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CONVERGENCE AND SUPERCONVERGENCE ANALYSIS OF LAGRANGE RECTANGULAR ELEMENTS WITH ANY ORDER ON ARBITRARY RECTANGULAR MESHES*

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Abstract

This paper is to study the convergence and superconvergence of rectangular finite elements under anisotropic meshes. By using of the orthogonal expansion method, an anisotropic Lagrange interpolation is presented. The family of Lagrange rectangular elements with all the possible shape function spaces are considered, which cover the Intermediate families, Tensor-product families and Serendipity families. It is shown that the *anisotropic interpolation error estimates* hold for *any order Sobolev norm.* We extend the convergence and superconvergence result of rectangular finite elements to *arbitrary rectangular meshes* in a unified way.

Mathematics subject classification: 65N12, 65N15, 65N30, 65N50.
Key words: Lagrange interpolation, Anisotropic error bounds, Arbitrary rectangular meshes, Orthogonal expansion, Superconvergence.

1. Introduction

The *nondegenerate assumption* or *regular assumption* on the meshes is a basic condition in the classical convergence analysis of the finite element methods, see [9,16]. Consider a bounded convex domain $\Omega \subset R^2$. Let $\mathcal{J}_h$ be a family of meshes of Ω. Denote the diameter of an element K and the diameter of the inscribed circle of K by h_K and ρ_K, respectively, $h = \max\limits_{K\in\mathcal{J}_h} h_K$. It is assumed in the classical finite element theory that

$$\frac{h_K}{\rho_K} \le C, \quad \forall \;\; K \in \mathcal{J}_h, \tag{1.1}$$

where C is a positive constant independent of K and the function considered.

We will consider the error estimates of finite elements in degenerate meshes. Then the regular assumption is no longer valid in this case. Conversely, *degenerate elements* (or *anisotropic elements*) are characterized by $\frac{h_K}{\rho_K} \to \infty$ as $h \to 0$. Error estimates for degenerate elements can go back to the works by Babuška and Aziz [6] and by Jamet [19]. Especially, anisotropic

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interpolation error estimates of Lagrange elements have been proved by several authors with different methods, interested readers are referred to [4, 5, 12–14, 20, 30, 34, 35, 38] and references therein. Especially, anisotropic triangle (tetrahedra) Lagrange finite elements have been extensively studied in the above mentioned references.

As we know, compared with triangular elements, the choice of shape functions of rectangular elements have much more possibilities than them and thus the constructions of rectangular finite elements are in many forms. Up to now, only a few elements with bi$-k$ tensor product polynomial space are treated by the above mentioned references, see e.g., [1,3–5,14,33,39], there are still some gaps in the anisotropic error estimates of the Lagrange rectangular elements. In fact, there are three popular polynomial bases for rectangular meshes in engineering practice: (1) Tensor-product spaces; (2) Serendipity families; (3) Intermediate elements. Most of these elements are still missing in anisotropic error estimates.

On the other hand, superconvergence in finite element methods has been highlighted for more than thirty years both in the engineering and scientific computing applications. It is a powerful tool in improving the accuracy, and plays an important role in the a posteriori error estimate, mesh refinement and adaptivity, see and references therein. For the literature, interested readers can refer to the recent works [8, 15, 18, 21–23, 28, 36, 41, 42], the books [7, 11, 24, 27, 43, 44] and the references therein. Note that all the superconvergence results obtained in the above references are studied on regular meshes. There are few superconvergence results are obtained on anisotropic meshes, see [26,31,32,39,40] etc. The problems left to us is: *Do the superconvergence results of rectangular finite elements with arbitrary order still hold under anisotropic meshes?*

In this paper, we show that the interpolation operators of Lagrange rectangular elements obtained by orthogonal expansions satisfy the anisotropic properties, and prove their anisotropic error estimates *on arbitrary rectangular meshes.* The analysis covers the Intermediate families, Tensor-product families and Serendipity families in a unified way. Furthermore, the superapproximation results between the interpolations and the finite element solutions are obtained for the Intermediate families, Tensor-product families on *arbitrary rectangular meshes.*

Lastly, we recall some notations and terminology (or refer to [10, 16]). For a bounded Lipschitz domain $\Omega \subset R^2$, let $(\cdot,\cdot)$ denote the usual L^2 -inner product and $\|u\|_{r,p,\Omega}$ (resp. $|u|_{r,p,\Omega}$) be the usual norm (resp. semi-norm) for the Sobolev space $W^{r,p}(\Omega)$, which are defined as

$$\|v\|_{m,p,\Omega}=\left(\int_\Omega\sum_{|\alpha|\le m}|D^\alpha v|^p\right)^{\frac{1}{p}},\quad |v|_{m,p,\Omega}=\left(\int_\Omega\sum_{|\alpha|=m}|D^\alpha v|^p\right)^{\frac{1}{p}}.$$

When $p=2$, denote $W^{2,r}(\Omega)$ by $H^r(\Omega)$. We shall also denote by $P_l(G)$ the space of polynomials on G of degrees no more than l. Throughout this paper, C is a positive constant independent of the mesh diameter, $\frac{h_K}{\rho_K}$ and the function considered.

2. Anisotropic Interpolations via Orthogonal Expansions

In this paper, we consider the following large family of rectangular elements defined on the reference square $\widehat{K}=\{\widehat{x}=(\xi,\eta)^T\in\mathbb{R}^2:\ -1<\xi,\eta<1\}$:

$$\mathcal{Q}_m(k)=\mathrm{Span}\{\xi^i\eta^j:(i,j)\in I_{k,m},1\le m\le k\},\tag{2.1}$$

$I_{k,m}$ is an index set satisfies

$$\{(i,j)|0\le i,j\le k,i+j\le m+k\}\subset I_{k,m}\subset\{(i,j)|0\le i,j\le k\}.\tag{2.2}$$

Obviously, the above family consists of all the possible shape spaces of rectangular elements. Based on [7], it can be divided into three groups:

i) Tensor-product families $\mathcal{Q}_2(k), k \geq 2$, it consists of the special case bi-$k-$th order finite element space $\mathcal{Q}_k(k)$;

ii) Intermediate families $\mathcal{Q}_1(k)$;

iii) Serendipity families, the shape space is between P_k and $\mathcal{Q}_1(k)$ with its index set

$$\mathcal{S}^k = \left\{ \widehat{p} | \widehat{p}(\xi,\eta) = \sum_{0\leq i+j\leq k} \alpha_{i,j}\xi^i\eta^j + \alpha_{k,1}\xi^k\eta + \alpha_{1,k}\xi\eta^k \right\}.$$

Taking the third order ($k = 3$) rectangular finite elements as an example, there are four different elements based on its structure (cf. [9]), i.e.,

$$\begin{aligned}
&\mathcal{Q}_3(3) = P_3(\widehat{K}) \oplus \mathrm{Span}\{\xi\eta^3, \xi^3\eta, \xi^2\eta^2, \xi^2\eta^3, \xi^3\eta^2, \xi^3\eta^3\},\\
&\mathcal{Q}_2(3) = P_3(\widehat{K}) \oplus \mathrm{Span}\{\xi\eta^3, \xi^3\eta, \xi^2\eta^2, \xi^2\eta^3, \xi^3\eta^2\},\\
&\mathcal{Q}_1(3) = P_3(\widehat{K}) \oplus \mathrm{Span}\{\xi\eta^3, \xi^3\eta, \xi^2\eta^2\},\\
&\mathcal{S}^3 = P_3(\widehat{K}) \oplus \mathrm{Span}\{\xi\eta^3, \xi^3\eta, \xi^2\eta^2\}.
\end{aligned} \tag{2.3}$$

The corresponding node type degree of freedoms $\mathcal{N}$ of them are illustrated in Fig. 2.1 and Fig. 2.2.

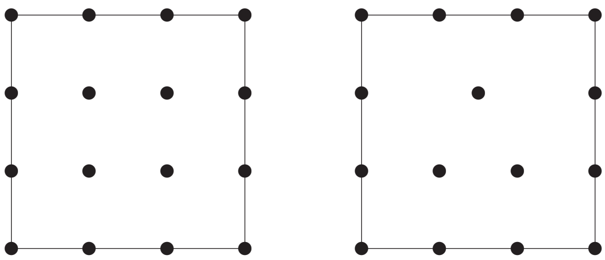

Fig. 2.1. The element $\mathcal{Q}_3(3)$ (left) and $\mathcal{Q}_2(3)$ (right).

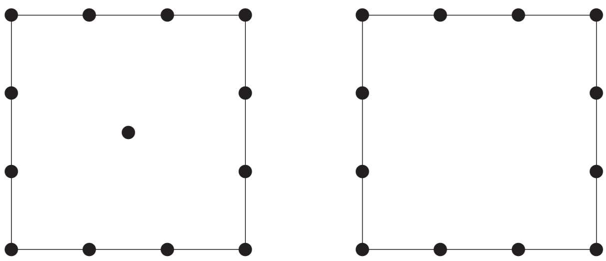

Fig. 2.2. The element $\mathcal{Q}_1(3)$ (left) and $\mathcal{S}^3$ (right).

If k is bigger, it will contain more finite elements with the same order. This phenomena is quite different from the triangular part. Note that the anisotropic interpolation error estimates for triangular finite elements with arbitrary order have been proved in [13, 35]. Concerning the rectangular finite elements, only anisotropic interpolation error estimates for a few tensor product elements of $\mathcal{Q}_k(k)$ have been obtained in the literatures.

For our subsequent analysis, we introduce the Legendre orthogonal polynomials defined on the interval $E=[-1,1]$

$$L_n(t)=c_1(n)\frac{d^n(t^2-1)^n}{dt},\quad c_1(n)=\frac{1}{2^n n!},\quad n=0,1,2,\cdots, \tag{2.4}$$

which has n vanishing points on E, which is the so-called Gauss points. On the two end-points of E, we have

$$L_n(\pm 1)=(\pm 1)^n,$$

see e.g., [29, 37].

Integral of the Legendre polynomials on $(-1,t)$ yields the so-called Lobatto polynomials

$$\phi_{n+1}(t)=\int_{-1}^{t}L_n(t)dt=c_1(n)\frac{d^{n-1}(t^2-1)^n}{dt},\quad n=1,2,\cdots \tag{2.5}$$

with $\phi_0(t)=1$ and $\phi_n(\pm 1)=0$ for $n\geq 2$.

It is known that $\phi_n(t)$ is a $n-th$ order polynomial and has n different zero points on E, which is also called as Lobatto points. Lobatto polynomials have the following quasi-orthogonal property:

$$\int_E \phi_m(t)\phi_n(t)dt=\begin{cases}\neq 0, & \text{if}\ \ m-n=0,\pm 2,\\ 0, & \text{else}\ \ m,n.\end{cases} \tag{2.6}$$

Consider a suitable smooth function u, we do an orthogonal expansion by Legendre polynomials for $\frac{du}{dt}$,

$$\frac{du}{dt}=\sum_{j=1}^{\infty}b_jL_{j-1}(t), \tag{2.7}$$

with

$$b_j=(j-\frac{1}{2})\int_E\frac{du}{dt}L_{j-1}(t)dt,\quad j=1,2,\cdots.$$

Integrating the both hand sides of (2.7) on $(-1,t)$, we get

$$u=b_0+\sum_{j=0}^{\infty}b_{j+1}\phi_{j+1}(t).$$

Now we can get a n order polynomial expansion of u, which is denoted by

$$u_n=\sum_{j=0}^{n}b_j\phi_j(t). \tag{2.8}$$

In order to determine the values of b_0, we assume u is C^0 continuous on the end-points $t=\pm 1$, then

$$u(1)=b_0+b_1,\quad u(-1)=b_0-b_1,$$

thus we can derive

$$b_0=\frac{1}{2}(u(1)+u(-1)),\quad b_1=\frac{1}{2}\int_E\frac{du}{dt}L_0(t)\,dt=\frac{1}{2}(u(1)-u(-1)).$$

Now we are in the position to present the anisotropic interpolation operators. For a suitable smooth function u, we can get an expansion with respect to the variable ξ by fixing the variable η on the reference element,

$$\widehat{u}(\xi,\eta)=\sum_{i=0}^{k+1} b_i(\eta)\phi_i(\xi)+R_{k+1}(\xi,\eta), \tag{2.9}$$

where

$$b_0(\eta)=\frac{\widehat{u}(1,\eta)+\widehat{u}(-1,\eta)}{2}, \qquad b_1(\eta)=\frac{\widehat{u}(1,\eta)-\widehat{u}(-1,\eta)}{2},$$
$$b_i(\eta)=(i-\frac{1}{2})\int_{-1}^{1}\frac{\partial\widehat{u}}{\partial\xi}L_i(\xi)d\xi, \quad i\geq 1, \quad R_{k+1}(\xi,\eta)=\sum_{i=k+2}^{\infty} b_i(\eta)\phi_i(\xi).$$

In the next step, by using of the Lobatto polynomials expansion of $b_i(\eta)$, we get

$$\begin{aligned} b_0(\eta)&=\sum_{j=0}^{k+1} b_{0,j}\phi_j(\eta)+R_{0,k+1}(\eta),\\ b_1(\eta)&=\sum_{j=0}^{k} b_{1,j}\phi_j(\eta)+R_{1,k}(\eta),\\ b_i(\eta)&=\sum_{(i,j)\in I_{k,m}} b_{i,j}\phi_j(\eta)+R_{i,k+m-i}(\eta), \quad 2\leq i\leq k, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} b_{0,0}&=\frac{\widehat{u}_1+\widehat{u}_2+\widehat{u}_3+\widehat{u}_4}{4},\\ b_{0,j}&=\frac{2j-1}{4}\int_{-1}^{1}\left(\frac{\partial\widehat{u}(1,\eta)}{\partial\eta}+\frac{\partial\widehat{u}(-1,\eta)}{\partial\eta}\right)L_{j-1}(\eta)d\eta, \quad j\geq 1,\\ b_{1,j}&=\frac{2j-1}{4}\int_{\widehat{K}}\frac{\partial^2\widehat{u}(\xi,\eta)}{\partial\xi\partial\eta}L_{j-1}(\eta)d\xi d\eta, \quad j\geq 1,\\ b_{i,0}&=\frac{2i-1}{4}\int_{-1}^{1}\left(\frac{\partial\widehat{u}(\xi,1)}{\partial\xi}+\frac{\partial\widehat{u}(\xi,-1)}{\partial\xi}\right)L_{i-1}(\xi)d\xi, \quad i\geq 1,\\ b_{i,1}&=\frac{2i-1}{4}\int_{\widehat{K}}\frac{\partial^2\widehat{u}(\xi,\eta)}{\partial\xi\partial\eta}L_{i-1}(\xi)d\xi d\eta, \quad i\geq 1,\\ b_{i,j}&=\frac{(2i-1)(2j-1)}{4}\int_{\widehat{K}}\frac{\partial^2\widehat{u}(\xi,\eta)}{\partial\xi\partial\eta}L_{i-1}(\xi)L_{j-1}(\eta)d\xi d\eta, \quad i,j\geq 1, \end{aligned} \tag{2.11}$$

where $\widehat{u}_i, i=1,2,3,4$, is the function values of $\widehat{u}$ on the four vertices. Then we can define the anisotropic interpolation on the reference element by

$$\widehat{\Pi}^k\widehat{u}=\sum_{(i,j)\in I_{k,m}} b_{i,j}(\widehat{u})\phi_i(\xi)\phi_j(\eta)\in\mathcal{Q}_m(k), \tag{2.12}$$

where we write $b_{i,j}(\widehat{u})=b_{i,j}$ for clarity that it is a functional of $\widehat{u}$.

It can be checked that the interpolated function constructed as above on the general rectangular mesh is a C^0 continuous function. In fact, we only need to show that the value of the above interpolated function on each edge of the elements only depends on the degrees of freedom of this edge. For simplicity, we only prove this fact on the edge $\eta=1$. Since

$$\widehat{\Pi}\widehat{u}(\xi,1)=\frac{1}{2}(\widehat{u}(1,1)+\widehat{u}(-1,1))+\sum_{i=1}^{k}\left(i-\frac{1}{2}\right)\phi_i(\xi)\int_{-1}^{1}\frac{\partial\widehat{u}(\xi,1)}{\partial\xi}L_{i-1}(\xi)d\xi,$$

is a $k-$th order polynomial with respect to the variable ξ, it can be seen easily that the function $\widehat{\Pi}\widehat{u}(\xi,1)$ is only depends on the nodal values defined on the two endpoints of this edge and the tangential derivative along the same edge between the two adjacent elements, which ensures the continuity of $\widehat{\Pi}\widehat{u}(\xi,1)$. Moreover, since $\phi_k(\pm1)=0$ for the case $k\geq 2$, we have $\widehat{\Pi}\widehat{u}(\widehat{a}_i)=\widehat{u}(\widehat{a}_i), i=1,2,3,4$, which means that it is continuous on the four vertexes.

3. Convergence Analysis on Arbitrary Rectangular Meshes

For the sake of convenience, let $\Omega\subset R^2$ be a convex polygon composed by a family of rectangular meshes $\mathcal{J}_h$ which need not satisfy the regularity assumptions. $\forall K\in\mathcal{J}_h$, denote the barycenter of element K by (x_K,y_K), the length of edges parallel to x-axis and y-axis by $2h_{K1},2h_{K2}$ respectively, $h_K=\max\{h_{K1},h_{K2}\}, h=\max_{K\in\mathcal{J}_h}h_K$, $\alpha=(\alpha_1,\alpha_2)$, $h_K^\alpha=h_{K1}^{\alpha_1}h_{K2}^{\alpha_2}$. There exists a unique mapping $F_K:\widehat{K}\longrightarrow K$ defined as

$$\begin{cases} x=x_K+h_{K1}\xi,\\ y=y_K+h_{K2}\eta. \end{cases} \tag{3.1}$$

From now on, we fix the integer $k\geq 1$ and define the $k-$th order rectangular Lagrange finite element space:

$$V_h^k=\{v_h^k\in C^0(\overline{\Omega});\quad v_h^k|_K\circ F_K\in\mathcal{Q}_m(k),\quad \forall K\in\mathcal{J}_h,\quad v_h^k|_{\partial\Omega}=0\}. \tag{3.2}$$

Define the interpolation operator $\Pi_h^k:C^0(\Omega)\longrightarrow V_h^k$ as

$$\Pi_h^k|_K=\Pi_K,\quad \Pi_K^k=\widehat{\Pi}^k\circ F_K^{-1},\quad \forall K\in\mathcal{J}_h.$$

We consider the following general second-order elliptic boundary value problem

$$\begin{cases} -\Delta u=f, & \text{in}\quad \Omega,\\ \quad u=0, & \text{on}\quad \partial\Omega, \end{cases} \tag{3.3}$$

with $f\in L^2(\Omega)$. Let $V=H_0^1(\Omega)$, then the weak form of (3.3) is:

$$\begin{cases} \text{Find } u\in V,\quad \text{such that}\\ a(u,v)=f(v),\quad \forall\ v\in V, \end{cases} \tag{3.4}$$

where

$$a(u,v)=\int_\Omega \nabla u\nabla v\,dxdy,\qquad f(v)=\int_\Omega fv\,dxdy.$$

The finite element approximation of (3.4) reads

$$\begin{cases} \text{Find } u_h^k\in V_h^k,\ \text{ such that}\\ a(u_h^k,v_h^k)=f(v_h^k),\quad \forall\ v_h^k\in V_h^k. \end{cases} \tag{3.5}$$

To analyze its convergence on anisotropic meshes, we first show that the interpolation operator defined in the last section satisfies the following anisotropic interpolation property.

Theorem 3.1. *Suppose $\alpha = (\alpha_1, \alpha_2) \in I_{k,m}$ is a multi-index with $|\alpha| > 0$ satisfies $0 \le |\alpha| < k+1$ and $\widehat{u} \in C^0(\widehat{K})$ satisfies $\widehat{D}^\alpha \widehat{u} \in H^{k+1-|\alpha|}(\widehat{K})$. Then there exists a generic constant $C > 0$ such that*

$$\|\widehat{D}^\alpha(\widehat{u} - \widehat{\Pi}^k \widehat{u})\|_{0,\widehat{K}} \le C|\widehat{D}^\alpha \widehat{u}|_{k+1-|\alpha|,\widehat{K}}. \tag{3.6}$$

If $|\alpha| \ge k+1$, then

$$\|\widehat{D}^\alpha(\widehat{u} - \widehat{\Pi}^k \widehat{u})\|_{0,\widehat{K}} \le C \left(\sum_{\beta \in D^\alpha I_{k,m}} \|\widehat{D}^{\alpha+\beta} \widehat{u}\|^2_{0,\widehat{K}} \right)^{\frac{1}{2}}, \tag{3.7}$$

where $D^\alpha I_{k,m} := \{(i-\alpha_1, j-\alpha_2) | (i,j) \in) I_{k,m}\}$.

Proof. We first consider the case $|\alpha| < k+1$. By (2.12), there holds

$$\widehat{D}^\alpha \widehat{\Pi}^k \widehat{u} = \frac{\partial^n \widehat{\Pi}^k \widehat{u}}{\partial \xi^{\alpha_1} \partial \eta^{\alpha_2}} = \sum_{(i,j) \in I_{k,m},\, i \ge \alpha_1, j \ge \alpha_2} b_{i,j}(\widehat{u}) \frac{d^{\alpha_1} \phi_i(\xi)}{d\xi^{\alpha_1}} \frac{d^{\alpha_2} \phi_j(\eta)}{d\eta^{\alpha_2}}. \tag{3.8}$$

In the following, we need to give a detailed estimate of the term $b_{i,j}$. Concerning the case $\alpha_1 = 0$, $\widehat{D}^\alpha \widehat{\Pi} \widehat{u}$ only consists of $b_{0,j}, j \ge \alpha_2$. Integrating it by parts and noticing that $\frac{d^{\alpha_2}(t^2-1)^{\alpha_1}}{dt^{\alpha_2}}|_{\pm 1} = 0$ when $\alpha_2 < \alpha_1$, we can derive

$$\begin{aligned}
|b_{0,j}(\widehat{u})| &= \left| \frac{c_1(j-1)(2j-1)}{4} \int_{-1}^{1} \left(\frac{\partial \widehat{u}(1,\eta)}{\partial \eta} + \frac{\partial \widehat{u}(-1,\eta)}{\partial \eta} \right) \frac{d^{j-1}(\eta^2-1)^{j-1}}{d\eta^{j-1}} d\eta \right| \\
&= \left| \frac{c_1(j-1)(2j-1)}{4} \int_{-1}^{1} \left(\frac{\partial^{\alpha_2} \widehat{u}(1,\eta)}{\partial \eta^{\alpha_2}} + \frac{\partial^{\alpha_2} \widehat{u}(-1,\eta)}{\partial \eta^{\alpha_2}} \right) \frac{d^{j-\alpha_2}(\eta^2-1)^{j-1}}{d\eta^{j-\alpha_2}} d\eta \right| \\
&\le C \left\| \frac{\partial^{\alpha_2} \widehat{u}}{\partial \eta^{\alpha_2}} \right\|_{0,\frac{1}{2},\partial \widehat{K}} \le C \left\| \frac{\partial^{\alpha_2} \widehat{u}}{\partial \eta^{\alpha_2}} \right\|_{k+1-|\alpha|,\widehat{K}} \\
&= C \|\widehat{D}^\alpha \widehat{u}\|_{k+1-|\alpha|,\widehat{K}},
\end{aligned} \tag{3.9}$$

where we have used the Cauchy-Schwarz inequality and the trace theorem [17, Theorem I.1.5].

The case $\alpha_2 = 0$ can be treated similarly. For the case $\alpha_1 \ne 0, \alpha_2 \ne 0$, we have

$$\begin{aligned}
|b_{i,j}(\widehat{u})| &= \left| \frac{c_1(i-1)c_1(j-1)(2i-1)(2j-1)}{4} \right. \\
&\qquad \left. \int_{\widehat{K}} \frac{\partial^2 \widehat{u}}{\partial \xi \partial \eta} \frac{d^{i-1}(\xi^2-1)^{i-1}}{d\xi^{i-1}} \frac{d^{j-1}(\eta^2-1)^{j-1}}{d\eta^{j-1}} d\xi d\eta \right| \\
&= \left| \frac{c_1(i-1)c_1(j-1)(2i-1)(2j-1)}{4} \right. \\
&\qquad \left. \int_{\widehat{K}} \frac{\partial^n \widehat{u}}{\partial \xi^{\alpha_1} \partial \eta^{\alpha_2}} \frac{d^{i-\alpha_1}(\xi^2-1)^{i-1}}{d\xi^{i-\alpha_1}} \frac{d^{j-\alpha_2}(\eta^2-1)^{j-1}}{d\eta^{j-\alpha_2}} d\xi d\eta \right| \\
&\le C \left\| \frac{\partial^{|\alpha|} \widehat{u}}{\partial \xi^{\alpha_1} \partial \eta^{\alpha_2}} \right\|_{k+1-|\alpha|,\widehat{K}} = C \|\widehat{D}^\alpha \widehat{u}\|_{k+1-|\alpha|,\widehat{K}}.
\end{aligned} \tag{3.10}$$

Then (3.6) can be obtained by the anisotropic interpolation theorem [4, 12].

Concerning (3.7), since $|\alpha| \ge k+1$, then we only need to treat the term $b_{i,j}$ with $\alpha_1 \ne 0, \alpha_2 \ne 0$. In a similar way as (3.10), (3.7) can be proved by minor adaption. $\square$

Remark 3.1. In fact, the proof in the above theorem using a similar argument as in [2,4,13,14], but only the bi$-k$ tensor product polynomial space and the case $|\alpha| \le k+1$ are studied in the above references.

Then we can derive the anisotropic interpolation results by using of Theorem 3.1.

Theorem 3.2. *Suppose $\mathcal{J}_h$ is an* ***arbitrary rectangular mesh****, then under the same hypothesises of Theorem 3.1, if $n < k+1$, we have*

$$|u - \Pi_h^k u|_{n,\Omega} \le C \left(\sum_{K\in\mathcal{J}_h} \sum_{|\beta|=k+1-n} h_K^{2\beta} |D^\beta u|_{n,K}^2 \right)^{\frac{1}{2}}. \tag{3.11}$$

Otherwise, if $n \ge k+1$, we have

$$|u - \Pi_h^k u|_{n,\Omega} \le C \left(\sum_{K\in\mathcal{J}_h} \sum_{\beta\in D^\alpha I_{k,m}} h_K^{2\beta} |D^\beta u|_{n,K}^2 \right)^{\frac{1}{2}}. \tag{3.12}$$

Proof. By (3.6), we have

$$\begin{aligned}
|u - \Pi_h^k u|_{n,\Omega} &= \left(\sum_{K\in\mathcal{J}_h} |u - \Pi_K^k u|_{n,K}^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{K\in\mathcal{J}_h} \sum_{|\alpha|=n} \|D^\alpha (u - \Pi_K^k u)\|_{0,K}^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{K\in\mathcal{J}_h} \sum_{|\alpha|=n} h_K^{-2\alpha} (h_{K1} h_{K2}) \|\widehat{D}^\alpha (\widehat{u} - \widehat{\Pi}^k \widehat{u})\|_{0,\widehat{K}}^2 \right)^{\frac{1}{2}} \\
&\le C \left(\sum_{K\in\mathcal{J}_h} \sum_{|\alpha|=n} h_K^{-2\alpha} (h_{K1} h_{K2}) |\widehat{D}^\alpha \widehat{u}|_{k+1-n,\widehat{K}}^2 \right)^{\frac{1}{2}} \\
&\le C \left(\sum_{K\in\mathcal{J}_h} \sum_{|\alpha|=n} \sum_{|\beta|=k+1-n} h_K^{2\beta} \|D^{\alpha+\beta} u\|_{0,K}^2 \right)^{\frac{1}{2}},
\end{aligned}$$

which implied the desired assertion (3.11). Concerning (3.12), it can be proved by a similar argument. $\square$

Remark 3.2. Theorems 3.1 and 3.2 show that the anisotropic property and interpolation error estimates of rectangular elements hold for any order derivative, which seems to be more impressive than that in the existing references, cf. [4, 5, 35].

Remark 3.3. In fact, it is possible to derive an explicit bounds of the constant C in the right-hand side of Theorem 3.2 by using of an explicit estimate of the expansion remainders, which is not the main scope in this paper and is omitted here.

The convergence results of the finite element methods is the following theorem.

Theorem 3.3. *Suppose* $u \in H^{k+1}(\Omega)$ *and* u_h^k *is the unique solution of* (3.4) *and* (3.5), *respectively, then under* arbitrary rectangular mesh, *we have*

$$|u - u_h^k|_{1,\Omega} \le Ch^k |u|_{k+1,\Omega}, \tag{3.13}$$
$$\|u - u_h^k\|_{0,\Omega} \le Ch^{k+1} |u|_{k+1,\Omega}. \tag{3.14}$$

Proof. According to the Cea's Lemma (cf. [10,16]) we have

$$|u - u_h^k|_{1,\Omega} \le C \inf_{v_h^k \in V_h^k} |u - v_h^k|_{1,\Omega} \le C|u - \Pi_h^k u|_{1,\Omega},$$

then the energy norm error estimate (3.13) can be obtained easily by applying the anisotropic interpolation error estimate (3.11) with $n = 1$. Concerning the L^2 norm error estimate (3.14), it can be proved by the usual Aubin-Nitsche-duality argument. Since the process is standard, we omit them. □

4. Superconvergence Analysis on Arbitrary Rectangular Meshes

In this section, we will show the superapproximation result of the Lagrange rectangular elements under arbitrary rectangular mesh. In fact, superconvergence result of some Lagrange rectangular elements on anisotropic meshes is not new in the literature. Superconvegence analysis of some low order tensor-product rectangular elements have been done in [25,26,39]. However, our analysis in the following will cover the tensor-product families and intermediate families of rectangular elements with arbitrary order, based on the anisotropic properties of the interpolations proved in the last section. To start with, we prove the following weak superconvergent estimates.

Lemma 4.1. *Suppose* $u \in H^{k+2}(\Omega) \cap H_0^1(\Omega)$, *then for the* $k-th$ *order interpolated function* $\Pi_h^k u$ *and* $\forall v_h^k \in V_h^k$, *we have the following superconvergent weak estimates under arbitrary rectangular meshes*

$$\left| \int_K \frac{\partial(u - \Pi_h^k u)}{\partial x} \frac{\partial v_h^k}{\partial x}\, dxdy \right| \le C \left(\sum_{|\beta|=k+1} h_K^{2\beta} |D^\beta u|_{1,K}^2 \right)^{\frac{1}{2}} \left| v_h^k \right|_{1,K}, \quad \forall K \in \mathcal{J}_h, \tag{4.1}$$

$$\left| \int_K \frac{\partial(u - \Pi_h^k u)}{\partial y} \frac{\partial v_h^k}{\partial y}\, dxdy \right| \le C \left(\sum_{|\beta|=k+1} h_K^{2\beta} |D^\beta u|_{1,K}^2 \right)^{\frac{1}{2}} \left| v_h^k \right|_{1,K}, \quad \forall K \in \mathcal{J}_h. \tag{4.2}$$

Proof. We only need to prove (4.1) and (4.2) can be proved in a similar way because of the symmetric structure of the finite element space.

A scaling argument of the left hand side of (4.1), we get

$$\int_K \frac{\partial(u - \Pi_h^k u)}{\partial x} \frac{\partial v_h^k}{\partial x}\, dxdy = \frac{h_{K1}}{4h_{K2}} \int_{\widehat{K}} \frac{\partial(\widehat{u} - \widehat{\Pi}^k \widehat{u})}{\partial \xi} \frac{\partial \widehat{v_h^k}}{\partial \xi}\, d\xi d\eta. \tag{4.3}$$

Without loss of generality, we may assume

$$\widehat{v_h^k} = \sum_{(i,j)\in I_{k,m}} c_{i,j} \phi_i(\xi)\phi_j(\eta) \in \mathcal{Q}_m(k,),$$

where $c_{i,j}$ is a functional of $\widehat{v_h^k}$, then we have

$$\frac{\partial \widehat{\Pi}^k \widehat{u}}{\partial \xi} = \sum_{(i,j)\in I_{k,m},\, i\geq 1} b_{i,j}(\widehat{u})\phi_i(\xi)\phi_j(\eta) = \sum_{(i,j)\in I_{k,m},\, i\geq 1} b_{i,j}(\widehat{u})L_{i-1}(\xi)\phi_j(\eta), \tag{4.4}$$

$$\frac{\partial \widehat{v_h^k}}{\partial \xi} = \sum_{(i,j)\in I_{k,m},\, i\geq 1} c_{i,j}L_{i-1}(\xi)\phi_j(\eta). \tag{4.5}$$

From the definition of $b_{i,j}$ in (2.11), we know that it is a functional of $\frac{\partial \widehat{u}}{\partial \xi}$. Setting $\widehat{W} = \frac{\partial \widehat{u}}{\partial \xi}$, then $\frac{\partial \widehat{\Pi}^k \widehat{u}}{\partial \xi}$ is a functional of $\widehat{W}$ and can be written as

$$\frac{\partial \widehat{\Pi}^k \widehat{u}}{\partial \xi} = \widehat{F}_1(\widehat{W}) = \sum_{(i,j)\in I_{k,m},\, i\geq 1} d_{i,j}(\widehat{W})L_{i-1}(\xi)\phi_j(\eta), \tag{4.6}$$

with

$$\begin{aligned} d_{1,0}(\widehat{W}) &= \frac{1}{4}\int_{-1}^{1}\Big(\widehat{W}(\xi,1)+\widehat{W}(\xi,-1)\Big)\,d\xi, \\ d_{i,j}(\widehat{W}) &= \frac{(2i-1)(2j-1)}{4}\int_{\widehat{K}} \frac{\partial \widehat{W}(\xi,\eta)}{\partial \eta} L_{i-1}(\xi)L_{j-1}(\eta)d\xi d\eta, \quad i,j\geq 1. \end{aligned} \tag{4.7}$$

In fact, $\widehat{F}_1(\widehat{W})$ can be regarded as a functional of $\widehat{W}$ on $H^1(\widehat{K})$ by the trace theorem, i.e., it satisfies that

$$|\widehat{F}_1(\widehat{W})| \leq C\|\widehat{W}\|_{1,\widehat{K}}. \tag{4.8}$$

Since

$$\widehat{u} - \widehat{\Pi}^k\widehat{u} = 0, \quad \forall \widehat{u}\in \mathcal{Q}_m(k),$$

with $m\geq 1$, we have

$$\widehat{W} - \widehat{F}_1(\widehat{W}) = 0, \quad \forall P_k(\widehat{K})\setminus\{L_k(\xi)\}.$$

In a further step, it can be checked easily that

$$\widehat{F}_1(\phi_k(\xi)) = \frac{1}{2}\int_{-1}^{1} L_k(\xi)d\xi = 0,$$

then by the orthogonal property of Legendre polynomials, we derive that

$$\int_{\widehat{K}} (\widehat{W} - \widehat{F}_1(\widehat{W}))\frac{\partial \widehat{v_h^k}}{\partial \xi}\, d\xi d\eta = 0, \tag{4.9}$$

when $\widehat{W} = L_k(\xi)$ and so (4.9) holds for all $\widehat{W}\in P_k(\widehat{K})$.

We define the functional

$$B_1\left(\widehat{W}, \frac{\partial \widehat{v_h^k}}{\partial \xi}\right) = \int_{\widehat{K}} (\widehat{W} - \widehat{F}_1(\widehat{W}))\frac{\partial \widehat{v_h^k}}{\partial \xi}\, d\xi d\eta. \tag{4.10}$$

It can be checked easily that

$$\left| B_1\left(\widehat{W}, \frac{\partial \widehat{v_h^k}}{\partial \xi}\right)\right| \leq C\,\|\widehat{W}\|_{1,\widehat{K}} \left\| \frac{\partial \widehat{v_h^k}}{\partial \xi}\right\|_{0,\widehat{K}}. \tag{4.11}$$

An application of the Bramble-Hilbert lemma, there exists a constant C only depends on $\widehat{K}$ such that

$$\begin{aligned}
\left|B_1\left(\widehat{W}, \frac{\partial \widehat{v_h^k}}{\partial \xi}\right)\right| &\le C \inf_{p\in P_k(\widehat{K})} \|\widehat{W}+p\|_{1,\widehat{K}} \left\|\frac{\partial \widehat{v_h^k}}{\partial \xi}\right\|_{0,\widehat{K}} \\
&\le C\left(\sum_{|\beta|=k+1} \|\widehat{D}^\beta \widehat{W}\|_{0,\widehat{K}}^2\right)^{\frac{1}{2}} \left\|\frac{\partial \widehat{v_h^k}}{\partial \xi}\right\|_{0,\widehat{K}} \\
&= C\left(\sum_{|\beta|=k+1} \left\|\widehat{D}^\beta \frac{\partial \widehat{u}}{\partial \xi}\right\|_{0,\widehat{K}}^2\right)^{\frac{1}{2}} \left\|\frac{\partial \widehat{v_h^k}}{\partial \xi}\right\|_{0,\widehat{K}},
\end{aligned} \tag{4.12}$$

which, together with the scaling argument, yields that

$$\begin{aligned}
\left|\int_{\widehat{K}} \frac{\partial(\widehat{u}-\widehat{\Pi}^k\widehat{u})}{\partial \xi} \frac{\partial \widehat{v_h^k}}{\partial \xi} d\xi d\eta\right| &= \left|B_1\left(\widehat{W}, \frac{\partial \widehat{v_h^k}}{\partial \xi}\right)\right| \\
&\le C\frac{h_{K2}}{h_{K1}}\left(\sum_{|\beta|=k+1} h_K^{2\beta}\left\|D^\beta \frac{\partial u}{\partial x}\right\|_{0,K}^2\right)^{\frac{1}{2}} \left\|\frac{\partial v_h^k}{\partial x}\right\|_{0,K} \\
&\le C\frac{h_{K2}}{h_{K1}}\left(\sum_{|\beta|=k+1} h_K^{2\beta}|D^\beta u|_{1,K}^2\right)^{\frac{1}{2}} \left|v_h^k\right|_{1,K}.
\end{aligned} \tag{4.13}$$

Then the desired result (4.1) follows from (4.3) and (4.12). The proof is completed. $\square$

Based on Lemma 4.1, we can prove the following anisotropic superclose result.

Theorem 4.1. *Suppose* $u \in H^{k+2}(\Omega)\cap H_0^1(\Omega)$ *and* $u_h^k \in V_h^k$ *are the solutions of* (3.4) *and* (3.5)*, respectively, under the* arbitrary rectangular meshes, *there holds*

$$|\Pi_h^k u - u_h^k|_{1,\Omega} \le C\left(\sum_{|\beta|=k+1} h_K^{2\beta}|D^\beta u|_{1,K}^2\right)^{\frac{1}{2}}. \tag{4.14}$$

Proof. Since $V_h^k \subset V$, then by (3.4) and (3.5), we get

$$a(u-u_h^k, v_h^k) = 0, \quad \forall \ \ v_h^k \in V_h^k, \tag{4.15}$$

which, together with the fact $\Pi_h^k u - u_h^k \in V_h^k$ and an application of Lemma 4.1 yields

$$\begin{aligned}
|\Pi_h^k u - u_h^k|_{1,\Omega}^2 &= a(\Pi_h^k u - u_h^k, \Pi_h^k u - u_h^k) \\
&= a(\Pi_h^k u - u, \Pi_h^k u - u_h^k) \\
&= \sum_{K\in\mathcal{J}_h}\left(\int_K \frac{\partial(u-\Pi_h^k u)}{\partial x}\frac{\partial(\Pi_h^k u - u_h^k)}{\partial x} dxdy \right. \\
&\qquad \left. + \int_K \frac{\partial(u-\Pi_h^k u)}{\partial y}\frac{\partial(\Pi_h^k u - u_h^k)}{\partial y} dxdy\right) \\
&\le C\sum_{K\in\mathcal{J}_h}\left(\sum_{|\beta|=k+1} h_K^{2\beta}|D^\beta u|_{1,K}^2\right)^{\frac{1}{2}} \left|\Pi_h^k u - u_h^k\right|_{1,K}
\end{aligned}$$

$$\leq C\sum_{K\in\mathcal{J}_h}\left(\sum_{|\beta|=k+1}h_K^{2\beta}|D^\beta u|_{1,K}^2\right)^{\frac{1}{2}}\left|\Pi_h^k u-u_h^k\right|_{1,\Omega}. \tag{4.16}$$

Then (4.14) follows from (4.16). □

Remark 4.1. The superconvergence result (4.14) is an anisotropic estimate in the following senses. Firstly, it dose ont impose any condition on the mesh and is valid on arbitrary rectangular mesh. The classical superconvergence order $O(h^{k+1})$ can be recovered easily from (4.14). Secondly, the convergence estimate is directional with respect to the two mesh diameters h_{K1} and h_{K2} and the different element scales are exploited.

5. Conclusions

In this paper, we give a unified analysis for the convergence of the large class of rectangular finite elements under arbitrary rectangular mesh. All the estimates in this work are obtained in anisotropic sense, which enrich the anisotropic error estimates for the finite elements of Intermediate families, Serendipity families and some of the tensor-product families. Furthermore, the anisotropic superapproximation results for the finite elements of tensor-product families and Intermediate families of arbitrary order are proved under arbitrary rectangular meshes. The global superconvergence results and the point superconvergent structures of these rectangular finite elements under anisotropic meshes are not addressed in this paper. These results, together with extensive numerical tests, will be reported in our future work.

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