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ON THE ERROR ESTIMATES OF A NEW OPERATE SPLITTING METHOD FOR THE NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, a new operator splitting scheme is introduced for the numerical solution of the incompressible Navier-Stokes equations. Under some mild regularity assumptions on the PDE solution, the stability of the scheme is presented, and error estimates for the velocity and the pressure of the proposed operator splitting scheme are given.

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1. Introduction

In this paper, we consider the numerical approximation of the unsteady Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \triangledown)\mathbf{u} + \triangledown p = \mathbf{f} \quad \text{in}\, \Omega \times [0,T], \\ \triangledown \cdot \mathbf{u} = 0 \quad \text{in}\, \Omega \times [0,T], \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^d with a sufficiently regular boundary $\partial\Omega$. $\mathbf{u}$, p are the velocity, pressure of the flow respectively, and $\nu = \frac{1}{Re}$ is the kinematic viscosity coefficient, Re is the Reynolds number.

For the well-posedness, the equations are supplemented with appropriate initial and boundary condition:

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 \quad \mathbf{x} \in \Omega\,, \mathbf{u}(\mathbf{x},t) = g(\mathbf{x},t) \quad (\mathbf{x},t) \in \partial\Omega \times [0,T]. \tag{2.2}$$

The difficulties for the numerical simulation of incompressible flows are mainly of two kinds: nonlinearity and incompressibility, the velocity and the pressure are coupled by the incompressibility constraint, which requires that the solution spaces satisfy the so called inf-sup condition. To overcome these difficulties, operator splitting methods and projection methods, which can be viewed as fractional step methods, are introduced. Fractional step methods allow to separate the effects of the different operators appearing in the equation by splitting each step into several sub-steps in order to reduce the cost of simulations.

The origin of this category of methods is due to the work of Chorin [1] and Temam [2], i.e., the so called projection method, in which the second step consists of the projection of

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an intermediate velocity field onto the space of solenoidal vector fields. The most attractive feature of projection methods is that, at each time step, one only needs to solve a sequence of decoupled elliptic equations for the velocity and the pressure, which makes the projection method very efficient for large scale numerical simulations. Guermond, Minev and Shen in [3] review theoretical and numerical convergence results available for projection methods. In[4-7], the analysis on first-order accurate schemes in time are presented. In [8,9], Shen derived a second-order error estimates for the projection method. However, several issues related to these methods still deserve further analysis, and perhaps the most important ones are the behavior of the computed pressure near boundaries and the stability of the pressure itself. The incompatibility of the projection boundary conditions may introduce a numerical boundary layer of size $O(\sqrt{\nu\Delta t})$ [10,11], where ν is the kinematic viscosity and Δt is the time step size. In addition, these methods have a main disadvantage that splitting error is inevitable unless the operators are commute.

In this paper, we will consider the non-stationary Navier-Stokes equations with Dirichlet boundary conditions and provide some error estimates for both velocity and pressure approximations by the operator splitting scheme. It is a two-step scheme, which allows to enforce the original boundary conditions of the problem in all substeps of the scheme[12].

The paper is organized as follows: In Section 2, we introduce some function and space notations and regularity assumptions for the PDE solution. In Section 3, we describe a new operator splitting method. In Section 4, the proof of the stability of the new method is given. In Section 5, we give an error analysis of this method. Error estimates for both velocity and pressure are obtained. Finally, numerical test results are presented in Section 6 to verify the theoretical results of Section 5.

2. Function Setting

In order to study approximation scheme for the problem (1.1). The following notations and assumptions are introduced.

we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm on $L^2(\Omega)$ or $L^2(\Omega)^d$. The space $H_0^1(\Omega)$ and $H_0^1(\Omega)^d$ are equipped with their usual norm, i.e.,

$$\|\mathbf{u}\|_1^2 = \int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x}.$$

The norm in $H^s(\Omega)$ will be denoted simply by $\|\cdot\|_s$. We will use $<\cdot,\cdot>$ to denote the duality between $H^{-s}(\Omega)$ and $H_0^s(\Omega)$ for all $s>0$.

The following subspace is also be introduced:

$$V = \Big\{\mathbf{u} \in H_0^1(\Omega)^d : \mathrm{div}\mathbf{u} = 0\Big\},$$
$$H = \Big\{\mathbf{u} \in L^2(\Omega)^d : \mathrm{div}\mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n} = 0)\Big\}.$$

For the treatment of the convective term, the following trilinear form is considered

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}d\mathbf{x}.$$

It is well known that $b(\cdot,\cdot,\cdot)$ is continuous in $H^{m_1}(\Omega)\times H^{m_2+1}(\Omega)\times H^{m_3}(\Omega)$, provided $m_1 + m_2 + m_3 \geq d/2$ if $m_i \neq d/2$, $i=1,2,3$, and this form is skew-symmetric with respect to its last two arguments, i.e.,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}) \quad \forall \mathbf{u} \in H, \mathbf{v},\mathbf{w} \in H_0^1(\Omega)^d.$$

In particular, we have

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \quad \forall \mathbf{u} \in H, \mathbf{v} \in H_0^1(\Omega)^d, \tag{2.1}$$

and for $d \leq 4$,

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) \leq \begin{cases} c\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_1, \\ c\|\mathbf{u}\|\|\mathbf{v}\|_2\|\mathbf{w}\|_1, \\ c\|\mathbf{u}\|_1\|\mathbf{v}\|_2\|\mathbf{w}\|, \\ c\|\mathbf{u}\|\|\mathbf{v}\|_1\|\mathbf{w}\|_2, \\ c\|\mathbf{u}\|_2\|\mathbf{v}\|_1\|\mathbf{w}\|, \\ c\|\mathbf{u}\|_{L^8(\Omega)}\|\mathbf{v}\|_1\|\mathbf{w}\|_{L^{\frac{8}{3}}(\Omega)}, \\ c\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|_{L^3(\Omega)}, \\ c\|\mathbf{u}\|_1\|\mathbf{v}\|_1\|\mathbf{w}\|^{\frac{1}{2}}\|\mathbf{w}\|_1^{\frac{1}{2}}, \\ c\|\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{u}\|_1^{\frac{1}{2}}\|\mathbf{v}\|_1\|\mathbf{w}\|_1. \end{cases} \tag{2.2}$$

We also define the Stokes operator

$$A\mathbf{u} = -P\Delta\mathbf{u}, \qquad \forall \mathbf{u} \in D(A) = V \cap H^2(\Omega)^d,$$

where P_H is an orthogonal projector in the Hilbert space $L^2(\Omega)^n$ onto its subspace H. The Stokes operator A is an unbounded positive self-adjoint closed operator in H with domain D(A). Its inverse A^{-1} is compact in H, and have following properties: there exist constant $c_1, c_2 > 0$, such that $\forall \mathbf{u} \in H$,

$$\begin{cases} \|A^{-1}\mathbf{u}\|_s \leq c_1\|\mathbf{u}\|_{s-2}, & \text{for} \quad s = 1,2, \\ c_2\|\mathbf{u}\|_{-1}^2 \leq (A^{-1}\mathbf{u},\mathbf{u}) \leq c_1^2\|\mathbf{u}\|_{-1}^2. \end{cases} \tag{2.3}$$

Further, from (2.3), we will use $(A^{-1}\mathbf{u},\mathbf{u})^{\frac{1}{2}}$ as an equivalent norm of $H^{-1}(\Omega)^d$ for $\mathbf{u} \in H$.

For the purpose of this paper, we also need the following regularity assumptions [9]:

- (A1) $\mathbf{u}_0 \in H^1(\Omega)^d \cap V$, $\mathbf{f} \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;H^1(\Omega)^d)$,
- (A2) $\sup_{t\in[0,T]} \|\mathbf{u}(t)\|_1 \leq M$,
- (A3) $f_t \in L^2(0,T;H^{-1}(\Omega))^d$,
- (A4) $\int_0^T \|\mathbf{u}_{tt}\|_{-1}^2 dt \leq M$,

where (A_2) is automatically satisfied with some appropriate constant M when d=2. However, when in the three-dimension case, we need assume additionally. Under the regularity assumption (A1-A2), one can show that[9]

(a) $\sup_{t\in[0,T]} \{\|\mathbf{u}(t)\|_2 + \|\mathbf{u}_t(t)\| + \|\nabla p(t)\|\} \leq M_1$,

(b) $\int_0^T \|\mathbf{u}_t(t)\|_1^2 \leq M_1$.

In addition, if the assumption (A1-A3)) hold, we have

(c) $\int_0^T \|\mathbf{u}_{tt}\|_{-1}^2 dt \le M_1$,

which will be used in the sequel. Next, we cite the following Lemma.

Lemma (Discrete Gronwall Lemma). Let y^n, h^n, g^n, f^n be nonnegative sequences satisfying

$$y^m + \Delta t \sum_{n=0}^{m} h^n \le B + \Delta t \sum_{n=0}^{m} \Big(g^n y^n + f^n\Big),$$
$$\text{with } \Delta t \sum_{n=0}^{[\frac{T}{\Delta t}]} g^n \le M, \quad \forall\, 0 \le m \le [\frac{T}{\Delta t}].$$

Assume $\Delta t g^n < 1$ and let $\sigma = \max_{0\le n \le [\frac{T}{\Delta t}]} (1-\Delta t^n)^{-1}$. Then

$$y^m + \Delta t \sum_{n=0}^{m} h^n \le \exp(\sigma M)\Big(B + \Delta t \sum_{n=0}^{m} f^n\Big) \quad \forall\, m \le [\frac{T}{\Delta t}].$$

Hereafter, we will use c to denote a generic constant which depends only on Ω, ν, T, and constants from various Sobolev inequalities.

3. New Operator Splitting Scheme

Eq. (1.1) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} = A_1 + A_2, \tag{3.1}$$

where

$$A_1 = -(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{2}\nu\Delta\mathbf{u}, \quad A_2 = \frac{1}{2}\nu\Delta\mathbf{u} - \nabla p + \mathbf{f}. \tag{3.2}$$

Based on the first-order accurate operator splitting theory, an algorithm can be formulated as follows, for $t \in [t_n, t_{n+1}]$

$$\left\{\begin{array}{l} \dfrac{\partial \tilde{\mathbf{u}}}{\partial t} = A_1, \\ \tilde{\mathbf{u}}(t_n, \mathbf{x}) = \mathbf{u}(t_n, \mathbf{x}), \end{array}\right. \longrightarrow \left\{\begin{array}{l} \dfrac{\partial \hat{\mathbf{u}}}{\partial t} = A_2, \\ \hat{\mathbf{u}}(t_n, \mathbf{x}) = \tilde{\mathbf{u}}(t_{n+1}, \mathbf{x}), \end{array}\right. \tag{3.3}$$

and take $\mathbf{u}(t_{n+1}, \mathbf{x}) \approx \hat{\mathbf{u}}$ as the approximate solution of the equation at time t_{n+1} (1.1).

The scheme (3.3) has a irreducible splitting error of order $O(\Delta t)$. Hence, using a higher-order time stepping scheme does not improve the overall accuracy. So, a semi-discretized version can be obtained as follows: let $\mathbf{u}^0 = \mathbf{u}_0$, we solve successively $\tilde{\mathbf{u}}^{n+1}$, and $\{\hat{\mathbf{u}}^{n+1}, \hat{p}^{n+1}\}$ by

$$\left\{\begin{array}{l} \dfrac{\tilde{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^n}{\Delta t} + (\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1} - \frac{1}{2}\nu\Delta\tilde{\mathbf{u}}^{n+1} = 0, \\ \tilde{\mathbf{u}}^{n+1}|_{\partial\Omega} = g(t_{n+1}, \mathbf{x}), \end{array}\right. \tag{3.4}$$

$$\left\{\begin{array}{l} \dfrac{\hat{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^{n+1}}{\Delta t} - \frac{1}{2}\nu\Delta\hat{\mathbf{u}}^{n+1} + \nabla\hat{p}^{n+1} = \mathbf{f}(t_{n+1}), \\ \mathrm{div}\hat{\mathbf{u}}^{n+1} = 0, \\ \hat{\mathbf{u}}^{n+1} = \mathbf{g}(t_{n+1}, \mathbf{x}). \end{array}\right. \tag{3.5}$$

Note that we have omitted the dependency to x of the function $\mathbf{f}$ to simplify our notations, we will do so for $\{\mathbf{u}$,p$\}$.

In the first step (3.4), we solve an intermediate velocity $\tilde{\mathbf{u}}^{n+1}$, which does not satisfy the incompressibility condition. Then in the second step we project $\tilde{\mathbf{u}}^{n+1}$ onto the divergence free space H to obtain an appropriate velocity approximation $\hat{\mathbf{u}}^{n+1}$.

The first step of method can be seen as a linearized Burger's problem. The second step is a stokes problem, the discretization of which leads to a symmetric system of linear equations. One defect of this method is that the discrete inf-sup compatibility condition must be satisfied. As can be seen in (3.5), the main difference between this method and the standard projection method is the introduction of a viscous term in the incompressible step, which allows the imposition of the original boundary condition (2.2) on the end-of-step velocity $\hat{\mathbf{u}}^{n+1}$. Similar ideas can be found in the $\theta-$ scheme [13] and in several other methods such as those of [14-16], all of which involve an incompressible step with part of the viscous term.

4. Stability Analysis

Although we consider that the first order, linearized form of the convective term $(\hat{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}^{n+1}$, similar error estimates can be obtained for other approaches, such as the full nonlinear form $(\tilde{\mathbf{u}}^{n+1}\cdot\nabla)\tilde{\mathbf{u}}^{n+1}$. For the sake of simplicity, we will only consider the homogeneous boundary condition $\mathbf{u}(t)|_{\partial\Omega}=0$, i.e.,**g(t,x)**=0 for the scheme (3.4-3.5).

Theorem 4.1. *If the regularity assumptions* (A1)-(A2) *hold, we have*

$$\tilde{\mathbf{u}}^n \in L^2(0,T,H^1(\Omega)^d)\cap L^\infty(0,T,L^2(\Omega)^d), \tag{4.1a}$$

$$\hat{\mathbf{u}}^n \in L^2(0,T,H^1(\Omega)^d)\cap L^\infty(0,T,L^2(\Omega)^d). \tag{4.1b}$$

Proof. We take the inner product of (3.4) with $2\Delta t\tilde{\mathbf{u}}^{n+1}$ to get

$$\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^n\|^2+\|\tilde{\mathbf{u}}^{n+1}\|^2-\|\hat{\mathbf{u}}^n\|^2+\nu\Delta t\|\tilde{\mathbf{u}}^{n+1}\|_1^2=0. \tag{4.2}$$

Next, taking the inner product of (3.5) with $2\Delta t\hat{\mathbf{u}}^{n+1}$, and using the condition $\operatorname{div}\hat{\mathbf{u}}^{\rm n+1}=0$, we obtain

$$\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^{n+1}\|^2+\|\hat{\mathbf{u}}^{n+1}\|^2-\|\tilde{\mathbf{u}}^{n+1}\|^2+\nu\Delta t\|\hat{\mathbf{u}}^{n+1}\|_1^2=2\Delta t\Big(f(t_{n+1}),\hat{\mathbf{u}}^{n+1}\Big). \tag{4.3}$$

Combing (4.2-4.3), the below inequality holds

$$\begin{aligned}&\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^{n+1}\|^2+\|\hat{\mathbf{u}}^{n+1}\|^2+\Delta t\nu\|\hat{\mathbf{u}}^{n+1}\|_1^2+\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^n\|^2+\nu\Delta t\|\tilde{\mathbf{u}}^{n+1}\|_1^2\\ \le\ &2\Delta t\Big(\mathbf{f}(t_{n+1}),\hat{\mathbf{u}}^{n+1}\Big)+\|\hat{\mathbf{u}}^n\|^2.\end{aligned} \tag{4.4}$$

Using the Young inequality, we have

$$\begin{aligned}&\|\hat{\mathbf{u}}^{n+1}\|^2+\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^{n+1}\|^2+\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^n\|^2+\nu\Delta t\Big(\|\tilde{\mathbf{u}}^{n+1}\|_1^2+\|\hat{\mathbf{u}}^{n+1}\|_1^2\Big)\\ \le\ &c\Delta t\|(\mathbf{f}(t_{n+1})\|^2+\Delta t\|\hat{\mathbf{u}}^{n+1}\|^2+\|\hat{\mathbf{u}}^n\|^2.\end{aligned}$$

Summing up the above inequality for n=0... $r\le N$, we obtain

$$\begin{aligned}&\|\hat{\mathbf{u}}^{r+1}\|^2+\sum_{n=0}^r\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^{n+1}\|^2+\sum_{n=0}^r\|\tilde{\mathbf{u}}^{n+1}-\hat{\mathbf{u}}^n\|^2+\Delta t\nu\sum_{n=0}^r\|\hat{\mathbf{u}}^{n+1}\|_1^2\\ &\quad+\nu\Delta t\sum_{n=0}^r\|\tilde{\mathbf{u}}^{n+1}\|_1^2\le c\sup_{t\in[0,T]}\|\mathbf{f}(t)\|^2+\|\mathbf{u}_0\|^2+\Delta t\sum_{n=0}^r\|\hat{\mathbf{u}}^{n+1}\|^2.\end{aligned} \tag{4.5}$$

Applying the discrete Gronwall lemma to the above inequality, we obtain

$$\|\hat{\mathbf{u}}^{r+1}\|^2 + \sum_{n=0}^{r} \|\tilde{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^{n+1}\|^2 + \sum_{n=0}^{r} \|\tilde{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^{n}\|^2 + \Delta t\nu \sum_{n=0}^{r} \|\hat{\mathbf{u}}^{n+1}\|_1^2$$
$$+ \nu\Delta t \sum_{n=0}^{r} \|\tilde{\mathbf{u}}^{n+1}\|_1^2 \leq c \sup_{t\in[0,T]} \|\mathbf{f}(t)\|^2 + \|\mathbf{u}_0\|^2. \tag{4.6}$$

Thus, using the regularity properties of the continuous solution $\mathbf{u}$, for arbitrary N, we have

$$\|\hat{\mathbf{u}}^{n+1}\|^2 + \sum_{n=0}^{N} \Big(\|\tilde{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^{n+1}\|^2 + \|\tilde{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^{n}\|^2\Big)$$
$$+ \nu\Delta t \sum_{n=0}^{N} \Big(\|\hat{\mathbf{u}}^{n+1}\|_1^2 + \|\tilde{\mathbf{u}}^{n+1}\|_1^2\Big) \leq c, \tag{4.7}$$

which means that

$$\tilde{\mathbf{u}}^n \in L^2(0,T,H^1(\Omega)^d), \quad \hat{\mathbf{u}}^n \in L^2(0,T,H^1(\Omega)^d) \cap L^\infty(0,T,L^2(\Omega)^d).$$

From (4.2),yields

$$\|\tilde{\mathbf{u}}^{n+1}\|^2 \leq \|\hat{\mathbf{u}}^n\|^2. \tag{4.8}$$

So we have $\tilde{\mathbf{u}}^n \in L^\infty(0,T,L^2(\Omega)^d)$. The proof is complete. □

Remark 1. This can be viewed as the discrete version of the classical energy estimate for the Navier-Stokes equations [19]. From (4.5), we have

$$\|\hat{\mathbf{u}}^{N+1}\|^2 \leq c\Delta t \sum_{n=0}^{N} \|\mathbf{f}(t_{n+1})\|_0^2 + \|\mathbf{u}_0\|^2.$$

This estimate deteriorates as N increases. Nevertheless, it provides a useful bound for $\|\hat{\mathbf{u}}^{N+1}\|^2$ for the first few time steps, that is for small T.

5. Error Analysis

In this section, we present some error analysis of the fractional method introduced in the previous section. Firstly, we define the semidiscrete velocity error as:

$$\hat{\mathbf{e}}^{n+1} = \mathbf{u}(t_{n+1}) - \hat{\mathbf{u}}^{n+1}, \quad \tilde{\mathbf{e}}^{n+1} = \mathbf{u}(t_{n+1}) - \tilde{\mathbf{u}}^{n+1}.$$

Theorem 5.1. *Under the regularity assumptions* (A1)-(A3)*, there exists a constant c, such that*

$$\|\hat{\mathbf{e}}^{N+1}\|^2 + \|\tilde{\mathbf{e}}^{N+1}\|^2 + \sum_{n=0}^{N} \Big(\|\tilde{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^{n}\|^2 + \|\hat{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^{n+1}\|^2$$
$$+ \nu\Delta t(\|\hat{\mathbf{e}}^{n+1}\|_1^2 + \|\tilde{\mathbf{e}}^{n+1}\|_1^2)\Big) \leq c\Delta t. \tag{5.1}$$

Proof. Let $\mathbf{R}^n$ be the truncation error defined by

$$\begin{aligned}&\frac{1}{\Delta t}\Big(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\Big)-\nu\Delta\mathbf{u}(t_{n+1})+(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})+\nabla p(t_{n+1})\\&=\mathbf{f}(t_{n+1})+R^n,\end{aligned}\tag{5.2}$$

where R^n is the integral residual of the Taylor series, i.e.

$$\mathbf{R}^n=\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}}(t-t_n)u_{tt}dt.$$

By subtracting (3.4) from (5.2), we obtain

$$\begin{aligned}&\frac{\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^n}{\Delta t}-\frac{1}{2}\nu\Delta\tilde{\mathbf{e}}^{n+1}-\frac{1}{2}\nu\Delta\mathbf{u}(t_{n+1})\\&=(\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}-(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})+R^n-\nabla p(t_{n+1})+f(t_{n+1}).\end{aligned}\tag{5.3}$$

The nonlinear terms on the right-side can be split up three terms:

$$\begin{aligned}&(\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}-(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})\\&=-(\hat{\mathbf{e}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}+((\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}))\cdot\nabla)\tilde{\mathbf{u}}^{n+1}-(\mathbf{u}(t_{n+1})\cdot\nabla)\tilde{\mathbf{e}}^{n+1}.\end{aligned}\tag{5.4}$$

Taking the inner product of (5.3) with $2\Delta t\tilde{\mathbf{e}}^{n+1}$, using the identity (a-b,2a)=$|a|^2+|a+b|^2-|b|^2$, we obtain

$$\begin{aligned}&\|\tilde{\mathbf{e}}^{n+1}\|^2+\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^n\|^2-\|\hat{\mathbf{e}}^n\|^2+\nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1^2-\nu\Delta t(\Delta\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1})\\&=2\Delta t<\mathbf{R}^n,\tilde{\mathbf{e}}^{n+1}>-2\Delta t(\nabla p(t_{n+1}),\tilde{\mathbf{e}}^{n+1})-2\Delta tb(\hat{\mathbf{e}}^n,\tilde{\mathbf{u}}^{n+1},\tilde{e}^{n+1})\\&\quad+2\Delta tb(\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}),\tilde{\mathbf{u}}^{n+1},\tilde{\mathbf{e}}^{n+1})-2\Delta tb(\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1})\\&\quad+2\Delta t(\mathbf{f}(t_{n+1}),\tilde{\mathbf{e}}^{n+1}).\end{aligned}\tag{5.5}$$

On the other hand, we derive from (3.5) that

$$\frac{\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}}{\Delta t}+\frac{1}{2}\nu\Delta\hat{\mathbf{u}}^{n+1}-\nabla\hat{p}^{n+1}+\mathbf{f}(t_{n+1})=0.$$

Taking the inner product of the above equality with $2\Delta t\hat{\mathbf{e}}^{n+1}$, and using $\operatorname{div}\hat{\mathbf{e}}^{n+1}=0$, we obtain

$$\|\hat{\mathbf{e}}^{n+1}\|^2+\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|^2-\|\tilde{\mathbf{e}}^{n+1}\|^2+\Delta t\nu(\Delta\hat{\mathbf{u}}^{n+1},\hat{\mathbf{e}}^{n+1})+2\Delta t(\mathbf{f}(t_{n+1}),\hat{\mathbf{e}}^{n+1})=0.\tag{5.6}$$

Combing (5.5) with (5.6), the below expression is derived

$$\begin{aligned}&\|\hat{\mathbf{e}}^{n+1}\|^2+\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^n\|^2+\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|^2+\Delta t\nu\|\hat{\mathbf{e}}^{n+1}\|_1^2+\nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1^2\\&=2\Delta t<\mathbf{R}^n,\tilde{\mathbf{e}}^{n+1}>-2\Delta t(\nabla p(t_{n+1}),\tilde{\mathbf{e}}^{n+1})-2\Delta tb(\hat{\mathbf{e}}^n,\tilde{\mathbf{u}}^{n+1},\tilde{\mathbf{e}}^{n+1})\\&\quad+2\Delta tb(\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}),\tilde{\mathbf{u}}^{n+1},\tilde{\mathbf{e}}^{n+1})-2\Delta tb(\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1})\\&\quad+2\Delta t(\mathbf{f}(t_{n+1}),\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1})+\Delta t\nu(\Delta\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1})+\|\hat{\mathbf{e}}^n\|^2.\end{aligned}\tag{5.7}$$

We bound each term in the right-hand side of (5.7) independently. Consider the Taylor residual term:

$$2\Delta t<\mathbf{R}^n,\tilde{\mathbf{e}}^{n+1}>\le 2\Delta t\|\mathbf{R}^n\|_{-1}\|\tilde{\mathbf{e}}^{n+1}\|_1\le\frac{\Delta t\nu}{6}\|\tilde{\mathbf{e}}^{n+1}\|_1^2+c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2dt.$$

For the pressure gradient term:

$$-2\Delta t(\nabla p(t_{n+1}),\tilde{\mathbf{e}}^{n+1}) = -2\Delta t(\nabla p(t_{n+1}),\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}) \le \frac{1}{4}\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n})\|^2 + c\Delta t^2\|\nabla p(t_{n+1})\|^2.$$

For the nonlinear terms:

$$-2\Delta t b(\hat{\mathbf{e}}^{n},\tilde{\mathbf{u}}^{n+1},\tilde{\mathbf{e}}^{n+1}) = -2\Delta t b(\hat{\mathbf{e}}^{n},\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1}) \le c\Delta t\|\hat{\mathbf{e}}^{n}\|\|\tilde{\mathbf{e}}^{n+1}\|_1\|\mathbf{u}(t_{n+1})\|_2 \le \frac{\Delta t\nu}{6}\|\tilde{\mathbf{e}}^{n+1}\|_1^2 + c\Delta t\|\hat{\mathbf{e}}^{n}\|^2,$$

$$\begin{aligned} 2\Delta t b(\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}),\tilde{\mathbf{u}}^{n+1},\tilde{\mathbf{e}}^{n+1}) &= -2\Delta t b(\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},\tilde{\mathbf{u}}^{n+1}) \\ &= -2\Delta t b(\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},\mathbf{u}(t_{n+1})) \\ &\le c\Delta t\|\mathbf{u}(t_n)-\mathbf{u}(t_{n+1})\|\|\tilde{\mathbf{e}}^{n+1})\|_1\|\mathbf{u}(t_{n+1})\|_2 \\ &\le \frac{\Delta t\nu}{6}\|\tilde{\mathbf{e}}^{n+1}\|_1^2 + c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t\|^2dt, \end{aligned}$$

$$-2\Delta t b(\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1}) = 0.$$

The external term:

$$2\Delta t(\mathbf{f}(t_{n+1}),\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}) \le \tfrac{1}{4}\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}\|^2 + c\Delta t^2\|\mathbf{f}(t_{n+1})\|^2.$$

The viscous term:

$$\begin{aligned} &2\Delta t(\nu\Delta\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}) \le \frac{1}{4}\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}\|^2 + c\Delta t^2\|\Delta\mathbf{u}(t_{n+1})\|^2 \\ &\le \frac{1}{4}\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}\|^2 + c\Delta t^2\|\mathbf{u}(t_{n+1})\|_2^2. \end{aligned}$$

Inserting the above estimate into (5.7), we obtain

$$\begin{aligned} &\|\hat{\mathbf{e}}^{n+1}\|^2 + \frac{1}{2}\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|^2 + \frac{1}{2}\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|^2 + \nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|_1^2 + \frac{1}{2}\nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1^2 \\ &\le c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2dt + c\Delta t^2\|\nabla p(t_{n+1})\|^2 + c\Delta t\|\hat{\mathbf{e}}^{n}\|^2 \\ &\quad + c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t\|^2dt + c\Delta t^2\|\mathbf{f}(t_{n+1})\|^2 + c\Delta t^2\|\mathbf{u}(t_{n+1})\|_2^2. \end{aligned} \tag{5.8}$$

Summing up the inequality (5.8) for n=0,1,$\cdots$ N, the below formula holds

$$\begin{aligned} &\|\hat{\mathbf{e}}^{N+1}\|^2 + \sum_{n=0}^{N}(\frac{1}{2}\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|^2 + \frac{1}{2}\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|^2 + \nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|_1^2 + \frac{1}{2}\nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1^2) \\ &\le c\Delta t(\Delta t\int_0^{t_T}\|\mathbf{u}_{tt}\|_{-1}^2dt + \sup_{t\in[0,T]}\|\nabla p(t)\|^2 + \Delta t\int_0^T\|\mathbf{u}_t\|^2dt \\ &\quad + \sup_{t\in[0,T]}\|\mathbf{f}(t)\|^2 + \sup_{t\in[0,T]}\|\mathbf{u}(t)\|_2^2) + c\Delta t\sum_{n=0}^{N}\|\hat{\mathbf{e}}^{n}\|^2. \end{aligned} \tag{5.9}$$

Applying the discrete Gronwall lemma to the above inequality, we derive

$$\|\hat{\mathbf{e}}^{N+1}\|^2 + \sum_{n=0}^{N}(\frac{1}{2}\|\tilde{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n\|^2 + \frac{1}{2}\|\hat{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^{n+1}\|^2 + \nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|_1^2 + \frac{1}{2}\nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1^2)$$
$$\leq c\Delta t(\Delta t\int_0^T \|\mathbf{u}_{tt}\|_{-1}^2 dt + \sup_{t\in[0,T]}\|\nabla p(t_{n+1})\|^2 + \Delta t\int_0^T \|\mathbf{u}_t\|^2 dt$$
$$+ \sup_{t\in[0,T]}\|\mathbf{f}(t)\|^2 + \sup_{t\in[0,T]}\|\mathbf{u}(t)\|_2^2). \tag{5.10}$$

Using the regularity properties, we obtain

$$\|\hat{\mathbf{e}}^{N+1}\|^2 + \sum_{n=0}^{n=N}\Big(\|\tilde{\mathbf{e}}^{n+1} - \hat{e}^n\|^2 + \|\hat{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^{n+1}\|^2 + \nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|_1^2$$
$$+ \nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1^2\Big) \leq c\Delta t. \tag{5.11}$$

Finally, the bounds for $\tilde{\mathbf{u}}^{n+1}$ follow from (5.11) and the triangle inequality. Theorem 5.1 is proved. ☐

Remark 2. Theorem 5.1 shows that $\|\tilde{\mathbf{u}}^{n+1}\|_1 \leq c, \|\hat{\mathbf{u}}^{n+1}\|_1 \leq c$, since $\|\hat{\mathbf{e}}^{n+1}\|_1 \leq c$, $\|\tilde{\mathbf{e}}^{n+1}\|_1 \leq c$. Moreover, we also have $\|\tilde{\mathbf{u}}^{n+1}\|_0 \leq c\Delta t^{\frac{1}{2}}, \|\hat{\mathbf{u}}^{n+1}\|_0 \leq c\Delta t^{\frac{1}{2}}$.

Next, we will use the previous result to improve the error estimates for the velocity and give an error estimate for the pressure as well.

Theorem 5.2. *Under the regularity assumptions* (A1)-(A3), *there exists a constant c, such that*

$$\|\hat{\mathbf{e}}^{N+1}\|_{-1}^2 + \sum_{n=0}^{N}\Big(\|\hat{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n\|_{-1}^2 + \nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|^2 + \nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|^2\Big) \leq c\Delta t^2, \tag{5.12}$$

$$\Delta t\sum_{n=0}^{N}\|p(t_{n+1}) - \hat{p}^{n+1}\|_{L_0^2(\Omega)}^2 \leq c\Delta t. \tag{5.13}$$

Proof. Taking the sum of (3.4) and (3.5), we obtain

$$\frac{\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^n}{\Delta t} + (\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1} - \frac{1}{2}\nu\Delta\tilde{\mathbf{u}}^{n+1} - \frac{1}{2}\nu\Delta\hat{\mathbf{u}}^{n+1} + \nabla\hat{p}^{n+1} = \mathbf{f}(t_{n+1}). \tag{5.14}$$

Let us denote

$$\hat{q}^{n+1} = p(t_{n+1}) - \hat{p}^{n+1}.$$

Subtracting (5.14) from (5.2), we obtain

$$\frac{\hat{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n}{\Delta t} - \frac{1}{2}\nu\Delta\tilde{\mathbf{e}}^{n+1} - \frac{1}{2}\nu\Delta\hat{\mathbf{e}}^{n+1} + \nabla\hat{q}^{n+1}$$
$$= (\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1} - (\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1}) + \mathbf{R}^n. \tag{5.15}$$

Taking the inner product of the Equation (5.15) with $2\Delta tA^{-1}\hat{e}^{n+1}$, we obtain

$$(\hat{\mathbf{e}}^{n+1}, A^{-1}\hat{\mathbf{e}}^{n+1}) - (\hat{\mathbf{e}}^n, A^{-1}\hat{\mathbf{e}}^n) + (\hat{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n, A^{-1}(\hat{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n))$$
$$- \nu\Delta t(\Delta\tilde{\mathbf{e}}^{n+1}, A^{-1}\hat{\mathbf{e}}^{n+1}) - \nu\Delta t(\Delta\hat{\mathbf{e}}^{n+1}, A^{-1}\hat{\mathbf{e}}^{n+1})$$
$$= 2\Delta tb(\hat{\mathbf{u}}^n, \tilde{\mathbf{u}}^{n+1}, A^{-1}\hat{\mathbf{e}}^{n+1}) - 2\Delta tb(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}, A^{-1}\hat{\mathbf{e}}^{n+1})$$
$$+ 2\Delta t(R^n, A^{-1}\hat{\mathbf{e}}^{n+1}). \tag{5.16}$$

Using

$$\begin{aligned}&-\nu\Delta t(\Delta\tilde{\mathbf{e}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})\geq\nu\Delta t\Big(\frac{1}{4}\|\hat{\mathbf{e}}^{n+1}\|^2-c\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}\|^2\Big),\\&-\nu\Delta t(\Delta\hat{\mathbf{e}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})=\nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|^2.\end{aligned}$$

and (5.16) yields

$$\begin{aligned}&\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2-\|\hat{\mathbf{e}}^{n}\|_{-1}^2+\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|_{-1}^2+\frac{5}{4}\nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|^2\\&\leq 2\Delta t b(\hat{\mathbf{u}}^n,\tilde{\mathbf{u}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})-2\Delta t b(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})\\&\quad+2\Delta t(\mathbf{R}^n,A^{-1}\hat{\mathbf{e}}^{n+1})+c\Delta t\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}\|^2,\end{aligned}\tag{5.17}$$

similar to (5.4) for the nonlinear term, and together with (5.17), yields

$$\begin{aligned}&\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2-\|\hat{\mathbf{e}}^{n}\|_{-1}^2+\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|_{-1}^2+\frac{5}{4}\nu\Delta t\|\hat{e}^{n+1}\|^2\\=&-2\Delta t b(\hat{\mathbf{e}}^n,\tilde{\mathbf{u}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})+2\Delta t b(\mathbf{u}(t_n)-u(t_{n+1}),\tilde{\mathbf{u}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})\\&-2\Delta t b(\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})+2\Delta t(\mathbf{R}^n,A^{-1}\hat{\mathbf{e}}^{n+1})+c\Delta t\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}\|^2.\end{aligned}\tag{5.18}$$

We will focus on the right-hand side as follows: For the Taylor residual term, we have

$$\begin{aligned}&2\Delta t(\mathbf{R}^n,A^{-1}\hat{\mathbf{e}}^{n+1})\\&\leq c\Delta t\|\mathbf{R}^n\|_{-1}\|A^{-1}\hat{\mathbf{e}}^{n+1}\|_1\leq\Delta t\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2+c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2dt.\end{aligned}\tag{5.19}$$

For the nonlinear term, we have

$$\begin{aligned}&-2\Delta t b(\hat{\mathbf{e}}^n,\tilde{\mathbf{u}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})=2\Delta t b(\hat{\mathbf{e}}^n,A^{-1}\hat{\mathbf{e}}^{n+1},\tilde{\mathbf{u}}^{n+1})\\&=2\Delta t b(\hat{\mathbf{e}}^n,A^{-1}\hat{\mathbf{e}}^{n+1},\mathbf{u}(t_{n+1}))-2\Delta t b(\hat{\mathbf{e}}^n,A^{-1}\hat{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1})=:T_1+T_2.\end{aligned}$$

It follows from (2.2), (2.3) and (A2) that

$$\begin{aligned}T_1&\leq c\Delta t\|\hat{\mathbf{e}}^n\|\|A^{-1}\hat{\mathbf{e}}^{n+1}\|_1\|\mathbf{u}(t_{n+1})\|_2\leq c\Delta t\|\hat{\mathbf{e}}^n\|\|\hat{\mathbf{e}}^{n+1}\|_{-1}\\&\leq c\Delta t\Big(\|\hat{\mathbf{e}}^{n+1}\|+\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|+\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|\Big)\|\hat{\mathbf{e}}^{n+1}\|_{-1}\\&\leq\frac{\nu\Delta t}{16}\|\hat{\mathbf{e}}^{n+1}\|^2+c\Delta t\Big(\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|^2+\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|^2\Big)+c\Delta t\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2.\end{aligned}\tag{5.20}$$

Furthermore, using Theorem 5.1 gives

$$\begin{aligned}T_2&\leq c\Delta t\|\hat{\mathbf{e}}^n\|\|A^{-1}\hat{\mathbf{e}}^{n+1}\|_2\|\tilde{\mathbf{e}}^{n+1}\|_1\leq c\Delta t\|\hat{\mathbf{e}}^n\|\|\hat{\mathbf{e}}^{n+1}\|\|\tilde{\mathbf{e}}^{n+1}\|_1\\&\leq c\Delta t^{\frac{3}{2}}\|\hat{\mathbf{e}}^{n+1}\|\|\tilde{\mathbf{e}}^{n+1}\|_1\leq\frac{\nu\Delta t}{16}\|\hat{\mathbf{e}}^{n+1}\|^2+c\Delta t^2\|\tilde{\mathbf{e}}^{n+1}\|_1,\end{aligned}\tag{5.21}$$

and

$$\begin{aligned}&-2\Delta t b(\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1})=2\Delta t b(\mathbf{u}(t_{n+1}),A^{-1}\hat{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1})\\&\leq c\Delta t\|\mathbf{u}(t_{n+1})\|_2\|A^{-1}\hat{\mathbf{e}}^{n+1}\|_1\|\tilde{\mathbf{e}}^{n+1}\|\leq c\Delta t\|\hat{\mathbf{e}}^{n+1}\|_{-1}\|\tilde{\mathbf{e}}^{n+1}\|\\&\leq c\Delta t\|\hat{\mathbf{e}}^{n+1}\|_{-1}\Big(\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|+\|\hat{\mathbf{e}}^{n+1}\|\Big)\\&\leq\frac{\nu\Delta t}{16}\|\hat{\mathbf{e}}^{n+1}\|^2+c\Delta t\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|^2+c\Delta t\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2.\end{aligned}\tag{5.22}$$

Similarly

$$\begin{aligned}&2\Delta t b\Big(\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}),\tilde{\mathbf{u}}^{n+1},A^{-1}\hat{\mathbf{e}}^{n+1}\Big)\\ \le\,& 2\Delta t\|\mathbf{u}(t_n)-\mathbf{u}(t_{n+1})\|\|\tilde{\mathbf{u}}^{n+1}\|_1\|A^{-1}\hat{\mathbf{e}}^{n+1}\|_2\\ \le\,&\frac{\nu\Delta t}{16}\|\hat{\mathbf{e}}^{n+1}\|^2+c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t\|^2dt.\end{aligned}$$

Combing the above inequality into (5.18), we obtain

$$\begin{aligned}&\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2-\|\hat{\mathbf{e}}^{n}\|_{-1}^2+\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|_{-1}^2+\nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|^2\\ =\,&c\Delta t^2\Big(\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2dt+\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t\|^2dt\Big)+c\Delta t\Big(\|\hat{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n+1}\|^2+\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|^2\Big)\\ &+c\Delta t^2\|\tilde{\mathbf{e}}^{n+1}\|_1+c\Delta t\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2.\end{aligned}$$

Taking the sum of the above inequality for n from 0 to N, and using the regularity assumption (A1)-(A3) and Theorem 5.1, we have

$$\|\hat{\mathbf{e}}^{N+1}\|_{-1}^2+\sum_{N=0}^{N}(\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|_{-1}^2+\nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|^2)\le c\Delta t^2+c\Delta t\sum_{N=0}^{N}\|\hat{\mathbf{e}}^{n+1}\|_{-1}^2.$$

By applying the discrete Gronwall lemma to the last inequality, we obtain

$$\|\hat{\mathbf{e}}^{N+1}\|_{-1}^2+\sum_{N=0}^{N}\Big(\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|_{-1}^2+\nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|^2\Big)\le c\Delta t^2. \tag{5.23}$$

For $\tilde{\mathbf{u}}^{n+1}$, we have

$$\nu\Delta t\sum_{N=0}^{N}\|\tilde{\mathbf{e}}^{n+1}\|^2\le\nu\Delta t\sum_{N=0}^{N}\Big(\|\hat{\mathbf{e}}^{n+1}\|+\|\tilde{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n+1}\|\Big)\le c\Delta t^2,$$

together with (5.23), we derive (5.12).

Next, we derive the estimate for the pressure, we recast (5.15) as

$$\begin{aligned}\nabla\hat{q}^{n+1}=\,&\frac{1}{2}\nu\Delta\tilde{\mathbf{e}}^{n+1}+\frac{1}{2}\nu\Delta\hat{\mathbf{e}}^{n+1}-\frac{\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}}{\Delta t}\\ &+(\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}-(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})+\mathbf{R}^n.\end{aligned}\tag{5.24}$$

Firstly, by using (2.2) and Theorem 5.1, for all $\mathbf{v}\in H_0^1(\Omega)^d$,

$$\begin{aligned}&(\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}-(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})\\ \le\,&c\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|\|\mathbf{u}(t_{n+1})\|_2\|\mathbf{v}\|_1+c\|\hat{\mathbf{e}}^n\|_1\|\mathbf{u}(t_{n+1})\|_1\|\mathbf{v}\|_1+c\|\hat{\mathbf{u}}^n\|_1\|\tilde{\mathbf{e}}^{n+1}\|_1\|\mathbf{v}\|_1\\ \le\,&\Big(\|\tilde{\mathbf{e}}^{n+1}\|_1+\|\hat{\mathbf{e}}^n\|_1+\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|\Big)\|\mathbf{v}\|_1.\end{aligned}$$

Using the Schwarz inequality, we have also, for all $\mathbf{v}\in H_0^1(\Omega)^d$,

$$\begin{aligned}&\Big(\frac{1}{2}\nu\Delta\tilde{\mathbf{e}}^{n+1}+\frac{1}{2}\nu\Delta\hat{\mathbf{e}}^{n+1}-\frac{\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}}{\Delta t}+\mathbf{R}^n,\mathbf{v}\Big)\\ \le\,&\Big(\frac{1}{\Delta t}\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^{n}\|_{-1}+\|\mathbf{R}^n\|_{-1}+\frac{1}{2}\nu\|\hat{\mathbf{e}}^{n+1}\|_1+\frac{1}{2}\nu\|\tilde{\mathbf{e}}^{n+1}\|_1\Big)\|\mathbf{v}\|_1,\end{aligned}$$

by the above inequality, we derive

$$\begin{aligned}\|\hat{q}^{n+1}\|_{L_0^2\Omega} &\le c \sup_{\mathbf{v}\in H_0^1(\Omega)^d} \frac{(\nabla \hat{q}^{n+1}, \mathbf{v})}{\|\mathbf{v}\|_1} \\ &\le \frac{c}{\Delta t}\|\hat{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n\|_{-1} + c\Big(\|\mathbf{R}^n\|_{-1} + \|\tilde{\mathbf{e}}^{n+1}\|_1 + \|\hat{\mathbf{e}}^{n+1}\|_1 + \|\hat{\mathbf{e}}^n\|_1 + \|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|\Big).\end{aligned}$$

Consequently, by using Theorem 5.1 and (5.23), we obtain (5.13). □

Theorem 5.3. *Assume that the regularity assumption* (A1)-(A3) *hold, and the domain* Ω *is of class* C^2 *(or is a convex polygon or polyhedron), then for small enough* Δt*:*

$$\Delta t \sum_{n=0}^{N} \left(\|\tilde{\mathbf{u}}^{n+1}\|_2^2 + \|\hat{\mathbf{u}}^{n+1}\|_2^2\right) \le c\Delta t \sum_{n=0}^{N} \|\hat{p}\|_1^2 \le c.$$

Proof. A similar argument to that of [18,Theorem 3] is presented. We recast (3.4) as

$$\nu \Delta \tilde{\mathbf{u}}^{n+1} = 2\left(\frac{\tilde{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^n}{\Delta t} + (\hat{\mathbf{u}}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1}\right). \tag{5.25}$$

Then

$$\begin{aligned}&\Delta t \sum_{n=0}^{N} \left\|\frac{\tilde{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^n}{\Delta t}\right\|^2 \\ \le& \frac{c}{\Delta t}\sum_{n=0}^{N}\left(\|\tilde{\mathbf{u}}^{n+1} - \mathbf{u}(t_{n+1})\|^2 + \|\mathbf{u}(t_{n+1} - \mathbf{u}(t_n)\| + \|\mathbf{u}(t_n) - \hat{\mathbf{u}}^n\|^2\right) \\ \le& \frac{c}{\Delta t}\sum_{n=0}^{N}\left(\|\tilde{\mathbf{e}}^{n+1}\|^2 + \Delta t\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t\|^2 dt + \|\hat{\mathbf{e}}^n\|^2\right) \le c,\end{aligned}$$

where we have used Theorem 5.1. Moreover

$$\|(\hat{\mathbf{u}}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1}\|_{L^{\frac{3}{2}}(\Omega)} = \sup_{\mathbf{v}\in L^3(\Omega)} \frac{((\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}, \mathbf{v})}{\|\mathbf{v}\|_{L^3(\Omega)}} \le c\|\hat{\mathbf{u}}^n\|_1\|\tilde{\mathbf{u}}^{n+1}\|_1 \le c,$$

due to (2.2) and Theorem 5.1. From (5.25), we conclude that $\Delta\tilde{\mathbf{u}}$ is bounded in $l^2(L^{\frac{3}{2}}(\Omega)^d)$.

Next the formula (3.5) can be written as

$$-\frac{1}{2}\nu\Delta\hat{\mathbf{u}}^{n+1} + \nabla\hat{p}^{n+1} = \mathbf{f}(t_{n+1}) - \frac{\hat{\mathbf{u}}^{n+1} - \tilde{\mathbf{u}}^{n+1}}{\Delta t}. \tag{5.26}$$

The last term above can be bounded in $l^2(L^2(\Omega)^d)$ by Theorem 4.1, As a result using the regularity of solutions of the Stokes problem (5.26) on regular domain, we can assert that $\hat{\mathbf{u}}^{n+1}$ is bounded in $l^2(H^{2,\frac{3}{2}}(\Omega)^d)$, and $\hat{p}^{n+1}$ is bounded in $l^2(H^{1,\frac{3}{2}}(\Omega)^d)$. According to Sobolev's compactness theorem, we conclude that $\hat{\mathbf{u}}^{n+1}$ is bounded in $l^2(L^8(\Omega)^d)$ for d=2 or d=3.

Furthermore

$$\|(\hat{\mathbf{u}}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1}\|_{L^{\frac{8}{5}}(\Omega)} = \sup_{\mathbf{v}\in L^{\frac{8}{3}}(\Omega)} \frac{((\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}, \mathbf{v})}{\|\mathbf{v}\|_{L^{\frac{8}{3}}(\Omega)}} \le c\|\hat{\mathbf{u}}^n\|_{\frac{8}{3}}\|\tilde{\mathbf{u}}^{n+1}\|_1 \le c.$$

We now turn back to (5.25). We enhance the regularity of $\Delta\tilde{\mathbf{u}}^{n+1}$ to $l^2(L^{\frac{8}{5}}(\Omega)^d)$, and that of $\hat{\mathbf{u}}^{n+1}$ to $l^2(H^{2,\frac{8}{5}}(\Omega)^d)$ and $\hat{p}^{n+1}$ to $l^2(H^{1,\frac{8}{5}}(\Omega)^d)$, as solutions of the Stokes problem (5.26). The fact that $\hat{u}^{n+1}$ is bounded in $l^2(L^\infty(\Omega)^d)$ by Sobolev's Theorem, together with Theorem 5.1, shows that $(\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}$ is bounded in $l^2(L^2(\Omega)^d)$. So the boundedness for $\Delta\tilde{\mathbf{u}}^{n+}$ also holds in $l^2(L^2(\Omega)^d)$, which is sufficient to bound $\tilde{\mathbf{u}}^{n+1}$ in $l^2(H^2(\Omega)^d)$, when Ω is regular enough. Finally, the bounds for $\hat{\mathbf{u}}^{n+1}$ and $\hat{p}^{n+1}$ follow from the regularity of the Stokes problem. □

The error estimates of Theorem 5.2 can be improved to first order in the norm of $l^\infty(L^2(\Omega)^d)$ and $l^2(H_0^1(\Omega)^d)$ for the velocity $\hat{\mathbf{u}}^{n+1}$.

Theorem 5.4. *Under the assumption* (A1)-(A3)*, for sufficiently small* Δt*, there exists a constant* c *such that*

$$\|\hat{\mathbf{e}}^{N+1}\|^2+\sum_{n=0}^{N}\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^n\|^2+\nu\Delta t\sum_{n=0}^{N}\Big(\|\tilde{\mathbf{e}}^{n+1}\|_1^2+\|\hat{\mathbf{e}}^{n+1}\|_1^2\Big)\le c\Delta t^2.$$

Proof. Taking the inner product of (5.15) with $2\Delta t\hat{e}^{n+1}$, we have

$$\begin{aligned}&\|\hat{\mathbf{e}}^{n+1}\|^2-\|\hat{\mathbf{e}}^n\|^2+\|\hat{\mathbf{e}}^{n+1}-\hat{\mathbf{e}}^n\|^2+\nu\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1^2+\nu\Delta t\|\hat{\mathbf{e}}^{n+1}\|_1^2\\ =\;&2\Delta tb(\hat{\mathbf{u}}^n,\tilde{\mathbf{u}}^{n+1},\hat{\mathbf{e}}^{n+1})-2\Delta tb(\mathbf{u}(t_{n+1}),\mathbf{u}(t_{n+1}),\hat{\mathbf{e}}^{n+1})+2\Delta t<\mathbf{R}^n,\hat{\mathbf{e}}^{n+1}>.\end{aligned}\tag{5.27}$$

The estimates below are obtained for the right-hand term of (5.27):

$$2\Delta t<\mathbf{R}^n,\hat{\mathbf{e}}^{n+1}>\le\frac{\nu\Delta t}{10}\|\hat{\mathbf{e}}^{n+1}\|_1^2+C\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_{tt}\|_{-1}^2dt.$$

For the nonlinear term in (5.27), similar to (5.4) gives the following estimate:

$$-2\Delta tb(\mathbf{u}(t_{n+1},\tilde{\mathbf{e}}^{n+1}.\hat{\mathbf{e}}^{n+1})\le\frac{\nu\Delta t}{10}\|\hat{\mathbf{e}}^{n+1}\|_1^2+c\Delta t\|\tilde{\mathbf{e}}^{n+1}\|^2.$$

For the remainder of nonlinear term, we have

$$\begin{aligned}&2\Delta tb(\mathbf{u}(t_n)-\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},\hat{\mathbf{e}}^{n+1})\le\|\mathbf{u}(t_n)-\mathbf{u}(t_{n+1})\|_1\|\tilde{\mathbf{e}}^{n+1}\|_1\|\hat{\mathbf{e}}^{n+1}\|_1\\ \le\;& c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t\|_1^2dt+\frac{\nu\Delta t}{10}\|\hat{\mathbf{e}}^{n+1}\|_1^2,\end{aligned}$$

and

$$-2\Delta tb(\hat{e}^n,\tilde{u}^{n+1},\hat{\mathbf{e}}^{n+1})=2\Delta tb(\hat{\mathbf{e}}^n,\tilde{\mathbf{e}}^{n+1},\hat{\mathbf{e}}^{n+1})-2\Delta tb(\hat{\mathbf{e}}^n,\mathbf{u}(t_{n+1}),\hat{\mathbf{e}}^{n+1}).$$

Thus,

$$\begin{aligned}&2\Delta tb(\hat{\mathbf{e}}^n,\tilde{\mathbf{e}}^{n+1},\hat{\mathbf{e}}^{n+1})\le c\Delta t\|\hat{\mathbf{e}}^n\|_1\|\hat{\mathbf{e}}^{n+1}\|_1\|\tilde{\mathbf{e}}^{n+1}\|^{\frac12}\|\tilde{\mathbf{e}}^{n+1}\|_1^{\frac12}\\ \le\;& c\Delta t\|\hat{\mathbf{e}}^n\|_1\|\hat{\mathbf{e}}^{n+1}\|_1\|\tilde{\mathbf{e}}^{n+1}\|^{\frac12}\le c\Delta t^{\frac54}\|\hat{\mathbf{e}}^n\|_1\|\hat{\mathbf{e}}^{n+1}\|_1\\ \le\;& c\Delta t^{\frac32}\|\hat{\mathbf{e}}^n\|_1^2+\frac{\nu\Delta t}{10}\|\hat{\mathbf{e}}^{n+1}\|_1^2,\end{aligned}$$

and

$$\begin{aligned}&-2\Delta tb(\hat{\mathbf{e}}^n,\mathbf{u}(t_{n+1}),\hat{\mathbf{e}}^{n+1})\le c\Delta t\|\hat{\mathbf{e}}^n\|_0\|\mathbf{u}(t_{n+1})\|_2\|\hat{\mathbf{e}}^{n+1}\|_1\\ \le\;& c\Delta t\|\hat{\mathbf{e}}^n\|_0\|\hat{\mathbf{e}}^{n+1}\|_1\le\frac{\nu\Delta t}{10}\|\hat{\mathbf{e}}^{n+1}\|_1^2+c\Delta t\|\hat{\mathbf{e}}^n\|^2.\end{aligned}$$

Taking the sum of the formula (5.27) for n from 0 to N, together with the above estimates, we obtain from (5.12) to get

$$\begin{aligned}
&\|\hat{\mathbf{e}}^{N+1}\|^2 + \sum_{n=0}^{N} \|\hat{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n\|^2 + \nu\Delta t \sum_{n=0}^{N} \Big(\|\tilde{\mathbf{e}}^{n+1}\|_1^2 + \frac{1}{2}\|\hat{\mathbf{e}}^{n+1}\|_1^2 \Big) \\
= {}& C\Delta t^2 \int_0^T \|\mathbf{u}_{tt}\|_{-1}^2 dt + c\Delta t \sum_{n=0}^{N} \|\tilde{\mathbf{e}}^{n+1}\|^2 + c\Delta t^2 \int_0^T \|\mathbf{u}_t\|_1^2 dt \\
&+ c\Delta t^{\frac{3}{2}} \sum_{n=0}^{N} \|\hat{\mathbf{e}}^n\|_1^2 + c\Delta t \sum_{n=0}^{N} \|\hat{\mathbf{e}}^n\|^2.
\end{aligned} \tag{5.28}$$

By virtue of the formula (5.12) and the regularity assumption of solution $\mathbf{u}$, we obtain

$$\begin{aligned}
&\|\hat{\mathbf{e}}^{N+1}\|^2 + \sum_{n=0}^{N} \|\hat{\mathbf{e}}^{n+1} - \hat{\mathbf{e}}^n\|^2 + \nu\Delta t \sum_{n=0}^{N} \Big(\|\tilde{\mathbf{e}}^{n+1}\|_1^2 + \frac{1}{2}\|\hat{\mathbf{e}}^{n+1}\|_1^2 \Big) \\
= {}& C\Delta t^2 + c\Delta t^{\frac{3}{2}} \sum_{n=0}^{N} \|\hat{\mathbf{e}}^n\|_1^2 + c\Delta t \sum_{n=0}^{N} \|\hat{\mathbf{e}}^{n+1}\|^2.
\end{aligned}$$

For sufficiently small Δt, we can take the last term to left side and apply the discrete Gronwall lemma to complete the proof. □

Theorem 5.5. *Under the assumption* (A1)-(A4), *for small enough* Δt, *there exists constants* c_0, *and* c *satisfying* $1 - 6\delta > c_0$ *and* $4c\sqrt{\Delta t}\|\hat{\mathbf{e}}^N\|^2\nu^{-1} < 1$, *such that*

$$\sum_{n=0}^{N} c_0 \|\tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n\|^2 \leq c\Delta t^2. \tag{5.29}$$

Proof. we shift the index $n+1$ to n in (3.5) to give

$$\frac{\hat{\mathbf{e}}^n - \tilde{\mathbf{e}}^n}{\Delta t} + \frac{1}{2}\nu\Delta\hat{\mathbf{u}}^n - \nabla\hat{p}^n + f(t_n) = 0 \tag{5.30}$$

and take the sum with (5.3), we obtain

$$\begin{aligned}
&\frac{\tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n}{\Delta t} - \frac{\nu}{2}\Delta\tilde{\mathbf{e}}^{n+1} \\
= {}& (\hat{\mathbf{u}}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1} - (\mathbf{u}(t_{n+1}) \cdot \nabla)\mathbf{u}(t_{n+1}) + R^n - \nabla(p(t_{n+1}) - \hat{p}^n) \\
&+ (\mathbf{f}(t_{n+1}) - \mathbf{f}(t_n)) + \frac{\nu}{2}\Delta\hat{e}^n + \frac{\nu}{2}\Delta(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)).
\end{aligned} \tag{5.31}$$

Taking the inner product with $\Delta t(\tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n)$, the left-hand term of above formula can be written as

$$\|\tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n\|^2 + \frac{\nu\Delta t}{4}\Big(\|\tilde{\mathbf{e}}^{n+1}\|_1^2 + \|\tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n\|_1^2 - \|\tilde{\mathbf{e}}^n\|_1^2 \Big). \tag{5.32}$$

Next, we give the estimates of the right-hand term of (5.31):

$$\begin{aligned}
&\frac{\nu\Delta t}{2}(\Delta(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)), \tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n) \\
= {}& -\frac{\nu\Delta t}{2}(\nabla(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)), \nabla(\tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n)) \\
\leq {}& c\Delta t\|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_1 \|\tilde{\mathbf{e}}^{n+1} - \tilde{\mathbf{e}}^n\|_1
\end{aligned}$$

$$
\begin{aligned}
&\leq c\Delta t\|\mathbf{u}(t_{n+1})-\mathbf{u}(t_n)\|_1^2+\frac{\nu\Delta t}{8}\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|_1^2\\
&\leq c\Delta t^2\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t\|_1^2dt+\frac{\nu\Delta t}{8}\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|_1^2, \qquad (5.33)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\nu\Delta t}{2}(\Delta\hat{\mathbf{e}}^n,\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\\
=&-\frac{\nu\Delta t}{2}(\nabla\hat{\mathbf{e}}^n,\nabla(\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n))\leq c\Delta t(\|\hat{\mathbf{e}}^n\|_1+\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|_1)\\
\leq& c\Delta t(\|\hat{\mathbf{e}}^n\|_1+\|\tilde{\mathbf{e}}^{n+1}\|_1+\|\tilde{\mathbf{e}}^n\|_1)\leq c\Delta t^2, \qquad (5.34)
\end{aligned}
$$

where we have used Theorem 5.4, and

$$
\Delta t(\mathbf{f}(t_{n+1})-\mathbf{f}(t_n),\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\leq c\Delta t^3\int_{t_n}^{t_{n+1}}\|\mathbf{f}_t\|^2dt+\delta\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^2. \qquad (5.35)
$$

For the Taylor residual term,we have

$$
\Delta t(\mathbf{R}^n,\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\leq\delta\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^2+c\Delta t^2\int_{t_n}^{t_{n+1}}t\|\mathbf{u}_{tt}\|^2dt. \qquad (5.36)
$$

For the pressure term, since div$\hat{\mathbf{e}}^{n+1}=0$, resp.div$\hat{\mathbf{e}}^n=0$, we obtain

$$
\begin{aligned}
&-\Delta t(\nabla(p(t_{n+1})-\hat{p}^n),\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\\
&=\Delta t(p(t_{n+1})-\hat{p}^n,\nabla(\tilde{\mathbf{e}}^n-\tilde{\mathbf{e}}^{n+1}))\leq\Delta t\|p(t_{n+1})-\hat{p}^n\|\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|_1\\
&\leq\Big(\|p(t_{n+1})\|+\|\hat{p}^n\|\Big)\Big(\Delta t\|\tilde{\mathbf{e}}^{n+1}\|_1+\Delta t\|\tilde{\mathbf{e}}^n\|_1\Big)\leq c\Delta t^2, \qquad (5.37)
\end{aligned}
$$

where Theorems 5.2 and 5.4 are used.

For the trilinear term, we consider the splitting below:

$$
\begin{aligned}
&(\hat{\mathbf{u}}^n\cdot\nabla)\tilde{\mathbf{u}}^{n+1}-(\mathbf{u}(t_{n+1})\cdot\nabla)\mathbf{u}(t_{n+1})\\
=&\,\hat{\mathbf{e}}^n\cdot\nabla\tilde{\mathbf{e}}^{n+1}-(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n))\cdot\nabla\tilde{u}^{n+1}-\mathbf{u}(t_{n+1})\cdot\nabla\tilde{\mathbf{e}}^{n+1}-\hat{e}^n\cdot\nabla\mathbf{u}(t_{n+1}).
\end{aligned}
$$

Using the formula (2.2) and regularity assumption yields

$$
\Delta tb(\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\leq\delta\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^2+c\Delta t^2\|\tilde{\mathbf{e}}^{n+1}\|_1^2, \qquad (5.38)
$$

$$
\begin{aligned}
&\Delta tb(\mathbf{u}(t_{n+1})-\mathbf{u}(t_n),\tilde{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\\
&\leq\delta\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^2+c\Delta t^3\|\tilde{\mathbf{u}}^{n+1}\|_1\int_{t_n}^{t_{n+1}}\|\mathbf{u}_t\|_2^2dt\\
&\leq\delta\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^2+c\Delta t^3, \qquad (5.39)
\end{aligned}
$$

$$
\Delta tb(\hat{\mathbf{e}}^n,\mathbf{u}(t_{n+1}),\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\leq\delta\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^2+c\Delta t^2\|\hat{\mathbf{e}}^n\|_1^2,
$$

and

$$
\begin{aligned}
&\Delta tb(\hat{\mathbf{e}}^n,\tilde{\mathbf{e}}^{n+1},\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n)\leq c\Delta t\|\hat{\mathbf{e}}^n\|_1\|\tilde{\mathbf{e}}^{n+1}\|_1\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|_1^{\frac{1}{2}}\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^{\frac{1}{2}}\\
&\leq c\Delta t^{\frac{3}{2}}\|\hat{\mathbf{e}}^n\|_1^2\|\tilde{\mathbf{e}}^{n+1}\|_1^2+\sqrt{\Delta t\nu\delta}\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|_1\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|\\
&\leq c\Delta t^{\frac{3}{2}}\|\hat{\mathbf{e}}^n\|_1^2\|\tilde{\mathbf{e}}^{n+1}\|_1^2+\frac{\nu\Delta t}{8}\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|_1^2+\delta\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^n\|^2. \qquad (5.40)
\end{aligned}
$$

Using (5.33)-(5.40), the regularity (A1-A3), and Theorem 5.1, we obtain

$$(1-6\delta)\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n}\|^2+\frac{\nu\Delta t}{4}\Big(\|\tilde{\mathbf{e}}^{n+1}\|_1^2-\|\tilde{\mathbf{e}}^{n}\|_1^2\Big)\le c\Big(\Delta t^2+\Delta t^{\frac{3}{2}}\|\hat{\mathbf{e}}^{n}\|_1^2\|\tilde{\mathbf{e}}^{n+1}\|_1^2\Big).$$

Summing up the above inequality for n from 0 to N gives

$$\sum_{n=0}^{N} c_0\|\tilde{\mathbf{e}}^{n+1}-\tilde{\mathbf{e}}^{n}\|^2+\frac{\nu\Delta t}{4}\|\tilde{\mathbf{e}}^{N+1}\|_1^2\le c\Big(\Delta t^2+\sum_{n=0}^{N}\Delta t^{\frac{3}{2}}\|\hat{\mathbf{e}}^{n}\|_1^2\|\tilde{\mathbf{e}}^{n+1}\|_1^2\Big),$$

where we have used the assumption $1-6\delta>c_0$.

Now, we assume that Δt is sufficiently small such that $4c\sqrt{\Delta t}\|\hat{\mathbf{e}}^N\|^2\nu^{-1}<1$ holds. Note that $\hat{\mathbf{e}}^N$ is uniformly bounded due to Theorem 5.1. Then by the discrete Gronwall inequality, the proof is complete. □

6. Numerical Results

In this section, we present some numerical tests to verify the theoretical results of the paper. Consider problem (2.1) in the domain $[0,1]\times[0,1]$ with exact solution

$$\begin{aligned}
&u(x,y)=(u_1(x,y),u_2(x,y)),\\
&p(x,y)=10(2x-1)(2y-1)\cos(t),\\
&u_1(x,y)=10x^2y(x-1)^2(y-1)(2y-1)\cos(t),\\
&u_2(x,y)=-10xy^2(x-1)(2x-1)(y-1)^2\cos(t).
\end{aligned}$$

The initial condition is set equal to the exact solution and f is computed by evaluating the momentum equation of problem (1.1) for the exact solution.

The uniform mesh partition of Ω into triangular element is obtained by dividing Ω into sub-squares of equal size and then drawing the diagonal in each sub-square, see Fig. 6.1.

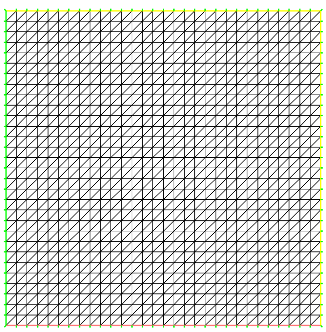

Fig. 6.1. The domain Ω.

We first compare the numerical solution with the exact solution Δt for ν=0.0005, $T=1$, and $\Delta t=0.001$. The results are resented in Figs. 6.2 and 6.3. We then present computed solutions using different mesh sizes for $\Delta t=0.005$, Re=500, T=1(see Table 6.1). Finally, computed solutions with different time step are presented for T=1, Re=2000, $h=\frac{1}{20}$. It is observed that the numerical results agree well with the theoretical results. Furthermore, compared with Chorn's projection method, our method produces slightly more accurate velocity.

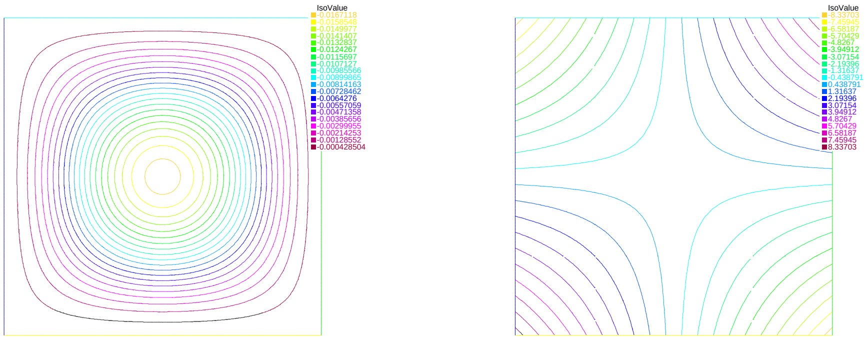

Fig. 6.2. Exact solution:P2-P1 element T=1 Δt = 0.001. Left: velocity streamlines, and Right: pressure contours.

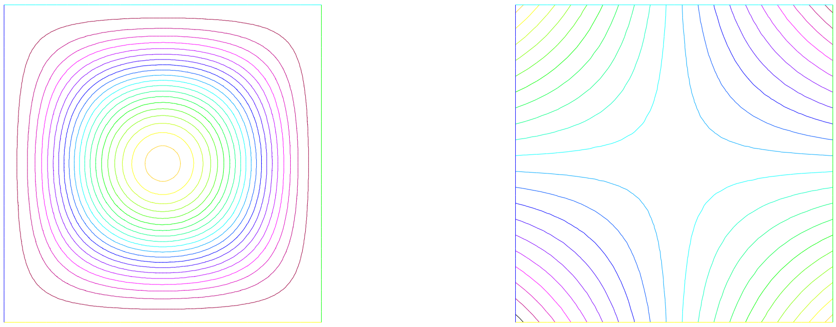

Fig. 6.3. Numerical solution:P2-P1 element T=1 and Δt = 0.001. Left: velocity streamlines, and Right: pressure contours.

Table 6.1: New operator splitting method with respect to different mesh sizes.

$\frac{1}{h}$	$\frac{\|\|u-u_h\|\|_0}{\|\|u\|\|_0}$	$\frac{\|\|u-u_h\|\|_1}{\|\|u\|\|_1}$	$\frac{\|\|p-p_h\|\|_0}{\|\|p\|\|_0}$	u_{L^2}rate	u_{H^1}rate	P_{L^2}rate	CPU(s)
5	0.0184268	0.1113150	0.03103180	/	/	/	37.750
10	0.0026112	0.0591252	0.00775794	2.8190	0.9128	2.0000	60.000
15	0.0011489	0.0383126	0.00348799	2.0248	1.0701	1.9715	86.031

Table 6.2: New operator splitting method with respect to different time step.

Δt	$\frac{\|\|u-u_h\|\|_0}{\|\|u\|\|_0}$	$\frac{\|\|u-u_h\|\|_1}{\|\|u\|\|_1}$	$\frac{\|\|p-p_h\|\|_0}{\|\|p\|\|_0}$	u_{L^2}rate	u_{H^1}rate	P_{L^2}rate	CPU(s)
0.1	0.0327224	0.435219	0.0019995	/	/	/	32.422
0.05	0.0512436	0.105637	0.00064242	0.9270	0.7079	0.5677	37.922
0.01	0.00010314	0.002637	0.0000815	0.7811	0.7381	0.4125	67.312

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References

[1] A.J. Chorin, Numerical solution of the Navier-Stokes equations, *Math. Comput.*, **22** (1968), 745-762.

[2] R. Temam, Sur lapproximation de la solution des equations de Navier-Stokes par la methode des pas fractionnaires ii, *Arch. Ration. Mech. Anal.*, **33** (1969), 377-385.

[3] J.L. Guermod, P Minev, J. Shen, An overview of projection methods for incompressible flows, *Comput. Methods Appl. Mech. Engrg.*, **195** (2006), 6011-6045.

[4] J.L. Guermond, L. Quartapelle, On the approximation of the unsteady Navier-Stokes equations by finite element projection methods, *Numer. Math.*, **80** (1998), 207-238.

[5] J. Shen, Remarks on the pressure error estimates for the projection method, *Numer. Math.*, **67** (1994), 513-520.

[6] J. Blasco, R Codina, Error estimates for an operator-splitting method for incompressible flows, *Appl. Numer. Math.*, **51** (2004), 1-17.

[7] J. Shen, On error estimates of the projection methods for the Navier-Stokes equations: First-order schemes, *SIAM J. Numer. Anal.*, **29** (1992), 57-77.

[8] J. Shen, On error estimates of the projection methods for the Navier-Stokes equations: Second-order schemes, *Math. Comput.*, **65**:215 (1996), 1039-1065.

[9] J. Shen, On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations, *Numer. Math.*, **62** (1992),49-73.

[10] S.A. Orzag, M.Israeli e.t., Boundary conditions for incompressible flows, *J. Sci. Comput.*, **1**:1 (1986), 75-111.

[11] R. Temam, Remark on the pressure boundary condition for projection method, *Theor. Comput. Fluid Dyn.*, **3** (1991), 181-184.

[12] J. Blasco, R. Codina, A fractional-step method for the incompressible Navier-Stokes equations related to a predictor-multicorrector algorithm, *Int. J. Numer. Methods Fluids.*, **28** (1997), 1391-1419.

[13] R. Glowinski, T.W. Pan, J. Periaux, A fictitious domain method for external incompressible viscous flow modeled by Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.*, **112** (1994), 133-148.

[14] L.A Ying, Viscosity-splitting scheme for the Navier-Stokes equations, *Numer. Methods Partial Differential Equations.*, **7** (1991), 317-338.

[15] R. Natarajan, A numerical method for incompressible viscous flow simulation, *J. Comput. Phys.*, **100** (1992), 384-395.

[16] L.P.J. Timmermans, P.D. Minev, F.N. van de Vosse, An approximate projection scheme for incompressible flow using spectral elements, *Internat. J. Numer. Methods Fluids.*, **22** (1996), 673-688.