}{2}}\|_h^2\Big)+\alpha\|\bar{\partial}_t\rho^n\|_0^2\\
\le& c\Big(\|\bar{\partial}_{tt}\sigma^n\|_0^2+\|\bar{\partial}_t\sigma^n\|_0^2\Big)+c\|\bar{\partial}_t\rho^n\|_0^2+c\tau^4+ch^2|u^{n,\frac{1}{4}}|_3^2\\
&+ch^4\Big(|u^{n+1}|_2^2+|u^n|_2^2+|u^{n-1}|_2^2\Big)+c\Big(\|\rho^{n+1}\|_0^2+\|\rho^n\|_0^2+\|\rho^{n-1}\|_0^2\Big)\\
\le& c\Big(\|\bar{\partial}_{tt}\sigma^n\|_0^2+\|\bar{\partial}_t\sigma^n\|_0^2\Big)+c\|\bar{\partial}_t\rho^n\|_0^2+c\tau^4+ch^2|u^{n,\frac{1}{4}}|_3^2\\
&+ch^4\Big(|u^{n+1}|_2^2+|u^n|_2^2+|u^{n-1}|_2^2\Big)+c\Big(\|\rho^{n+1}\|_h^2+\|\rho^n\|_h^2+\|\rho^{n-1}\|_h^2\Big).
\end{aligned}\tag{3.15}$$

On the other hand

$$\begin{aligned}
|\bar{\partial}_{tt}\sigma^n\|_0^2=&\left\|\frac{1}{\tau^2}(\sigma^{n+1}-2\sigma^n-\sigma^{n-1})\right\|_0^2=\left\|\frac{1}{\tau^2}\left(\int_{t_n}^{t_{n+1}}\sigma_t dt-\int_{t_{n-1}}^{t_n}\sigma_t dt\right)\right\|_0^2\\
=&\left\|\frac{1}{\tau^2}\left(\int_{t_n}^{t^{n+1}}\sigma_t dt-\tau\sigma_t^n+\tau\sigma_t^n-\int_{t_{n-1}}^{t_n}\sigma_t dt\right)\right\|_0^2\\
=&\left\|\frac{1}{\tau^2}\left(\int_{t_n}^{t_{n+1}}\sigma_{tt}(t_{n+1}-t)dt+\int_{t_{n-1}}^{t_n}\sigma_{tt}(t-t_{n-1})dt\right)\right\|_0^2\\
\le&\frac{1}{\tau^4}\bigg[\int_K\left(\int_{t_n}^{t_{n+1}}\sigma_{tt}^2dt\cdot\int_{t_n}^{t_{n+1}}(t_{n+1}-t)^2\right)dxdy\\
&+\int_K\left(\int_{t_{n-1}}^{t_n}\sigma_{tt}^2dt\cdot\int_{t_{n-1}}^{t_n}(t-t_{n-1})^2\right)\bigg]dxdy\\
\le&\frac{c}{\tau}\int_{t_{n-1}}^{t_n}\|\sigma_{tt}\|_0^2dt\le\frac{ch^4}{\tau}\int_{t_{n-1}}^{t_n}|u_{tt}|_2^2dt,
\end{aligned}\tag{3.16}$$

and similarly,

$$\|\bar{\partial}_t\sigma^n\|_0^2\le\frac{c}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|\sigma_t\|_0^2dt\le\frac{ch^4}{\tau}\int_{t_{n-1}}^{t_{n+1}}|u_t|_2^2dt.\tag{3.17}$$

By the definition of $\bar{\partial}_t\rho^n$, there holds

$$\|\bar{\partial}_t\rho^n\|_0^2\le c\Big(\|\bar{\partial}_t\rho^{n+\frac{1}{2}}\|_0^2+\|\bar{\partial}_t\rho^{n-\frac{1}{2}}\|_0^2\Big).$$

Summing from $n=1$ to $M-1$ and noting $\alpha\|\bar{\partial}_t\rho^n\|_0^2 \geq 0$, we obtain

$$\begin{aligned}
&\|\bar{\partial}_t\rho^{M-\frac{1}{2}}\|_0^2+\gamma\|\rho^{M-\frac{1}{2}}\|_h^2 \\
\leq\ &\|\bar{\partial}_t\rho^{\frac{1}{2}}\|_0^2+\gamma\|\rho^{\frac{1}{2}}\|_h^2+c\tau\sum_{n=1}^{M-1}\Big(\|\bar{\partial}_t\rho^{n-\frac{1}{2}}\|_0^2+\|\rho^n\|_h^2\Big)+c\tau\Big(\|\bar{\partial}_t\rho^{M-\frac{1}{2}}\|_0^2+\|\rho^M\|_h^2\Big) \\
&+c\tau^4+c\tau h^2\sum_{n=1}^{M-1}|u^{n,\frac{1}{4}}|_3^2+c\tau h^4\sum_{n=1}^{M}|u^n|_2^2+ch^4\int_0^T\Big(|u_{tt}|_2^2+|u_t|_2^2\Big)dt, \qquad (3.18)
\end{aligned}$$

where

$$\begin{aligned}
c\tau h^2\sum_{n=1}^{M}|u^{n,\frac{1}{4}}|_3^2 &\leq ch^2\cdot M\tau\cdot 4\max\{|u^n|_3^2\}\leq ch^2, \\
c\tau h^4\sum_{n=1}^{M}|u^n|_2^2 &\leq ch^4\cdot M\tau\cdot 4\max\{|u^n|_2^2\}\leq ch^4.
\end{aligned}$$

At the same time, by Taylor's formula, we have

$$\begin{aligned}
\|\bar{\partial}_t\rho^{\frac{1}{2}}\|_0^2 &= \|\frac{1}{\tau}\rho^1\|_0^2=\frac{1}{\tau^2}\|U^1-I_h^1u^1\|_0^2 \\
&\leq \frac{1}{\tau^2}\|U^1-I_h^1(u_0(X)+\tau u_1(X)+\frac{\tau^2}{2}u_{tt}(X,0)+\frac{\tau^3}{6}u_{ttt}(X,\delta))\|_0^2 \\
&\leq c\tau^4\|I_h^1u_{ttt}(X,\delta)\|_0^2, \qquad (\delta\in(0,t_1)),
\end{aligned}$$

and

$$\|\rho^{\frac{1}{2}}\|_h^2=\|\frac{1}{2}\rho^1\|_h^2\leq c\tau^6.$$

On the other hand, by $\varepsilon-$Young inequality, we have

$$\begin{aligned}
\|\rho^{M-\frac{1}{2}}\|_h^2 &= \frac{1}{4}\Big(\|\rho^M\|_h^2+\|\rho^{M-1}\|_h^2+2(\nabla\xi^M,\nabla\rho^{M-1})\Big) \\
&\geq \frac{1}{4}\Big(\|\rho^M\|_h^2-\varepsilon\|\rho^M\|_h^2-C(\varepsilon)\|\rho^{M-1}\|_h^2\Big).
\end{aligned}$$

Then (3.18) can be rewritten as

$$\begin{aligned}
&(1-c\tau)\|\bar{\partial}_t\rho^{M-\frac{1}{2}}\|_0^2+(\frac{1-\varepsilon}{4}\gamma-c\tau)\|\rho^M\|_h^2 \qquad (3.19)\\
\leq\ &c(h^2+\tau^4)+c\tau\sum_{n=1}^{M-2}\Big(\|\bar{\partial}_t\rho^{n-\frac{1}{2}}\|_0^2+\|\rho^n\|_h^2\Big)+(c\tau+\frac{C(\varepsilon)}{4})\Big(\|\bar{\partial}_t\rho^{M-\frac{3}{2}}\|_0^2+\|\rho^{M-1}\|_h^2\Big).
\end{aligned}$$

Choos sufficiently small τ and ε such that $1-c\tau,\ \frac{1-\varepsilon}{4}\gamma-c\tau>0$. Applying discrete Gronwall's lemma gives

$$\|\bar{\partial}_t\rho^{n-\frac{1}{2}}\|_0^2+\|\rho^n\|_h^2\leq c(\tau^4+h^4). \qquad (3.20)$$

Similarly with (3.16), we have

$$\begin{aligned}
\|\bar{\partial}_t\sigma^{n-\frac{1}{2}}\|_0^2 &\leq \frac{c}{\tau}\int_{t_{n-1}}^{t_n}\|\sigma_t\|_0^2dt\leq\frac{ch^4}{\tau}\int_{t_{n-1}}^{t_n}|u_t|_2^2dt \\
&\leq \frac{ch^4}{\tau}\cdot\tau\cdot\sup|u_t|_2^2\leq ch^4, \qquad (3.21)\\
\|\sigma^n\|_h^2 &= \|\sigma^n-\sigma^0\|_h^2\leq c\int_0^{t_n}\|\sigma_t\|_h^2dt\leq ch^4\int_0^{t_n}|u_t|_3^2dt. \qquad (3.22)
\end{aligned}$$

By (3.20-3.22) and triangular inequality, we can get the desired result of (3.4). Moreover, if $u \in H^4(\Omega)$, T_h is rectangular partition, from (2.9) and (2.12) we know that (3.12) and (3.13) can be rewritten as

$$\gamma(\nabla_h \sigma^{n,\frac{1}{4}}, \nabla_h \bar{\partial}_t \rho^n) \le ch^2 |u^{n,\frac{1}{4}}|_4 \|\bar{\partial}_t \rho^n\|_0 \le ch^4 |u^{n,\frac{1}{4}}|_4^2 + c\|\bar{\partial}_t \rho^n\|_0^2, \tag{3.23}$$

$$\begin{aligned}\gamma \sum_K \int_{\partial K} \frac{\partial u^{n,\frac{1}{4}}}{\partial n} \bar{\partial}_t \rho^n ds &\le ch^3 |u^{n,\frac{1}{4}}|_4 \|\bar{\partial}_t \rho^n\|_h \le ch^2 |u^{n,\frac{1}{4}}|_4 \|\bar{\partial}_t \rho^n\|_0 \\ &\le ch^4 |u^{n,\frac{1}{4}}|_4^2 + c\|\bar{\partial}_t \rho^n\|_0^2. \end{aligned}\tag{3.24}$$

Thus if the estimates of (3.12) and (3.13) are replaced by (3.23) and (3.24), respectively, we can get (3.5) similarly. This completes the proof. □

4. Concluding Remarks

We conclude this paper by making some remarks. First, it can be checked that the result of (3.5) is as same as [5] for problem (1.1) in 1D with two implicit difference schemes and [6] in 2D with ADI scheme of bilinear finite element.

For the well known nonconforming Q_1^{rot} element, EQ_1^{rot} element, CNQ_1^{rot} element, quasi-Carey element ([21]), P^1-nonconforming rectangular element ([22]) and modified Crouzeix-Raviart type element ([23]), the estimation of semi-discrete is also valid if they are applied to problem (1.1). However, the result of (3.5) can not be achieved by the above finite elements as they do not satisfy characters (2.9) and (2.12) in Lemma 2.2 and Lemma 2.3, i.e., (3.13) and (3.14) can not be improved to (3.23) and (3.24), respectively, although they have been widely applied to many problems (see [21-25]). This further indicates the significance of the choice of special quasi-Wilson element in this paper.

We also point out that, for quasi-Wilson element, $\int_F [v_h] ds \neq 0,\ \forall F \subset \partial K, v_h \in V^h$, which does not satisfy the requirement in the usual nonconforming finite element analysis, and the techniques used in the present work are very different from the existing literature (see [7,13-25]).

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