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THE COUPLING OF NBEM AND FEM FOR QUASILINEAR PROBLEMS IN A BOUNDED OR UNBOUNDED DOMAIN WITH A CONCAVE ANGLE*

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Abstract

Based on the Kirchhoff transformation and the natural boundary element method, we investigate a coupled natural boundary element method and finite element method for quasi-linear problems in a bounded or unbounded domain with a concave angle. By the principle of the natural boundary reduction, we obtain natural integral equation on circular arc artificial boundaries, and get the coupled variational problem and its numerical method. Moreover, the convergence of approximate solutions and error estimates are obtained. Finally, some numerical examples are presented to show the feasibility of our method. Our work can be viewed as an extension of the existing work of H.D. Han et al..

Mathematics subject classification: 65N30, 35J65.
Key words: Quasilinear elliptic equation, Concave angle domain, Natural integral equation.

1. Introduction

The standard procedure of the coupling of boundary element and finite element can be described as follows. We introduce an artificial boundary to divide the original domain into two regions, an unbounded domain and a bounded one on which the boundary element method and finite element method are used, respectively.

In this paper, the coupling of natural boundary element method (NBEM) [4, 5, 19, 20] and finite element method (FEM) which is also called artificial boundary method [7–9] or DtN method [6, 13] is applied to solve boundary value problems in a bounded or unbounded domain with a concave angle.

Let Ω be a bounded and simple connected domain with sufficiently smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_\alpha \cup \Gamma$, where

$$\Gamma_0 = \big\{(r,0) \mid 0 \le r \le a\big\},\ \Gamma_\alpha = \big\{(r,\alpha) \mid 0 \le r \le b\big\},$$
$$\Gamma = \big\{(r,\theta) \mid r = \psi(\theta) \ge 0,\ 0 \le \theta \le \alpha,\ \psi(0) = a, \psi(\alpha) = b\big\},$$

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and Γ is a smooth curve. Here α is a concave angle (Fig. 1.1). Particularly, when $\alpha = 2\pi$, Ω is a cracked domain. And Ω^c refers to the unbounded domain with boundary $\partial\Omega^c = \Gamma_0 \cup \Gamma_\alpha \cup \Gamma$, where Γ is defined as above and Γ_0 and Γ_α are changed by

$$\Gamma_0 = \{(r, 0) \mid 0 \le a \le r\},\ \Gamma_\alpha = \{(r, \alpha) \mid 0 \le b \le r\}.$$

The problem can be described as follows [3,8,10,11,16].

$$\begin{cases} -\nabla \cdot (a(x, u)\nabla u) = f, & \text{in } \Omega \text{ or } \Omega^c, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_0 \cup \Gamma_\alpha, \\ u = 0, & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $a(\cdot, \cdot)$ and f are given functions with various properties which will be ranked in the following. When we study the domain Ω^c, problem (1.1) is not well posed. We need an appropriate boundary condition at infinity

$$u(x) \text{ is bounded, as } |x| \to \infty. \tag{1.2}$$

Problem (1.1)–(1.2) has many physical applications in, e.g., the field of magnetostatics, where a is the magnetic permeability and u is the magnetic scalar potential; the field of compressible flow, where a is the density and u is the velocity potential. See [2,15] etc. for more numerical results about problems of this kind with bounded domains. And note that, when we consider problem (1.1)-(1.2) in the unbounded domain Ω^c and get rid of the boundary conditions on Γ_0 and Γ_α, this is the right problem which was discussed in [8] by the artificial boundary method. Hence, our work can be viewed as a continuation of [8]. Moreover, we give an error analysis in Section 3, which was not presented in [8].

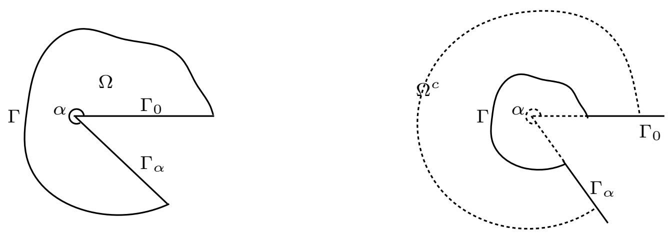

Fig. 1.1. The illustration of domains: the left is Ω, and the right is Ω^c.

Following [8,11], suppose that the given function $a(\cdot, \cdot)$ satisfies

$$0 < C_0 \le a(x, u) \le C_1,\ \forall\ u \in \mathbb{R},\ \text{and for almost all } x \in \Omega \text{ or } x \in \Omega^c, \tag{1.3}$$

where C_0, $C_1 \in \mathbb{R}$ are two constants, and

$$|a(x, u) - a(x, v)| \le C_L|u - v|,\ \forall\ u, v \in \mathbb{R} \text{ and for almost all } x \in \Omega \text{ or } x \in \Omega^c, \tag{1.4}$$

with a constant $C_L > 0$. We also assume that $\dfrac{\partial a}{\partial s}, \dfrac{\partial^2 a}{\partial s^2}$ are continuous.

In the following, we suppose that the function$f \in L^2(\Omega)$ or $f \in L^2(\Omega^c)$ has compact support, i.e., there exists a constant $R_0 > 0$, such that

$$\text{supp} f \subset \Omega_{R_0} = \Big\{x \in \mathbb{R}^2 \mid |x| \ge R_0\Big\} \text{ or } \Big\{x \in \mathbb{R}^2 \mid |x| \le R_0\Big\}, \tag{1.5}$$

which corresponds to the domains Ω and Ω^c, respectively. We also assume that

$$a(x,u) \equiv a_0(u) \text{ when } |x| \le R_0 \text{ or } |x| \ge R_0. \tag{1.6}$$

Now taking the vertex of the angle α as the origin of coordinates and put Γ_0 on the x–axis. Drawing an arc as below

$$\Gamma_R = \{(R,\theta) \mid 0 \le \theta \le \alpha\}, \text{ in } \Omega \text{ or } \Omega^c. \tag{1.7}$$

Then, Γ_R divides the original domain Ω or Ω^c into two sections. And if continuous conditions (1.10) of [8] on Γ_R are satisfied, then the original problem can be solved. Particularly, when $a(x,u) = a$ is independent of x and u, [4,19] have obtained the natural integral equation. One can also refer to the book [20] for more details. In the paper, we shall derive natural integral equations for more general quasilinear elliptic equations in a bounded or unbounded domain with a concave angle. We introduce the so-called *Kirchhoff transformation* [12]

$$w(x) = \int_0^{u(x)} a_0(\xi)d\xi, \text{ for } x \in \Omega_e, \tag{1.8}$$

then we have

$$\nabla w = a_0(u)\nabla u. \tag{1.9}$$

The rest of the paper is organized as follows. In section 2, we obtain the natural integral equation for problem (1.1) in a bounded or unbounded domain. In section 3, we give the equivalent variational problems and the finite element approximations. We also discuss the reduced problem's well-posedness and the convergence result. What's more, we give an error analysis to show how the errors can be affected by the order of artificial boundary condition, the mesh of the domain and the location of the artificial boundary. At last, in section 4, we present some numerical examples to illustrate the efficiency and feasibility of our method. Our paper can be actually considered as a sequel of [8].

Throughout this paper, we denote C as a general positive constant independent of R, N and h, where h and N are defined in section 3.

2. Natural Boundary Reduction

In this section, by virtue of the Poisson integral formula and the natural integral equation for the linear problem, we shall obtain the corresponding results for the quasilinear problem in Ω and Ω^c.

2.1. The problem in the bounded domain Ω

Let us introduce an artificial boundary Γ_R which divides Ω into two parts Ω_i and Ω_e, where Ω_e is a sector. The domain Ω_e can be described as follows

$$\Omega_e \triangleq \{(r,\theta) \mid 0 < r < R, \theta \in (0,\alpha)\},$$
$$\Gamma_R \triangleq \{(R,\theta) \mid \theta \in (0,\alpha)\},$$

Γ_{0e}, $\Gamma_{\alpha e}$ is the restriction of Γ_0, Γ_α of section 1 in Ω_e (Fig. 2.1).

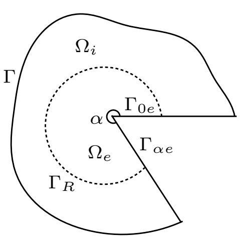

Fig. 2.1. The illustration of domain Ω.

Then by (1.1), (1.8), (1.9), the problem confines in Ω_e can be described as below:

$$\begin{cases} -\Delta w = 0, & \text{in } \Omega_e, \\ \dfrac{\partial w}{\partial n} = 0, & \text{on } \Gamma_{0e} \cup \Gamma_{\alpha e}, \\ \dfrac{\partial w}{\partial n} = w_n, & \text{on } \Gamma_R. \end{cases} \tag{2.1}$$

Then, there are the Poisson integral formula

$$\begin{aligned} w(r,\theta) = \frac{1}{2\alpha}\left(R^{\frac{2\pi}{\alpha}} - r^{\frac{2\pi}{\alpha}}\right)\int_0^\alpha \Big[&\frac{1}{r^{\frac{2\pi}{\alpha}} + R^{\frac{2\pi}{\alpha}} - 2(Rr)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta-\theta')} \\ &+\frac{1}{r^{\frac{2\pi}{\alpha}} + R^{\frac{2\pi}{\alpha}} - 2(Rr)^{\frac{\pi}{\alpha}}\cos\frac{\pi}{\alpha}(\theta+\theta')}\Big] w(R,\theta')d\theta', \end{aligned} \tag{2.2}$$

with $0 < r < R$, and the natural integral equation

$$\frac{\partial w}{\partial n} = -\frac{\pi}{4\alpha^2 R}\int_0^\alpha \Big(\frac{1}{\sin^2\frac{\theta-\theta'}{2\alpha}\pi} + \frac{1}{\sin^2\frac{\theta+\theta'}{2\alpha}\pi}\Big) w(R,\theta')d\theta'. \tag{2.3}$$

Eqs. (2.2) and (2.3) can also be expressed in the following Fourier series forms. One can refer to [4] and Appendix of [18] for more details.

$$w(r,\theta) = \frac{1}{\alpha}\sum_{n=0}^{+\infty}\varepsilon_n\Big(\frac{r}{R}\Big)^{\frac{n\pi}{\alpha}}\int_0^\alpha w(R,\theta')\cos\frac{n\pi\theta}{\alpha}\cos\frac{n\pi\theta'}{\alpha}d\theta', \tag{2.4}$$

with $r < R$, and

$$\frac{\partial w}{\partial r} = \frac{\pi}{R\alpha^2}\sum_{n=0}^{+\infty}\varepsilon_n n\int_0^\alpha w(R,\theta')\cos\frac{n\pi\theta}{\alpha}\cos\frac{n\pi\theta'}{\alpha}d\theta', \tag{2.5}$$

where ε_n refers to: when $n = 0$, $\varepsilon_n = 1$; when $n > 0$, $\varepsilon_n = 2$. Equation (2.5) can also be changed to the following equivalent form

$$\frac{\partial w}{\partial r} = -\frac{1}{R\alpha}\sum_{n=0}^{+\infty}\varepsilon_n\int_0^\alpha \frac{\partial w(R,\theta')}{\partial\theta'}\cos\frac{n\pi\theta}{\alpha}\sin\frac{n\pi\theta'}{\alpha}d\theta'. \tag{2.6}$$

From (1.9), we have

$$\frac{\partial w}{\partial n} = a_0(u)\frac{\partial u}{\partial n}. \tag{2.7}$$

By (2.5), (2.7) and $\dfrac{\partial w}{\partial n} = \dfrac{\partial w}{\partial r}$, we obtain the exact artificial boundary condition of u on Γ_R

$$a_0(u)\frac{\partial u}{\partial n} = \frac{\pi}{R\alpha^2}\sum_{n=0}^{+\infty}\varepsilon_n n\int_0^\alpha\Big(\int_0^{u(R,\theta')}a_0(y)dy\Big)\cos\frac{n\pi\theta}{\alpha}\cos\frac{n\pi\theta'}{\alpha}d\theta' \triangleq \mathscr{K}_1(u(R,\theta)). \tag{2.8}$$

Then by (1.1), (2.8), the original problem confines in Ω_i can be defined as follows:

$$\begin{cases} -\nabla\cdot(a(x,u)\nabla u) = f, & \text{in } \Omega_i, \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{0i}\cup\Gamma_{\alpha i}, \\ u = 0, & \text{on } \Gamma, \\ a_0(u)\dfrac{\partial u}{\partial n} = \mathscr{K}_1(u(R,\theta)), & \text{on } \Gamma_R, \end{cases} \tag{2.9}$$

with $\Gamma_{0i} = \Gamma_0 \cap \overline{\Omega}_i$, $\Gamma_{\alpha i} = \Gamma_\alpha \cap \overline{\Omega}_i$.

Therefore, the solution of problem (2.9) is the restriction of the solution of problem (1.1) in the bounded domain Ω_i.

2.2. The problem in the unbounded domain Ω^c

Firstly, we introduce an artificial boundary Γ_R which divides Ω^c into two parts Ω_i and Ω_e, where Ω_e is the unbounded domain. The domain Ω_e can be described as follows

$$\Omega_e \triangleq \{(r,\theta) \mid 0 < R < r, \theta \in (0,\alpha)\},$$
$$\Gamma_R \triangleq \{(R,\theta) \mid \theta \in (0,\alpha)\},$$

Γ_{0e}, $\Gamma_{\alpha e}$ are similarly with Γ_0, Γ_α of section 1. (Fig. 2.2).

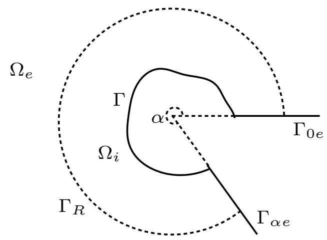

Fig. 2.2. The illustration of domain Ω^c.

By (1.1)–(1.2), (1.8)–(1.9), the problem confines in Ω_e can be described as follows

$$\begin{cases} -\Delta w = 0, & \text{in } \Omega_e, \\ \dfrac{\partial w}{\partial n} = 0, & \text{on } \Gamma_{0e} \cup \Gamma_{\alpha e}, \\ \dfrac{\partial w}{\partial n} = w_n, & \text{on } \Gamma_R, \\ w(x) = \mathcal{O}(1), & \text{as } |x| \to +\infty. \end{cases} \tag{2.10}$$

Then, there are the Poisson integral formula

$$\begin{aligned} w(r,\theta) = -\frac{1}{2\alpha}\Big(R^{\frac{2\pi}{\alpha}} - r^{\frac{2\pi}{\alpha}}\Big) \int_0^\alpha \Big[&\frac{1}{r^{\frac{2\pi}{\alpha}} + R^{\frac{2\pi}{\alpha}} - 2(Rr)^{\frac{\pi}{\alpha}} \cos\frac{\pi}{\alpha}(\theta-\theta')} \\ &+\frac{1}{r^{\frac{2\pi}{\alpha}} + R^{\frac{2\pi}{\alpha}} - 2(Rr)^{\frac{\pi}{\alpha}} \cos\frac{\pi}{\alpha}(\theta+\theta')}\Big] w(R,\theta')d\theta', \end{aligned} \tag{2.11}$$

with $0 < R < r$, and the natural integral equation

$$\frac{\partial w}{\partial n} = -\frac{\pi}{4\alpha^2 R}\int_0^\alpha \Big(\frac{1}{\sin^2\frac{\theta-\theta'}{2\alpha}\pi} + \frac{1}{\sin^2\frac{\theta+\theta'}{2\alpha}\pi}\Big) w(R,\theta')d\theta'. \tag{2.12}$$

Eqs. (2.11) and (2.12) can also be expressed in the following Fourier series forms

$$w(r,\theta) = \frac{1}{\alpha}\sum_{n=0}^{+\infty} \varepsilon_n \Big(\frac{R}{r}\Big)^{\frac{n\pi}{\alpha}} \int_0^\alpha w(R,\theta') \cos\frac{n\pi\theta}{\alpha} \cos\frac{n\pi\theta'}{\alpha} d\theta', \tag{2.13}$$

with $R < r$, and

$$\frac{\partial w}{\partial r} = -\frac{\pi}{R\alpha^2}\sum_{n=0}^{+\infty} \varepsilon_n n \int_0^\alpha w(R,\theta') \cos\frac{n\pi\theta}{\alpha} \cos\frac{n\pi\theta'}{\alpha} d\theta', \tag{2.14}$$

where ε_n refers to: when $n = 0$, $\varepsilon_n = 1$; when $n > 0$, $\varepsilon_n = 2$. Equation (2.14) can also be changed to the following equivalent form

$$\frac{\partial w}{\partial r} = \frac{1}{R\alpha}\sum_{n=0}^{+\infty} \varepsilon_n \int_0^\alpha \frac{\partial w(R,\theta')}{\partial \theta'} \cos\frac{n\pi\theta}{\alpha} \sin\frac{n\pi\theta'}{\alpha} d\theta'. \tag{2.15}$$

From (1.9), we have

$$\frac{\partial w}{\partial n} = a_0(u)\frac{\partial u}{\partial n}. \tag{2.16}$$

Combining (2.14) and (2.16) and $\frac{\partial w}{\partial n} = -\frac{\partial w}{\partial r}$, we obtain the exact artificial boundary condition of u on Γ_R,

$$a_0(u)\frac{\partial u}{\partial n} = \frac{\pi}{R\alpha^2}\sum_{n=0}^{+\infty}\varepsilon_n n\int_0^\alpha\Big(\int_0^{u(R,\theta')}a_0(y)dy\Big)\cos\frac{n\pi\theta}{\alpha}\cos\frac{n\pi\theta'}{\alpha}d\theta' \triangleq \mathscr{K}_1(u(R,\theta)). \tag{2.17}$$

Then by (1.1), (1.2), (2.17), the original problem confines in Ω_i can be defined as follows

$$\begin{cases} -\nabla\cdot(a(x,u)\nabla u) = f, & \text{in } \Omega_i,\\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{0i}\cup\Gamma_{\alpha i},\\ u = 0, & \text{on } \Gamma,\\ a_0(u)\dfrac{\partial u}{\partial n} = \mathscr{K}_1(u(R,\theta)), & \text{on } \Gamma_R, \end{cases} \tag{2.18}$$

with $\Gamma_{0i} = \Gamma_0\cap\overline{\Omega}_i$ and $\Gamma_{\alpha i} = \Gamma_\alpha\cap\overline{\Omega}_i$.

Therefore, the solution of problem (2.18) is the restriction of the solution of problem (1.1) and (1.2) in the bounded domain Ω_i.

3. Finite Element Approximation

3.1. The equivalent variational problems

Now we consider the problems (2.9) and (2.18). We shall use $W^{m,p}$ denoting the standard Sobolev spaces. $\|\cdot\|$ and $|\cdot|$ refer to the corresponding norm and semi-norm, respectively. Especially, we define $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_{m,\Omega} = \|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,\Omega} = |\cdot|_{m,2,\Omega}$. Let us introduce the space

$$V = \big\{v\in H^1(\Omega_i)\mid v|_\Gamma = 0\big\}, \tag{3.1}$$

and the corresponding norms

$$\|v\|_{0,\Omega_i}^2 = \int_{\Omega_i}|v|^2dx,\ \ \|v\|_{1,\Omega_i}^2 = \int_{\Omega_i}(|v|^2+|\nabla v|^2)dx.$$

The boundary value problems (2.9) and (2.18) are equivalent to the following variational problem: Find $u\in V$, such that

$$D(u;u,v) + \widehat{D}(u;u,v) = F(v),\ \ \forall\, v\in V, \tag{3.2}$$

with

$$D(w;u,v) = \int_{\Omega_i}a(x,w)\nabla u\nabla v dx, \tag{3.3}$$

$$\widehat{D}(w;u,v) = \frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{\varepsilon_n}{n}\int_0^\alpha\!\!\int_0^\alpha a_0(w(R,\theta'))\frac{\partial u(R,\theta')}{\partial\theta'}\frac{\partial v(R,\theta)}{\partial\theta}\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha}d\theta' d\theta, \tag{3.4}$$

$$F(v) = \int_{\Omega_i}f(x)v(x)dx. \tag{3.5}$$

Following [4] and [20], for any real number s, we have the equivalent definition of Sobolev spaces $H^s(\Gamma_R)$ as follows

$$\forall\ f \in H^s(\Gamma_R) \Leftrightarrow f(R,\theta) = \sum_{n=-\infty}^{+\infty} \Big(e^{i\frac{n\pi}{\alpha}\theta} + e^{-i\frac{n\pi}{\alpha}\theta}\Big) f_n, \quad \sum_{n=-\infty}^{+\infty} \Big[1 + (\frac{n\pi}{\alpha})^2\Big]^s \cdot |f_n|^2 < \infty.$$

with $f_n = \dfrac{1}{2\alpha}\displaystyle\int_0^\alpha \Big(e^{-i\frac{n\pi}{\alpha}\theta} + e^{i\frac{n\pi}{\alpha}\theta}\Big) f(R,\theta)d\theta$. The norm of $H^s(\Gamma_R)$ can be defined as follows

$$\|f(R,\theta)\|^2_{s,\Gamma} \triangleq \sum_{n=-\infty}^{+\infty} \Big[1 + (\frac{n\pi}{\alpha})^2\Big]^s \cdot |f_n|^2.$$

Particularly, when $s = 0$, we have

$$\|f(R,\theta)\|_{0,\Gamma} \triangleq \Big[\sum_{n=-\infty}^{+\infty} |f_n|^2\Big]^{\frac{1}{2}} = \|f(R,\theta)\|_{L^2(\Gamma_R)}.$$

Similar with [19], we have the following result

Lemma 3.1. *There exists a positive constant $C > 0$ which has different meaning in different place, such that*

$$|\widehat{D}(w;u,v)| \le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},\ \ \widehat{D}(u;u,u) \ge C_0|u|^2_{1,\Omega_i},\ \ \forall\ u,v,w \in V.$$

Proof. For $u, v \in V$, we assume that

$$u(R,\theta) = \sum_{n=-\infty}^{+\infty} u_n\Big(e^{i\frac{n\pi}{\alpha}\theta} + e^{-i\frac{n\pi}{\alpha}\theta}\Big), \quad v(R,\theta) = \sum_{n=-\infty}^{+\infty} v_n\Big(e^{i\frac{n\pi}{\alpha}\theta} + e^{-i\frac{n\pi}{\alpha}\theta}\Big).$$

Then we have

$$\begin{aligned}
\frac{\partial u}{\partial\theta}(R,\theta) &= \sum_{n=-\infty}^{+\infty} \frac{in\pi}{\alpha} u_n\Big(e^{i\frac{n\pi}{\alpha}\theta} - e^{-i\frac{n\pi}{\alpha}\theta}\Big),\\
\frac{\partial v}{\partial\theta}(R,\theta) &= \sum_{n=-\infty}^{+\infty} \frac{in\pi}{\alpha} v_n\Big(e^{i\frac{n\pi}{\alpha}\theta} - e^{-i\frac{n\pi}{\alpha}\theta}\Big).
\end{aligned}$$

By property (1.3), Cauchy inequality and the trace theorem, we have

$$\begin{aligned}
|\widehat{D}(w;u,v)| &\le C\Big(\sum_{n=-\infty}^{+\infty} \frac{n\pi^2}{\alpha^2}\cdot|u_n|^2\Big)^{\frac{1}{2}}\Big(\sum_{n=-\infty}^{+\infty} \frac{n\pi^2}{\alpha^2}\cdot|v_n|^2\Big)^{\frac{1}{2}}\\
&\le C\|u\|_{1/2,\Gamma_R}\|v\|_{1/2,\Gamma_R} \le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i}, \quad \forall\ u,v \in V.
\end{aligned}$$

Next, we show that $\widehat{D}(u;u,u) \ge C_0|u|^2_{1,\Omega_i}$, for any $u \in V$. Firstly, for problem (2.9), for any given $v \in V$, let us consider the following auxiliary problem

$$\begin{cases}
-\nabla\cdot(a(x,u)\nabla u) = 0, & \text{in } \Omega_i \cap \Omega,\\
\dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma_{0i} \cup \Gamma_{\alpha i},\\
u = 0, & \text{on } \Gamma,\\
u = v, & \text{on } \Gamma_R,
\end{cases} \tag{3.6}$$

with $\Gamma_{0i}=\Gamma_0\cap\overline{\Omega}_i$, $\Gamma_{\alpha i}=\Gamma_\alpha\cap\overline{\Omega}_i$. From the analysis in Section 2.1, we know that the solution u of the above problem (3.6) satisfies

$$a_0(u)\frac{\partial u}{\partial n}=\mathscr{K}_1(u(R,\theta)).$$

We multiply (3.6) by u and integrate over $\Omega_i\cap\Omega$, we have

$$\widehat{D}(u;u,u)=\int_{\Omega_i}a_0(u)|\nabla u|^2dx\ge C_0|u|^2_{1,\Omega_i}.$$

Secondly, for problem (2.18), for any given $v\in V$, we consider the following auxiliary problem

$$\begin{cases}-\nabla\cdot(a(x,u)\nabla u)=0, & \text{in } \Omega_i\cap\Omega^c,\\ \dfrac{\partial u}{\partial n}=0, & \text{on } \Gamma_{0i}\cup\Gamma_{\alpha i},\\ u=v, & \text{on } \Gamma_R,\\ u(x)=\mathcal{O}(1), & \text{as } |x|\to+\infty,\end{cases}\tag{3.7}$$

where Γ_{0i}, $\Gamma_{\alpha i}$ are denoted by problem (2.18). From the analysis in Section 2.2, we know that the solution u of the above problem (3.7) satisfies

$$a_0(u)\frac{\partial u}{\partial n}=\mathscr{K}_1(u(R,\theta)).$$

We multiply (3.7) by u and integrate over $\Omega_i\cap\Omega$, then we can obtain the desired result. This completes the proof. □

In practice, we need to truncate the series in (2.8) and (2.17) for some nonnegative integer N, i.e.,

$$\Big(a_0(u)\frac{\partial u}{\partial n}\Big)\Big|_{\Gamma_R}=\mathscr{K}_1^N(u),\tag{3.8}$$

with

$$\mathscr{K}_1^N(u)=\frac{\pi}{R\alpha^2}\sum_{n=0}^N\varepsilon_n n\int_0^\alpha\Big(\int_0^{u(R,\theta')}a_0(y)dy\Big)\cos\frac{n\pi\theta}{\alpha}\cos\frac{n\pi\theta'}{\alpha}d\theta'.\tag{3.9}$$

That is, we shall use the summation of the first N terms in (2.8) and (2.17). Now we begin to consider the approximate problems of (2.9) and (2.18), respectively

$$\begin{cases}-\nabla\cdot(a(x,u^N)\nabla u^N)=0, & \text{in } \Omega_i\cap\Omega \text{ or } \Omega_i\cap\Omega^c,\\ \dfrac{\partial u^N}{\partial n}=0, & \text{on } \Gamma_{0i}\cup\Gamma_{\alpha i},\\ u^N=0, & \text{on } \Gamma,\\ a_0(u^N)\dfrac{\partial u^N}{\partial n}=\mathscr{K}_1^N(u^N), & \text{on } \Gamma_R,\end{cases}\tag{3.10}$$

where Γ_{0i}, $\Gamma_{\alpha i}$ are defined as (2.9) and (2.18). And problem (3.10) also has the following equivalent variational problem: Find $u^N\in V$, such that

$$D(u^N;u^N,v)+\widehat{D}_N(u^N;u^N,v)=F(v),\ \forall\, v\in V,\tag{3.11}$$

with

$$\widehat{D}_N(w;u,v)=\frac{1}{\pi}\sum_{n=0}^N\frac{\varepsilon_n}{n}\int_0^\alpha\int_0^\alpha a_0(w(R,\theta'))\frac{\partial u(R,\theta')}{\partial\theta'}\frac{\partial v(R,\theta)}{\partial\theta}\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha}d\theta'd\theta.\tag{3.12}$$

Similar with Lemma 3.1, we have

Lemma 3.2. *There exists a positive constant C, such that*

$$|\widehat{D}_N(w;u,v)| \le C\|u\|_{1,\Omega_i}\|v\|_{1,\Omega_i},\ \widehat{D}_N(u;u,u) \ge C_0|u|^2_{1,\Omega_i},\ \forall\ u,v,w \in V.$$

3.2. Finite element approximation

Divide the arc Γ_R into N_1 parts and take a finite element subdivision in Ω_i such that their nodes on Γ_R are coincident. That is, we make a regular and quasi-uniform triangulation $\mathscr{T}_h$ on Ω_i, such that

$$\Omega_i = \bigcup_{K\in\mathscr{T}_h} K, \tag{3.13}$$

where K is a (curved) triangle and h is the maximal diameter of the triangles. Let

$$V_h = \Big\{v_h \in V \mid v|_K \text{ is a linear polynomial, } \forall\ K \in \mathscr{T}_h\Big\}. \tag{3.14}$$

Then the approximate problem of (3.11) can be written as

$$\begin{cases} \text{Find } u_h^N \in V_h, \text{ such that} \\ D(u_h^N;u_h^N,v_h) + \widehat{D}_N(u_h^N;u_h^N,v_h) = F(v_h),\ \forall\ v_h \in V_h. \end{cases} \tag{3.15}$$

Similar with Proposition 6.1 in [20] and existence and uniqueness in [10], we have

Lemma 3.3. *The variational problems* (3.2), (3.11) *and* (3.15) *are uniquely solvable.*

3.2.1. Convergence Theorems

In this section, we obtain the convergence result of the problems discussed above. We let $u, u^N \in H^2(\Omega_i)$ and $u_h^N \in V_h$ be the solution of problems (3.2), (3.11) and (3.15), respectively. We also assume that

$$V_h \subset V \cap W^{1,2+\varepsilon} \text{ for some } \varepsilon \in (0,1). \tag{3.16}$$

And we require that $\{V_h\}_{h\to 0}$ is a family of finite-dimensional subspaces of $V \cap C(\Omega_i)$, which satisfies for any

$$v \in V \cap C(\Omega_i),\ \text{there exists } \{v_h\} : v_h \in V_h,\ \|v - v_h\|_{1,\Omega_i} \to 0,\ \text{as } h \to 0, \tag{3.17}$$

$$\|v_h\|_{1,2+\varepsilon,\Omega_i} \le C(v) \text{ for any } h, \tag{3.18}$$

where $C(v) > 0$ is independent of h.

Remark 3.1. The continuous piecewise polynomial spaces, such as (3.14), satisfy the condition (3.16). And if we let $v_h = \Pi_h v$, where $\Pi_h : V \to V_h$ is the interpolation operator, then by (3.18), we have

$$\|v_h\|_{1,2+\varepsilon,\Omega_i} \le \|\Pi_h v - v\|_{1,2+\varepsilon,\Omega_i} + \|v\|_{1,2+\varepsilon,\Omega_i} \le C(v).$$

Following the convergence theory in [10, 20], we have the following result

$$\lim_{h\to 0}\|u_h^N - u^N\|_{1,\Omega_i} = 0 \text{ and } u^N \in V \cap W^{1,2+\varepsilon},\ \forall\ N \ge 0. \tag{3.19}$$

Moreover, we can obtain the following result.

Lemma 3.4. *Let u^N be the solution of* (3.11) *and u be the solution of* (3.2)*. Then we have*

$$\lim_{N\to\infty}\|u-u^N\|_{1,\Omega_i}=0. \tag{3.20}$$

Proof. From (1.3), (3.11) and Lemma 3.2, we have

$$\begin{aligned}\|u^N\|^2_{1,\Omega_i}&\le C\big[D(u^N;u^N,u^N)+\widehat{D}(u^N;u^N,u^N)\big]\\&=C\big[F(u^N)+\widehat{D}(u^N;u^N,u^N)-\widehat{D}_N(u^N;u^N,u^N)\big]\\&\le C\big[\|f\|_{0,\Omega_i}\cdot\|u^N\|_{1,\Omega_i}+|\widehat{D}(u^N;u^N,u^N)-\widehat{D}_N(u^N;u^N,u^N)|\big].\end{aligned}$$

For $u^N\in V$, we assume that

$$\begin{aligned}&w^N(r,\theta')=\int_0^{w^N(r,\theta')}a_0(y)dy=\sum_{n=0}^{+\infty}w_n\Big(\frac{R_0}{r}\Big)^{\frac{n\pi}{\alpha}}\cos\frac{n\pi\theta'}{\alpha},\quad\forall\ r\ge R_0,\\&u^N(R,\theta)=\sum_{n=0}^{+\infty}u_n\cos\frac{n\pi\theta}{\alpha}.\end{aligned}$$

Then by

$$\int_0^\alpha\sin\frac{n\pi\theta}{\alpha}\sin\frac{m\pi\theta}{\alpha}d\theta=\begin{cases}\dfrac{\alpha}{2},& m=n\ne0,\\-\dfrac{\alpha}{2},& m=-n\ne0,\\0,&\text{otherwise.}\end{cases}$$

We have the following result

$$\begin{aligned}&|\widehat{D}(u^N;u^N,u^N)-\widehat{D}_N(u^N;u^N,u^N)|\\&=\Big|\frac{1}{\pi}\sum_{n=N+1}^{+\infty}\frac{\varepsilon_n}{n}\int_0^\alpha\!\!\int_0^\alpha\frac{\partial w^N(R,\theta')}{\partial\theta'}\frac{\partial u^N(R,\theta)}{\partial\theta}\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha}d\theta'd\theta\Big|\\&=\Big|\sum_{n=N+1}^{+\infty}\frac{n\pi\varepsilon_n}{4}\Big(\frac{R_0}{R}\Big)^{\frac{n\pi}{\alpha}}w_nu_n\Big|\\&\le C\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\Big[\sum_{n=N+1}^{+\infty}\Big(1+(\frac{n\pi}{\alpha})^2\Big)^{\frac12}|w_n|^2\Big]^{\frac12}\Big[\sum_{n=N+1}^{+\infty}\Big(1+(\frac{n\pi}{\alpha})^2\Big)^{\frac12}|u_n|^2\Big]^{\frac12}\\&\le C\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\|w^N\|_{\frac12,\Gamma_{R_0}}\|u^N\|_{\frac12,\Gamma_R}\\&\le C\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\|u^N\|_{1,\Omega_i}.\end{aligned}$$

From $R\ge R_0$, we obtain that $\{u^N\}$ is bounded in V. Therefore, there exists a subsequence $\{u^{N_n}\}$ such that $u^{N_n}\rightharpoonup\overline{u}\in V$. Then similar with the proof of Lemma 3.4 of [8], we obtain (3.20). □

By the above lemmas, we get the following convergence result.

Theorem 3.1. *Let $u\in H^2(\Omega_i)$, and the assumptions* (3.16)–(3.18) *be satisfied. Then we have*

$$\lim_{h\to0,N\to\infty}\|u-u_h^N\|_{1,\Omega_i}=0. \tag{3.21}$$

Remark 3.2. Noticing that, the above convergence theorem is obtained for the unbounded domain with a concave angle. Without any difficulty we can obtain the similar convergence result for the bounded domain with a concave angle.

3.2.2. Error Analysis

In the following, we shall get error estimates for the approximate solution obtained from the FEM-NBEM discrete scheme. We assume that the solution u of problem (1.1) satisfies

$$u|_{\Omega_i} \in V \cap W^{k,2+\varepsilon}(\Omega_i), \ \varepsilon > 0, \ k \geq 2.$$

For simplicity, let us define the following notation

$$\begin{aligned}
&\overline{A}(u; u, v) \triangleq D(u; u, v) + \widehat{D}(u; u, v),\\
&\overline{A}_N(u^N; u^N, v) \triangleq D(u^N; u^N, v) + \widehat{D}_N(u^N; u^N, v),\\
&\overline{A}_N(u_h^N; u_h^N, v) \triangleq D(u_h^N; u_h^N, v_h) + \widehat{D}_N(u_h^N; u_h^N, v_h).
\end{aligned}$$

Then problems (3.2), (3.11) and (3.15) can be replaced by the corresponding simple forms, respectively. Now we introduce the bilinear form $A'(u; \cdot, \cdot)$ and $A'_N(u^N; \cdot, \cdot)$ defined by

$$\begin{aligned}
A'(u; v, z) = &\int_{\Omega_i} \frac{\partial a}{\partial s}(x, u) v \nabla u \cdot \nabla z dx + \int_{\Omega_i} a(x, u) \nabla v \cdot \nabla z dx\\
&+ \int_0^\alpha\!\!\int_0^\alpha \frac{\partial a_0}{\partial s}(x, u) v \frac{\partial u}{\partial \theta'}(R, \theta') \frac{\partial z}{\partial \theta}(R, \theta) \sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n\pi} \sin \frac{n\pi\theta'}{\alpha} \sin \frac{n\pi\theta}{\alpha} d\theta' d\theta\\
&+ \int_0^\alpha\!\!\int_0^\alpha a_0(x, u) \frac{\partial v}{\partial \theta'}(R, \theta') \frac{\partial z}{\partial \theta}(R, \theta) \sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n\pi} \sin \frac{n\pi\theta'}{\alpha} \sin \frac{n\pi\theta}{\alpha} d\theta' d\theta,\\
A'_N(u^N; v, z) = &\int_{\Omega_i} \frac{\partial a}{\partial s}(x, u^N) v \nabla u^N \cdot \nabla z dx + \int_{\Omega_i} a(x, u^N) \nabla v \cdot \nabla z dx\\
&+ \int_0^\alpha\!\!\int_0^\alpha \frac{\partial a_0}{\partial s}(x, u^N) v \frac{\partial u^N}{\partial \theta'}(R, \theta') \frac{\partial z}{\partial \theta}(R, \theta) \sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n\pi} \sin \frac{n\pi\theta'}{\alpha} \sin \frac{n\pi\theta}{\alpha} d\theta' d\theta\\
&+ \int_0^\alpha\!\!\int_0^\alpha a_0(x, u^N) \frac{\partial v}{\partial \theta'}(R, \theta') \frac{\partial z}{\partial \theta}(R, \theta) \sum_{n=1}^{+\infty} \frac{\varepsilon_n}{n\pi} \sin \frac{n\pi\theta'}{\alpha} \sin \frac{n\pi\theta}{\alpha} d\theta' d\theta.
\end{aligned}$$

Let V' be the dual space of V. By (1.3) and continuity of $\frac{\partial a}{\partial s}(\cdot, u(\cdot))$, we obtain that $A'(u; \cdot, \cdot)$ is bounded in Ω_i. Then there exists an operator $T: \ V \to V'$ such that

$$(Tv, z) = A'(u; v, z), \ \forall \ v, z \in V. \tag{3.22}$$

Similar with the proof of [14], we have the lemma as follows

Lemma 3.5. *The bilinear form (Tv, v) defined by $A'(u; v, v)$ satisfies the following inequality*

$$(Tv, z) + K(\|v\|^2_{0,\Omega_i} + \|v\|^2_{1/2,\Gamma_R}) \geq C\|v\|^2_{1,\Omega_i}, \ \forall \ v \in V, \tag{3.23}$$

where $K \geq 0$ is a sufficient large constant.

We assume that

$$A'(u; v, z) = 0, \quad \forall \ z \in V \Longrightarrow v = 0. \tag{3.24}$$

Let $I: \ V \to V'$ be the canonical injection. Since V is compactly embedded in $L^2(\Omega_i)$, we have that the operator $J: \ V \to V'$ defined by $J(v) = (I(v), 0)$ is also compact. By (3.22) and (3.24) and T satisfies the property of J, we obtain that $T: \ V \to V'$ is an isomorphism.

By the conditions (3.2), (3.23), (3.24) and Theorem 10.1.2 of [1], one can get that there exists $h_0 \in (0,1]$, such that the following inequality is satisfied

$$\sup_{x\in V_h} \frac{A'(u;v,z)}{\|z\|_{1,\Omega_i}} \geq \alpha_1 \|v\|_{1,\Omega_i}, \ \forall\ v \in V_h, \tag{3.25}$$

for some constant α_1 independent of h ($h < h_0$).

We define the Galerkin projection with respect to $A'(u;\cdot,\cdot)$, $P_h:\ V \to V_h$

$$A'(u; P_h v, z) = A'(u; v, z), \quad \forall\ z \in V_h.$$

Then the operator P_h satisfies

$$\|v - P_h v\|_{1,p,\Omega_i} \leq C \inf_{v_h \in V_h} \|v - v_h\|_{1,p,\Omega_i} \leq Ch^\sigma, \quad 2 \leq p \leq \infty,\ 0 < \sigma < 1. \tag{3.26}$$

We define the set

$$B_h \triangleq \Big\{ v \in V_h \mid \|v - P_h v\|_{1,\infty,\Omega_i} \leq Ch^\sigma \Big\}. \tag{3.27}$$

Lemma 3.6. *$u_h^N \in V_h$ is a solution of (3.15) if and only if the following equation*

$$A'_N(u^N; u^N - u_h^N, v) = R(u^N; u^N, v), \quad \forall\ v \in V_h,$$

holds, where

$$\begin{aligned}
&R(u^N;u^N,v)\\
\triangleq& \int_{\Omega_i}\Big(\int_0^1 \Big[\frac{\partial^2 a}{\partial s^2}(x,w_h^N)\nabla w_h^N \nabla v\Big](1-t)dt\Big)(d_h^N)^2 dx\\
&+2\int_{\Omega_i}\Big(\int_0^1 \Big[\frac{\partial a}{\partial s}(x,w_h^N)\nabla d_h^N \nabla v\Big](1-t)dt\Big)d_h^N dx\\
&+\int_0^\alpha\int_0^\alpha\Big(\int_0^1 \Big[\frac{\partial^2 a_0}{\partial s^2}(x,w_h^N)\frac{\partial w_h^N}{\partial\theta'}\frac{\partial v}{\partial\theta}\sum_{n=1}^N \frac{\varepsilon_n}{n\pi}\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha}\Big](1-t)dt\Big)(d_h^N)^2 d\theta' d\theta\\
&+2\int_0^\alpha\int_0^\alpha\Big(\int_0^1 \Big[\frac{\partial a_0}{\partial s}(x,w_h^N)\frac{\partial d_h^N}{\partial\theta'}\frac{\partial v}{\partial\theta}\sum_{n=1}^N \frac{\varepsilon_n}{n\pi}\sin\frac{n\pi\theta'}{\alpha}\sin\frac{n\pi\theta}{\alpha}\Big](1-t)dt\Big)d_h^N d\theta' d\theta,
\end{aligned}$$

with $w_h^N = u^N + t(u_h^N - u^N)$, $d_h^N = u_h^N - u^N$.

Proof. Let $\eta(t) \triangleq \overline{A}_N(w_h^N; w_h^N, v)$. Then by

$$\begin{aligned}
&\eta(1) = \eta(0) + \eta'(0) + \int_0^1 \eta''(t)(1-t)dt,\\
&\overline{A}_N(u^N;u^N,v) = \overline{A}_N(u_h^N;u_h^N,v) = F(v), \quad \forall\ v \in V_h,
\end{aligned}$$

we can get the desired result. □

Let

$$M_h \triangleq \Big\{ v \in V_h \mid \|v\|_{1,\infty,\Omega_i} \leq 1 + \|u^N\|_{1,\infty,\Omega_i} \Big\}. \tag{3.28}$$

Then following [14] and [16], we have

Lemma 3.7. *There exists a positive constant C independent of h, such that*

$$|R(u^N;v,z)|\le C\Big(\|u^N-v\|_{1,\Omega_i}^2+\|u^N-v\|_{1,\Omega_i}\Big)\|z\|_{1,\Omega_i},\ \forall\ v\in M_h,\quad \forall\ z\in V_h.$$

We also have the following result

Lemma 3.8. *Let B_h and M_h be defined by* (3.27) *and* (3.28)*, respectively. Then $B_h\subset M_h$.*

Proof. For any $v\in B_h$, we only need to show that $v\in M_h$.

$$\|v\|_{1,\infty,\Omega_i}\le\|u^N-v\|_{1,\infty,\Omega_i}+\|u^N\|_{1,\infty,\Omega_i}, \tag{3.29a}$$
$$\|u^N-v\|_{1,\infty,\Omega_i}\le\|u^N-P_hu^N\|_{1,\infty,\Omega_i}+\|P_hu^N-v\|_{1,\infty,\Omega_i}, \tag{3.29b}$$
$$\|u^N-P_hu^N\|_{1,\infty,\Omega_i}\le\|u^N-\Pi_hu^N\|_{1,\infty,\Omega_i}+\|\Pi_hu^N-P_hu^N\|_{1,\infty,\Omega_i}. \tag{3.29c}$$

Since $\mathscr{T}_h$ is regular and quasi-uniform, referring to [17], we obtain the following inverse inequality

$$\|w\|_{1,\infty,\Omega_i}\le C\Big(\log\frac{1}{h}\Big)^{\frac12}\|w\|_{1,\Omega_i},\quad \forall\ w\in V_h. \tag{3.30}$$

Combining the above inequalities with the definition of B_h and (3.26), we obtain

$$\|u^N-v\|_{1,\infty,\Omega_i}\le 1.$$

By the definition of M_h, we get the desired result. □

Theorem 3.2. *Assume $u\in V\cap W^{k,2+\varepsilon}(\Omega_i)$ be the solution of* (1.1)*, with $\varepsilon>0$, $k\ge 2$. And we also assume that $u|_{\Gamma_{R_0}}\in H^{k-1/2}(\Gamma_{R_0})$ and u satisfies* (3.24)*. With sufficiently small h, the finite element equation* (3.15) *has the approximate solution $u_h^N\in V_h$ such that*

$$\|u-u_h^N\|_{1,\Omega_i}\le C\Big[h^\sigma+\frac{1}{(N+1)^{k-1}}\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\|u\|_{k-1/2,\Gamma_{R_0}}\Big]. \tag{3.31}$$

Proof. Firstly, for any $u^N\in V$, from (3.20), we have

$$\begin{aligned}
&|\widehat{D}(u^N;u^N,v)-\widehat{D}_N(u^N;u^N,v)|\\
&\le C\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\Big[\sum_{n=N+1}^{+\infty}\Big(1+\Big(\frac{n\pi}{\alpha}\Big)^2\Big)^{\frac12}|w_n|^2\Big]^{\frac12}\Big[\sum_{n=N+1}^{+\infty}\Big(1+\Big(\frac{n\pi}{\alpha}\Big)^2\Big)^{\frac12}|v_n|^2\Big]^{\frac12}\\
&\le\frac{C}{(N+1)^{k-1}}\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\Big[\sum_{n=N+1}^{+\infty}\Big(1+\Big(\frac{n\pi}{\alpha}\Big)^2\Big)^{k-\frac12}|w_n|^2\Big]^{\frac12}\Big[\sum_{n=N+1}^{+\infty}\Big(1+\Big(\frac{n\pi}{\alpha}\Big)^2\Big)^{\frac12}|v_n|^2\Big]^{\frac12}\\
&\le\frac{C}{(N+1)^{k-1}}\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\|u\|_{k-\frac12,\Gamma_{R_0}}\|v\|_{1,\Omega_i}.
\end{aligned}$$

Then by (3.11), we have

$$\overline{A}(u^N;u^N,v)=D(u^N;u^N,v)+\widehat{D}(u^N;u^N,v)=F(v)+\widehat{D}(u^N;u^N,v)-\widehat{D}_N(u^N;u^N,v).$$

Let $\eta(t)=\overline{A}(u+t(u^N-u),v)$. We have

$$\int_0^1A'(u+t(u^N-u);u^N-u,v)dt=\overline{A}(u^N;u^N,v)-\overline{A}(u;u,v).$$

From (3.2), (3.23), (3.24) and [1], we obtain

$$\begin{aligned}\|u-u^N\|_{1,\Omega_i} &\le C\sup_{v\in V}\Big(\frac{1}{\|v\|_{1,\Omega_i}}\int_0^1 A'(u+t(u^N-u);u^N-u,v)dt\Big)\\ &\le C\frac{|\widehat{D}(u^N;u^N,v)-\widehat{D}_N(u^N;u^N,v)|}{\|v\|_{1,\Omega_i}}\\ &\le \frac{C}{(N+1)^{k-1}}\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\|u\|_{k-\frac{1}{2},\Gamma_{R_0}}. \end{aligned}\tag{3.32}$$

We denote a nonlinear mapping $\phi:\ V_h\to V_h$, which satisfies that for any given $v\in V_h$, $\phi(v)$ is the unique solution of

$$A'(u,\phi(v),z)=A'(u,u,z)-R(u,v,z),\ \forall\ z\in V_h. \tag{3.33}$$

Therefore, we have

$$A'(u,\phi(v)-\phi(v_n),z)=R(u,v_n,z)-R(u,v,z).$$

Combining the above equation with (3.25), we obtain the operator ϕ is continuous, i.e.,

$$\lim_{v_n\to v}\phi(v_n)=\phi(v).$$

Next, we assume that $v\in B_h$, then by Lemma 3.8, we have that $v\in M_h$. By the definition of P_h, equation (3.33) can be rewritten as

$$A'(u^N,\phi(v)-P_hu^N,z)=-R(u^N,v,z),\ \forall\ z\in V_h.$$

Then, from (3.25), Lemma 3.6 and Lemma 3.7, we have

$$\begin{aligned}&\|\phi(v)-P_hu^N\|_{1,\Omega_i}\le C\sup_{z\in V_h}\frac{|A'(u,\phi(v)-P_hu^N,z)|}{\|z\|_{1,\Omega_i}}\\ &\le C\Big(\|u^N-v\|_{1,\Omega_i}^2+\|u^N-v\|_{1,\Omega_i}\Big)\\ &\le C\Big(\|u^N-P_hu^N\|_{1,\Omega_i}^2+\|P_hu^N-v\|_{1,\Omega_i}^2+\|u^N-P_hu^N\|_{1,\Omega_i}+\|P_hu^N-v\|_{1,\Omega_i}\Big)\le Ch^\sigma.\end{aligned}$$

This implies that $\phi:\ B_h\to B_h$. And since ϕ is also continuous, following from Brouwer's fixed theorem, one can obtain that there exists $u_h^N\in V_h$, such that $\phi(u_h^N)=u_h^N$. From Lemma 3.6, we deduce that u_h^N is the solution of (3.15). What's more, by (3.26) and the fact $u_h^N\in B_h$, we obtain

$$\|u^N-u_h^N\|_{1,\Omega_i}\le\|u^N-P_hu^N\|_{1,\Omega_i}+\|P_hu^N-u_h^N\|_{1,\Omega_i}\le Ch^\sigma,\quad 0<\sigma<1. \tag{3.34}$$

Combining (3.32) with (3.34), one can obtain

$$\begin{aligned}\|u-u_h^N\|_{1,\Omega_i}&\le\|u-u^N\|_{1,\Omega_i}+\|u^N-u_h^N\|_{1,\Omega_i}\\ &\le C\Big(\frac{1}{(N+1)^{k-1}}\Big(\frac{R_0}{R}\Big)^{\frac{(N+1)\pi}{\alpha}}\|u\|_{k-\frac{1}{2},\Gamma_{R_0}}+h^\sigma\Big).\end{aligned}$$

This completes the proof. □

Remark 3.3. The above conclusions are obtained for the unbounded domain with a concave angle. Without any difficulty we can obtain corresponding error analysis for the bounded domain problem. Therefore, we have the following results.

Theorem 3.3. *If the assumptions of Theorem 3.2 are satisfied, then with sufficiently small h, the finite element equation* (3.15) *has the approximate solution* $u_h^N \in V_h$ *such that*

$$\|u-u_h^N\|_{1,\Omega_i} \le C\Big[h^\sigma + \frac{1}{(N+1)^{k-1}}\Big(\frac{R}{R_0}\Big)^{\frac{(N+1)\pi}{\alpha}} \|u\|_{k-1/2,\Gamma_{R_0}}\Big]. \tag{3.35}$$

4. Numerical Examples

Since the problems discussed in the bounded or unbounded domains possess similar convergence results, we only need to give some examples for unbounded domains to confirm our theoretical results. In the following, we choose the finite element space as given in (3.14). For simplicity, we let

$$\Delta r = \frac{1}{m}, \quad \Delta\theta = \frac{\alpha}{M-1}, \quad e_0(h,N) = \|u-u_h^N\|_{L^2(\Omega_i)}, \quad e_\infty(h,N) = \|u-u_h^N\|_{L^\infty(\Omega_i)}.$$

Example 4.1. We take $\Omega^c = \big\{(x,y) \mid x,y\in\mathbb{R},\ r=\sqrt{x^2+y^2} > 1.5\big\}$ and its boundary $\partial\Omega = \Gamma_0\cup\Gamma_\alpha\cup\Gamma_R$, with $\Gamma_0 = \big\{(r,0) \mid r\ge 1.5\big\}$, $\Gamma_\alpha = \big\{(r,\frac{7\pi}{4}) \mid r \ge 1.5\big\}$ and $\Gamma_R = \big\{(3,\theta) \mid 0\le\theta\le\frac{7\pi}{4}\big\}$. We show our numerical results for problem (1.1), with

$$a(x,u) = \begin{cases} 9-r^2+\frac{1}{1+u^2}, & 1.5\le r\le 3,\\ \frac{1}{1+u^2}, & r>3,\end{cases} \tag{4.1}$$

$$f(x) = \begin{cases} 9-r^2, & 1.5\le r\le 3,\\ 0, & r>3.\end{cases} \tag{4.2}$$

The numerical results are given in Table 4.1.

Example 4.2. Similar with Example 4.1, $a(x,u)$ is replaced by

$$a(x,u) = \begin{cases} 9-r^2+\frac{1}{\sqrt{1-u^2}}, & 1.5\le r\le 3,\\ \frac{1}{\sqrt{1-u^2}}, & r>3.\end{cases} \tag{4.3}$$

The numerical results are given in Table 4.2.

Notice that, for Examples 4.1 and 4.2, we use $N=0$ to get the approximation results and the exact $'u'$ is solved with $N=0$ and $m=64, M=257$.

Table 4.1: The errors with $N=0$ for Example 4.1.

m	M	$e_0(h,N)$	ratio	$e_\infty(h,N)$	ratio
2	9	4.3198E-01	–	1.6711E-01	–
4	17	1.5342E-01	2.8157	6.3018E-02	2.6519
8	33	5.4614E-02	2.8091	2.4175E-02	2.6067
16	65	1.6398E-02	3.3306	7.5890E-03	3.1856

Table 4.2: The errors with $N=0$ for Example 4.2.

m	M	$e_0(h,N)$	ratio	$e_\infty(h,N)$	ratio
2	9	2.3333E-01	–	8.7970E-02	–
4	17	7.8699E-02	2.9649	3.0874E-02	2.8494
8	33	2.9048E-02	2.7092	1.2144E-02	2.5423
16	65	1.1956E-02	2.4296	5.3168E-03	2.2841

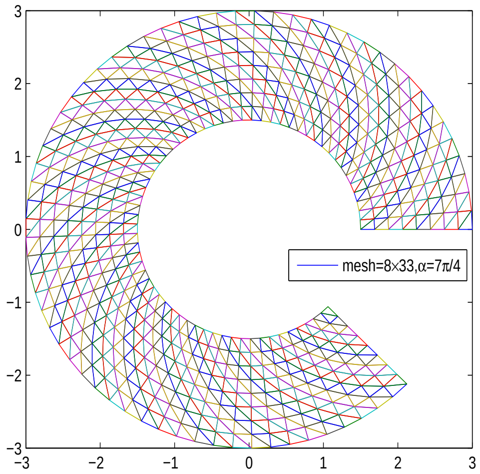

Fig. 4.1. Mesh=8×33 for Examples 4.1 and 4.2.

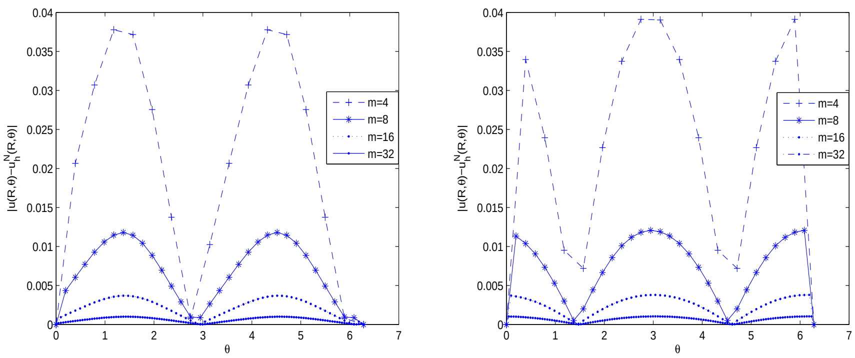

Fig. 4.2. The errors on the artificial boundary with different mesh sizes. Left: Example 4.3 with $N = 100$. Right: Example 4.4 with $N = 5$.

Example 4.3. We take $\Omega^c = \big\{(x,y) \mid x,y \in \mathbb{R}, r = \sqrt{x^2+y^2} > 1.5\big\}$ and its boundary $\partial\Omega = \Gamma_0 \cup \Gamma_\alpha \cup \Gamma_R$, with $\Gamma_0 = \big\{(r,0) \mid r \geq 1.5\big\}$, $\Gamma_\alpha = \big\{(r,2\pi) \mid r \geq 1.5\big\}$ and $\Gamma_R = \big\{(3,\theta) \mid 0 \leq \theta \leq 2\pi\big\}$. We show our numerical results for problem (1.1), with

$$a(x,u) = \begin{cases} 9 - r^2 + \frac{1}{1+u^2}, & 1.5 \leq r \leq 3, \\ \frac{1}{1+u^2}, & r > 3, \end{cases} \tag{4.4}$$

$$f(x) = \begin{cases} -2\Big(1+\tan^2\left(\frac{y}{r^2}\right)\Big)\Big(\frac{9-r^2}{r^4}\tan\left(\frac{y}{r^2}\right) + \frac{y}{r^2}\Big), & 1.5 \leq r \leq 3, \\ 0, & r > 3. \end{cases} \tag{4.5}$$

The exact solution of Example 4.3 is $u(x) = \tan(y/r^2)$. The numerical results are given in Fig. 4.2, Fig. 4.3 and Table 4.3.

Example 4.4. We take $\Omega^c = \big\{(x,y) \mid x,y \in \mathbb{R}, r = \sqrt{x^2+y^2} > 1.5\big\}$ and its boundary $\partial\Omega = \Gamma_0 \cup \Gamma_\alpha \cup \Gamma_R$, with $\Gamma_0 = \big\{(r,0) \mid r \geq 1.5\big\}$, $\Gamma_\alpha = \big\{(r,2\pi) \mid r \geq 1.5\big\}$ and $\Gamma_R =$

Table 4.3: The errors with $N = 100$ for Example 4.3.

m	M	$e_0(h,N)$	ratio	$e_\infty(h,N)$	ratio
2	9	2.3042E-01	–	1.2804E-01	–
4	17	5.8681E-02	3.9275	3.7770E-02	3.3902
8	33	1.6166E-02	3.6299	1.1791E-02	3.2034
16	65	4.6622E-03	3.4675	3.6876E-03	3.1974
32	129	1.1925E-03	3.9095	1.0023E-03	3.6793

Table 4.4: The errors with $N = 5$ for Example 4.4.

m	M	$e_0(h,N)$	ratio	$e_\infty(h,N)$	ratio
2	9	3.9146E-01	–	1.3430E-01	–
4	17	9.9956E-02	3.9164	3.9109E-02	3.4341
8	33	2.6848E-02	3.7230	1.2063E-02	3.2420
16	65	7.4989E-03	3.5803	3.7906E-03	3.1824
32	129	1.8592E-03	4.0333	1.0416E-03	3.6391

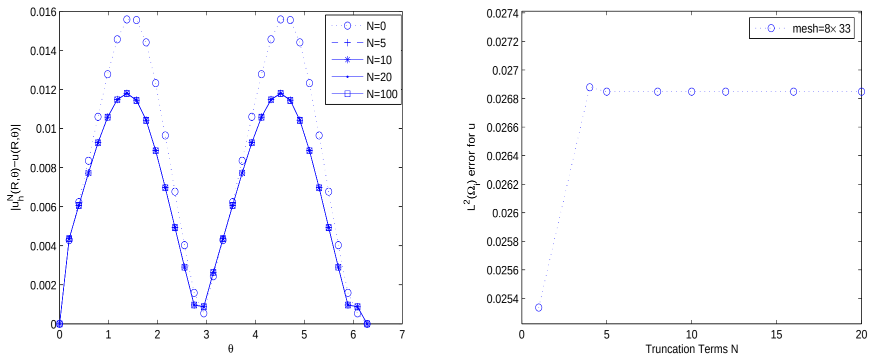

Fig. 4.3. Absolute errors and $L^2(\Omega_i)$ errors against N for Examples 4.3 and 4.4, respectively. Left: Example 4.3: the errors on the artificial boundary with different N. Here we let $(m, M) = (8, 33)$. Right: Example 4.4: $L^2(\Omega_i)$ errors for u with different N.

$\{(4,\theta) \mid 0 \le \theta \le 2\pi\}$. We present our numerical results for problem (1.1), with

$$a(x,u) = \begin{cases} 16 - r^2 + \frac{1}{\sqrt{1-u^2}}, & 1.5 \le r \le 4, \\ \frac{1}{\sqrt{1-u^2}}, & r > 4, \end{cases} \tag{4.6}$$

$$f(x) = \begin{cases} \frac{16-r^2}{r^4} \sin(\frac{x}{r^2}) - \frac{2x}{r^2} \cos(\frac{x}{r^2}), & 1.5 \le r \le 4, \\ 0, & r > 4. \end{cases} \tag{4.7}$$

The exact solution of Example 4.4 is $u(x) = \sin(x/r^2)$. The numerical results are given in Fig. 4.2, Fig. 4.3 and Table 4.4.

From the numerical results, one obtains that the numerical errors can be affected by the order of artificial boundary condition, the mesh of the domain and the location of the artificial boundary. And they can be reduced by increasing the order of the artificial boundary condition and refining the mesh. When a finer mesh cannot produce a much more accurate numerical solution, the errors originated from the series truncating is dominating. The numerical results are in agreement with the error analysis we obtain and show the efficiency of the coupling method.

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