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GRADED MESHES FOR HIGHER ORDER FEM*

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Abstract

A singularly perturbed one-dimensional convection-diffusion problem is solved numerically by the finite element method based on higher order polynomials. Numerical solutions are obtained using S-type meshes with special emphasis on meshes which are graded (based on a mesh generating function) in the fine mesh region. Error estimates in the ε-weighted energy norm are proved. We derive an 'optimal' mesh generating function in order to minimize the constant in the error estimate. Two layer-adapted meshes defined by a recursive formulae in the fine mesh region are also considered and a new technique for proving error estimates for these meshes is presented. The aim of the paper is to emphasize the importance of using optimal meshes for higher order finite element methods. Numerical experiments support all theoretical results.

Mathematics subject classification: 65L11, 65L50, 65L60, 65L70.
Key words: Singular perturbation, Boundary value problem, Layer-adapted meshes, Graded meshes, Finite element method.

1. Introduction

We consider the singularly perturbed boundary value problem

$$-\varepsilon u'' - b(x)u' + c(x)u = f(x), \quad u(0) = u(1) = 0, \tag{1.1}$$

where $0 < \varepsilon << 1$, $b(x) > 1$ for $x \in [0,1]$ (we avoid some additional parameter β from the assumption $b(x) > \beta > 0$). Let the functions b, c, f be sufficiently smooth. Because (1.1) is characterized by a boundary layer at $x = 0$, the Galerkin finite element method on a standard mesh works unsatisfactory. Therefore 20 years ago exponentialy fitted splines attracted many researchers (see Section 2.2.5 in Part I of [10]). But later it turned out that standard splines on layer-adapted meshes were better. Today layer adapted meshes are the main ingredient in handling problems with boundary or interior layers.

The first, uniformly with respect to ε, convergence result for *linear* finite elements on a Shishkin mesh in the ε-weighted H^1- norm was proved in [12] (for the precise definition of

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norms and meshes see Section 2). That mesh is equidistant and fine in a small interval $[0,\tau]$, where the choice of $\tau = \tau_0 \varepsilon \ln N$ is important, and equidistant but coarse in the large subinterval $[\tau, 1]$ with a mesh size of order $O(N^{-1})$. The resulting error estimate

$$\|u - u^N\|_\varepsilon \le C(N^{-1} \ln N)$$

(by C we denote a generic constant independent of ε) is not optimal due to the logarithmic factor.

In [7] the authors introduced the class of S-type meshes, where in the fine subinterval $[0,\tau]$ the mesh is not necessarily equidistant but graded based on a mesh generating function. Then, for certain mesh generating functions the optimal estimate

$$\|u - u^N\|_\varepsilon \le C N^{-1}$$

follows. While for linear elements the influence of the logarithmic factor is not that large, the expressions $(N^{-1} \ln N)^k$ and N^{-k} differ significantly for larger k. Therefore, for higher order finite element methods it is important to use a graded mesh and not a piecewise equidistant mesh. In Section 2 we generalize the results of [7] and prove for finite elements based on polynomials of order k

$$\|u - u^N\|_\varepsilon \le C(N^{-1} \max |\psi'|)^k.$$

Here ψ is the so-called mesh characterizing function. For some meshes (for instance, Bakhvalov-Shishkin or Vulanović meshes) it follows

$$\|u - u^N\|_\varepsilon \le C N^{-k}.$$

In Section 3 we use some heuristic arguments to optimize the mesh generating function, i.e. to minimize the constant in the error estimate. Although the analysis is incomplete the numerical results in Section 5 confirm the nice properties of the 'optimal' mesh generating function.

Layer-adapted meshes can also be defined by a *recursive* formula. Examples are the Gartland-Shishkin meshes [9] or meshes analysed in [2]. We modify these meshes and use the recursive formula only in the subinterval $[0, \tilde{\tau}]$, where $\tilde{\tau}$ is the smallest mesh point not smaller than τ generated by the recursive formula for the mesh points. In Section 4 we present a new technique to prove error estimates for these meshes. Finally, in Section 5 we present a detailed numerical study for the comparison of all these meshes, especially for the polynomials of degree $k = 2, 3$. The numerical results confirm the obtained estimates and demonstrate the important role of Bakhvalov-Shiskin and Gartland-Shishkin meshes.

2. S-type Meshes Based on Mesh Generating Functions

It is well known that for the problem (1.1) with smooth data the following solution decomposition into a smooth part S, and a layer part E exists [10]:

$$u = S + E,$$

with

$$|S^{(k)}| \le C, \quad |E^{(k)}| \le C\varepsilon^{-k} e^{-x/\varepsilon} \quad k = 0, 1, \dots, q \quad \text{(for any prescribed } q). \tag{2.1}$$

On a given mesh $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$, we denote by V^N the finite element space of continuous piecewise polynomials of degree k. The weak formulation of (1.1) is based on the bilinear form

$$a(w, v) := \varepsilon(w', v') + (-bw' + cw, v) \tag{2.2}$$

for $w, v \in H_0^1(0,1)$. We want to study the following finite element method:
Find $u^N \in V^N$ such that

$$a(u^N, v) = (f, v) \quad \text{for all } v \in V^N. \tag{2.3}$$

Assuming $b > 1$ we can always ensure $c + \frac{1}{2}b' > \alpha^* > 0$, therefore we have coercivity of the bilinear form with respect to the ε-weighted H^1 norm:

$$a(v,v) \geq \alpha \|v\|_\varepsilon^2 \quad \text{where} \quad \|v\|_\varepsilon^2 := \varepsilon |v|_1^2 + \|v\|_0^2$$

Here $\|\cdot\|_0$ denotes the L_2-norm and $|\cdot|_1$ the H^1-seminorm, later we shall also use the L_∞-norm $\|\cdot\|_\infty$ and the L_1-norm $\|\cdot\|_{L_1}$. As a consequence, the discrete problem (2.3) admits a unique solution.

We are interested in the case of dominating convection. Then, it is standard to assume

$$\varepsilon \leq CN^{-1}. \tag{2.4}$$

From this point on, C is a generic constant independent of both ε and N. S-type meshes are characterized by a fine mesh in the layer region and a coarse mesh away from the layer. We define the transition point from the fine to the coarse mesh by (see [7] for $k = 1$):

$$\tau = (k+1)\varepsilon \ln N. \tag{2.5}$$

Then $|E^{(k)}(\tau)| \leq C\varepsilon^{-k} N^{-(k+1)}$.

Here $\tau = x_{N/2}$ and on the interval $[0, x_{N/2}]$ the mesh is defined by a mesh generating function φ which has the following properties: $\varphi(0) = 0, \varphi(\frac{1}{2}) = \ln N, \varphi$ is continuous, monotonically increasing and piecewise continuously differentiable. On the interval $[x_{N/2}, 1]$, the mesh is equidistant. Thus we have

$$x_i = \begin{cases} (k+1)\varepsilon\varphi(t_i), & t_i = i/N,\ i = 0, 1, \ldots, N/2, \\ 1 - (1 - x_{N/2})\dfrac{2(N-i)}{N}, & i = N/2 + 1, \ldots, N. \end{cases} \tag{2.6}$$

Additionally, we define the mesh characterizing function ψ by

$$\varphi = -\ln \psi \quad \left(\Rightarrow\ \psi(t_i) = \exp\left(-\frac{x_i}{(k+1)\varepsilon} \right) \right).$$

The analysis of finite element methods on S-type meshes is based on the following:

Assumption H1

$$\max \varphi' \leq C\, N. \tag{2.7}$$

On the fine mesh, we have

$$h_i = x_i - x_{i-1} = (k+1)\varepsilon(\varphi(t_i) - \varphi(t_{i-1})) \leq (k+1)\varepsilon N^{-1} \max \varphi',$$

and thus

$$h_i \leq C\varepsilon \leq CN^{-1}. \tag{2.8}$$

Next we analyze the interpolation error on the fine mesh for the layer component E (for the smooth component, the analysis is standard).

Lemma 2.1. *Assuming H1* (2.7)*, we have*

$$\|E - E^I\|_{\infty,[0,x_{N/2}]} \le C(N^{-1} \max|\psi'|)^{k+1}, \tag{2.9}$$

$$\varepsilon^{1/2}|E - E^I|_{1,[0,x_{N/2}]} \le C(N^{-1} \max|\psi'|)^{k}. \tag{2.10}$$

Proof. In the L_∞-norm we obtain by standard interpolation error estimates

$$\|E - E^I\|_{\infty,[x_{i-1},x_i]} \le Ch_i^{k+1}\|E^{(k+1)}\|_{\infty,[x_{i-1},x_i]}$$

and with

$$h_i \le (k+1)\varepsilon N^{-1} \max \varphi' \le (k+1)\varepsilon N^{-1} \max|\psi'| e^{x_i/((k+1)\varepsilon)}$$

we have

$$\|E - E^I\|_{\infty,[x_{i-1},x_i]} \le C(N^{-1}\max|\psi'|)^{k+1} e^{x_i/\varepsilon - x_{i-1}/\varepsilon} \le C(N^{-1}\max|\psi'|)^{k+1}.$$

In the H^1-seminorm we get similarly

$$\begin{aligned} |E - E^I|^2_{1,[x_{i-1},x_i]} &\le Ch_i^{2k}\int_{x_{i-1}}^{x_i}(E^{(k+1)})^2 \le Ch_i^{2k}\varepsilon^{-(2k+1)}(e^{-2x_{i-1}/\varepsilon} - e^{-2x_i/\varepsilon}) \\ &\le C\varepsilon^{-1}(N^{-1}\max|\psi'|)^{2k} e^{\frac{2k}{k+1}\frac{x_i}{\varepsilon}} e^{-2x_{i-1/2}/\varepsilon}\sinh\frac{h_i}{\varepsilon}. \end{aligned}$$

Next we use

$$\sinh\frac{h_i}{\varepsilon} \le C\frac{h_i}{\varepsilon} = C\int_{t_{i-1}}^{t_i}\varphi' = C\int_{t_{i-1}}^{t_i}\left(-\frac{\psi'}{\psi}\right) \le Ce^{x_i/((k+1)\varepsilon)}\int_{t_{i-1}}^{t_i}(-\psi').$$

Based on

$$e^{x_i/(k+1)\varepsilon} e^{\frac{2k}{k+1}\frac{x_i}{\varepsilon}} e^{-2x_{i-1/2}/\varepsilon} \le C,$$

we obtain

$$\varepsilon|E - E^I|^2_{1,[0,x_{N/2}]} \le C(N^{-1}\max|\psi'|)^{2k},$$

because

$$\int_0^{x_{N/2}}(-\psi') = \psi(0) - \psi\left(\frac{1}{2}\right) \le 1.$$

Theorem 2.1. *Assuming H1, the error of our finite element method on an S-type mesh satisfies the uniform, with respect to* ε*, error estimate*

$$\|u - u^N\|_\varepsilon \le C(N^{-1}\max|\psi'|)^k.$$

Remark that Shishkin meshes with $\max|\psi'| = \mathcal{O}(\ln N)$ are not optimal. Numerically for increasing k the logarithmic factor gets more and more influence. A sufficient condition for an optimal mesh is $\max|\psi'| \le C$ and examples are

$$\psi(t) = 1 - 2(1 - N^{-1})t \qquad \text{(Bakhvalov-Shishkin mesh)}$$

or

$$\psi(t) = e^{-t/(q-t)} \text{ with } q = \frac{1}{2} + \frac{1}{2\ln N} \qquad \text{(Vulanović mesh)}.$$

Examples of mesh generating functions for Shishkin, Bakhvalov-Shishkin and Vulanović mesh are given in Fig. 1.

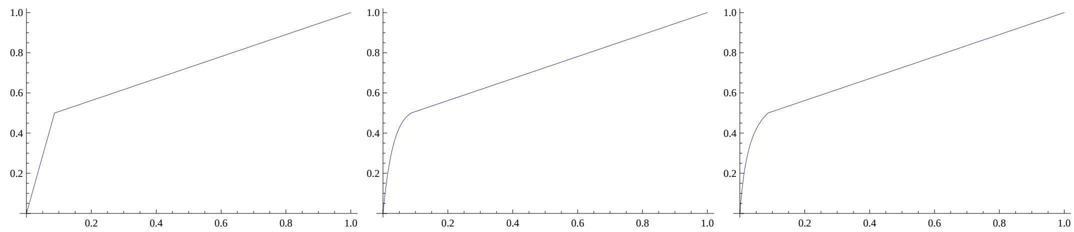

Fig. 2.1. Mesh generating functions for Shishkin, Bakhvalov-Shishkin and Vulanović mesh

Proof of Theorem 2.1. We denote $v^N = u^N - u^I$. Standard arguments lead to

$$\begin{aligned}\alpha\|u^N - u^I\|_\varepsilon^2 &= a(u-u^I, v^N)\\ &= \varepsilon((u-u^I)', (v^N)') + (c(u-u^I), v^N) - (b(S-S^I)', v^N)\\ &\quad + (b'(E-E^I), v^N) + (b(E-E^I), (v^N)').\end{aligned}$$

Only the last term causes some trouble:

$$\begin{aligned}|(b(E-E^I), (v^N)')|_{[x_{N/2},1]} &\le C\|E-E^I\|_{\infty,[x_{N/2},1]} N\|v^N\|_0\\ &\le CN^{-k}\|v^N\|_\varepsilon\end{aligned}$$

(using the smallness of the layer term), otherwise

$$\begin{aligned}|(b(E-E^I), (v^N)')|_{[0,x_{N/2}]} &\le C\|E-E^I\|_{\infty,[0,x_{N/2}]}\|(v^N)'\|_{L_1(0,x_{N/2})}\\ &\le C\,(N^{-1}\max|\psi'|)^{k+1}(meas(0,x_{N/2}))^{1/2}\|(v^N)'\|_0\\ &\le C\,(N^{-1}\max|\psi'|)^{k}\|v^N\|_\varepsilon\end{aligned}$$

where we assumed $N^{-1}\max|\psi'|(\ln N)^{1/2} \le C$. Thus, Theorem 1 is proved. □

Remark that it is alternatively possible to avoid the last condition if for the convection term on the fine mesh one uses integration by parts and the interpolation error estimate

$$\|E-E^I\|_{0,[0,x_{N/2}]} \le C\,\varepsilon^{1/2}\,(N^{-1}\max|\psi'|)^{k+1}.$$

3. Optimization of the Mesh Characterizing Function

We expect that on the fine mesh $|E-E^I|_1$ is the dominating part of the error, and we try to minimize $|E-E^I|_1$ heuristically. Starting from

$$\begin{aligned}|E-E^I|^2_{1,[x_{i-1},x_i]} &\le C(\varepsilon N^{-1}\varphi'(\xi_i))^{2k}\varepsilon^{-(2k+1)}(e^{-2x_{i-1}/\varepsilon} - e^{-2x_i/\varepsilon})\\ &= C\varepsilon^{-1}N^{-2k}(\varphi'(\xi_i))^{2k}(\psi^{2(k+1)}(t_{i-1}) - \psi^{2(k+1)}(t_i)),\end{aligned}$$

it makes sense to minimize

$$K = \sum_i (\psi'(\xi_i))^{2k}(\psi(\xi_i))^{-2k}\frac{\psi^{2(k+1)}(t_{i-1}) - \psi^{2(k+1)}(t_i)}{t_i - t_{i-1}}(t_i - t_{i-1})$$

or

$$\widetilde{K}(\psi) = -\int_0^{1/2} \psi(\psi')^{2k+1}. \tag{3.1}$$

Remark 3.1. If $|\psi'|$ is bounded, $\tilde{K}$ is bounded as well. For instance, for a Bakhvalov-Shishkin mesh we obtain

$$\widetilde{K} = 2^{2k-1} + \mathcal{O}(N^{-1}).$$

The Euler equation related to the minimization of $\widetilde{K}$ reads

$$(\psi')^{2k+1} - \frac{d}{dt}((2k+1)(\psi')^{2k}\psi) = 0$$

or

$$(2k+1)\psi\psi'' + \psi'^2 = 0.$$

Its general solution is

$$\psi = (C_1 t + C_2)^{\frac{2k+1}{2k+2}}.$$

With the boundary conditions $\psi(0) = 1$, $\psi(1/2) = N^{-1}$, we obtain

$$\psi_{opt} = \left[1 - 2t(1 - N^{-\frac{2k+2}{2k+1}})\right]^{\frac{2k+1}{2k+2}}. \tag{3.2}$$

Remark 3.2. For that 'optimal' ψ_{opt}, the quantity $\max|\psi'|$ is of order $\mathcal{O}(N^{\frac{1}{2k+1}})$. We conjecture that the boundness of $\tilde{K}$ implies the validity of the error estimate of Theorem 1 and that the boundness of $\max|\psi'|$ is a sufficient, but not necessary condition. For ψ_{opt} we get

$$\widetilde{K}(\psi_{opt}) = 2^{2k}\left(\frac{2k+1}{2k+2}\right)^{2k+1} + \mathcal{O}(N^{-1}).$$

Our numerical results of Section 5 shall confirm that the optimal mesh generating function yields the same convergence rate as the Bakhvalov-Shishkin mesh but with a smaller error constant.

Next we try to minimize $\|E - E^I\|_{\infty,[0,x_{N/2}]}$. We have

$$\|E - E^I\|_{\infty,[x_{i-1},x_i]} \leq CN^{-(k+1)}(\varphi'(\xi))^{k+1}\psi^{k+1}(t_{i-1})$$

and therefore one should just minimize $\max|\psi'|$. Because $\psi(0) = 1$ and $\psi(1/2) = N^{-1}$, the optimal solution is the Bakhvalov-Shishkin mesh with

$$\psi(t) = 1 - 2(1 - N^{-1})t.$$

Remark that for the streamline diffusion finite element method with linear elements it holds

$$\|u - u^N\|_\infty \leq C \inf_{v^N \in V^N} \|u - v^N\|_\infty,$$

(see [1]). For that method a Bakhvalov-Shishkin mesh should be optimal.

4. Recursively Graded Meshes

In this section we generate S-type meshes too, but define the fine mesh, in some interval close to the interval $[0, \tau]$, in a different way. Set (with some positive M)

$$\begin{aligned} x_1 &= \varepsilon N^{-1}, \\ x_i &= x_{i-1} + g(\varepsilon, N, x_{i-1}), \quad i = 2, \ldots, M. \end{aligned} \tag{4.1}$$

More precisely: we choose the smallest M such that $x_M \geq \tau$, hoping for the relation $M = \mathcal{O}(N)$, and use an equidistant mesh with $N/2$ subintervals in $[x_M, 1]$. We use also the notation $x_M = \tilde{\tau}$.

One idea to come to a recursive formula consists of equidistribution of the pointwise error requiring

$$h_i^{k+1} e^{-x_{i-1}/\varepsilon} = h_{i+1}^{k+1} e^{-x_i/\varepsilon},$$

which leads to

$$h_{i+1} = h_i e^{h_i/((k+1)\varepsilon)}$$

or

$$x_i = x_{i-1} + \varepsilon N^{-1} e^{x_{i-1}/((k+1)\varepsilon)}. \tag{4.2}$$

We call that mesh Gartland-Shishkin mesh (see [5], [9]).

Remark 4.1. The choice $x_1 = \varepsilon N^{-1}$ is motivated by the necessary property $x_1 = o(\varepsilon)$ for obtaining uniform convergence, see Remark 2.85 in [10].

A simplified version of (4.2) is

$$x_1 = \varepsilon N^{-1}, \quad x_i = x_{i-1} + 2N^{-1} x_{i-1} \tag{4.3}$$

(we call that mesh Duran-Shishkin mesh, see [2]).

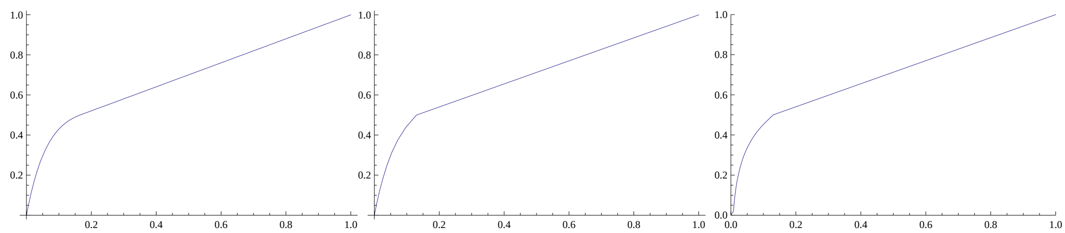

Fig. 4.1. Mesh generating functions for Bakhvalov-Shishkin, Gartland-Shishkin and Duran-Shishkin mesh

Now we define the mesh generating function φ (see Fig. 2) by its nodal values

$$\varphi\left(\frac{i}{M}\right) = \frac{x_i}{(k+1)\varepsilon}, \quad i = 0, 1, \ldots, M,$$

φ is linear on every subinterval $\left[\frac{i-1}{M}, \frac{i}{M}\right]$. To guarantee assumption H1, we need specific features of g. On $\left[\frac{i-1}{M}, \frac{i}{M}\right]$, we have

$$\varphi' = M\, \frac{x_i - x_{i-1}}{\varepsilon(k+1)} = N \frac{M}{N} \frac{g(\varepsilon, N, x_{i-1})}{\varepsilon(k+1)}. \tag{4.4}$$

Therefore H1 is satisfied if the following condition $\widetilde{H_1}$ holds:

$$\widetilde{H_1}: \quad g(\varepsilon, N, x) \le C\,\frac{\varepsilon N}{M} \quad \text{for} \quad x \le x_M. \tag{4.5}$$

Let us check condition (4.5) for meshes (4.2) and (4.3), where we have $M = \mathcal{O}(N)$ and $M = \mathcal{O}(N \ln\ln N)$ respectively (we shall prove that later).

For the mesh (4.2):

$$g(\varepsilon, N, x) = \varepsilon N^{-1} e^{x/((k+1)\varepsilon)} \le \varepsilon \qquad \text{for} \qquad x \le x_M.$$

For the mesh (4.3):

$$g(\varepsilon, N, x) = 2N^{-1}x \le CN^{-1}\varepsilon \ln N \le C\frac{\varepsilon N}{N \ln\ln N},$$

because

$$N^{-1}(\ln N)(\ln\ln N) \le C.$$

Next we have to investigate $\max|\psi'|$. Because $|\psi'| = \varphi' e^{-\varphi}$, using (4.4), on $\left[\frac{i-1}{M}, \frac{i}{M}\right]$ we get:

$$|\psi'| \le M\,\frac{g(\varepsilon, N, x_{i-1})}{\varepsilon}\,e^{-\frac{x_{i-1}}{(k+1)\varepsilon}}. \tag{4.6}$$

Hence, for the Gartland-Shishkin mesh (4.2) we have

$$|\psi'| \le MN^{-1} \le C, \tag{4.7}$$

but for the Duran-Shishkin mesh (4.3)

$$|\psi'| \le MN^{-1}\frac{x_{i-1}}{\varepsilon}e^{-\frac{x_{i-1}}{(k+1)\varepsilon}} \le CMN^{-1} \le C\ln\ln N. \tag{4.8}$$

Finally, following [5] and [2], we estimate the number M of mesh points in the fine mesh. We start from εN^{-1}, and thus we need not more than $\mathcal{O}(N)$ points to come to ε. From ε to τ, the minimum meshpoints M^* can be estimated by

$$M^* \le C\int_{\varepsilon}^{(k+1)\varepsilon\ln N} \frac{dx}{g(\varepsilon, N, x)},$$

assuming that g is monotonically increasing, and

$$g(\varepsilon, N, x_i) \le Cg(\varepsilon, N, x_{i-1}). \tag{4.9}$$

For the meshes defined by (4.2) and (4.3), condition (4.9) is satisfied. Therefore, for the mesh (4.2)

$$M_1^* \le C\varepsilon^{-1}N\int_{\varepsilon}^{(k+1)\varepsilon\ln N} e^{-x/(k+1)\varepsilon}dx \le CN(e^{-\frac{1}{k+1}} - N^{-1}) \le CN, \tag{4.10}$$

while for the mesh (4.3)

$$M_2^* \le CN\int_{\varepsilon}^{(k+1)\varepsilon\ln N}\frac{dx}{x} \le CN\ln((k+1)\ln N). \tag{4.11}$$

Consequently, the Gartland-Shishkin mesh belongs to the class of meshes with optimal convergence rates, while for the Duran-Shishkin mesh we additionally have a very weak dependence on logarithmic factors.

5. Numerical Results

This section presents numerical experiment to illustrate our theoretical results. Our test problem is

$$-\varepsilon u'' - u' + 2u = e^{x-1}, \quad x \in I, \quad u(0) = u(1) = 0. \tag{5.1}$$

The test problems is identically with Linß test problem in [6, p. 162]. Linß presented results for linear elements for three meshes: Shishkin, Bakhvalov-Shishkin and a mesh with a rational mesh generating function.

We consider first five different S-type meshes: the original Shishkin mesh, a mesh with a rational mesh characterizing function ψ, the Vulanović mesh, the Bakhvalov-Shishkin mesh and the mesh with 'optimal' mesh characterizing function ψ_{opt}. A rational mesh characterizing function ψ (from [7]) is given by

$$\psi(t) = \frac{1}{1 + (N-1)(2t)^m}, \qquad m > 1.$$

We used $m = 2$ in our numerical experiments. In the tables below we present errors $E(N)$ for two values of $\varepsilon = 2^{-12}$ and $\varepsilon = 2^{-25}$, and different number of mesh points N. Set

$$E(N) = \|u - u^N\|_\varepsilon,$$

where u^N is the numerical solution of the problem (5.1) obtained using quadratic or cubic elements.

In the Tables 5.1–5.8 are the results for these meshes. We also present numerical order of convergence:

$$Ord(N) = \frac{\ln E(N) - \ln E(2N)}{\ln 2}.$$

Table 5.1: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-12}$

	mesh with rational ψ		Shishkin mesh		Vulanović mesh	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$3.307e-3$	0.948	$2.534e-3$	1.454	$3.818e-4$	1.902
2^6	$1.715e-3$	0.973	$9.251e-4$	1.548	$1.022e-4$	1.935
2^7	$8.734e-4$	0.986	$3.164e-4$	1.612	$2.674e-5$	1.946
2^8	$4.409e-4$	0.993	$1.035e-4$	1.659	$6.939e-6$	1.953
2^9	$2.216e-4$	0.996	$3.277e-5$	1.696	$1.792e-6$	1.960
2^{10}	$1.111e-4$	0.998	$1.012e-5$	1.725	$4.605e-7$	1.967
2^{11}	$5.559e-5$	–	$3.060e-6$	–	$1.178e-7$	–

Table 5.2: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-12}$

	Bakhvalov-Shishkin mesh		mesh with ψ_{opt}	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$3.607e-4$	1.963	$3.350e-4$	1.979
2^6	$9.253e-5$	1.995	$8.494e-5$	2.009
2^7	$2.322e-5$	1.999	$2.111e-5$	2.009
2^8	$5.807e-6$	1.998	$5.244e-6$	2.003
2^9	$1.454e-6$	1.998	$1.308e-6$	2.001
2^{10}	$3.640e-7$	1.999	$3.269e-7$	2.000
2^{11}	$9.107e-8$	–	$8.170e-8$	–

Table 5.3: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-25}$

	mesh with rational ψ		Shishkin mesh		Vulanović mesh	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$3.305e-3$	0.948	$2.532e-3$	1.454	$3.817e-4$	1.896
2^6	$1.713e-3$	0.973	$9.243e-4$	1.548	$1.026e-4$	1.922
2^7	$8.728e-4$	0.986	$3.162e-4$	1.612	$2.707e-5$	1.940
2^8	$4.407e-4$	0.993	$1.034e-4$	1.659	$7.055e-6$	1.952
2^9	$2.214e-4$	0.996	$3.274e-5$	1.696	$1.823e-6$	1.961
2^{10}	$1.110e-4$	0.998	$1.011e-5$	1.725	$4.682e-7$	1.968
2^{11}	$5.556e-5$	–	$3.058e-6$	–	$1.197e-7$	–

Table 5.4: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-25}$

	Bakhvalov-Shishkin mesh		mesh with ψ_{opt}	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$3.604e-4$	1.956	$3.344e-4$	1.974
2^6	$9.291e-5$	1.978	$8.512e-5$	1.989
2^7	$2.359e-5$	1.989	$2.144e-5$	1.995
2^8	$5.943e-6$	1.994	$5.379e-6$	1.998
2^9	$1.492e-6$	1.997	$1.347e-6$	1.999
2^{10}	$3.737e-7$	1.999	$3.371e-7$	1.999
2^{11}	$9.351e-8$	–	$8.434e-8$	–

Table 5.5: Galerkin FEM with cubic elements, $\varepsilon = 2^{-12}$

	mesh with rational ψ		Shishkin mesh		Vulanović mesh	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$4.727e-4$	1.440	$3.208e-4$	2.165	$1.667e-5$	2.834
2^6	$1.743e-4$	1.456	$7.153e-5$	2.315	$2.338e-6$	2.878
2^7	$6.353e-5$	1.478	$1.437e-5$	2.416	$3.179e-7$	2.906
2^8	$2.281e-5$	1.489	$2.693e-6$	2.488	$4.240e-8$	2.926
2^9	$8.129e-6$	1.494	$4.800e-7$	2.543	$5.580e-9$	2.940
2^{10}	$2.885e-6$	1.497	$8.234e-8$	2.587	$7.273e-10$	2.834
2^{11}	$1.022e-6$	–	$1.370e-8$	–	$1.020e-10$	–

Table 5.6: Galerkin FEM with cubic elements, $\varepsilon = 2^{-12}$

	Bakhvalov-Shishkin mesh		mesh with ψ_{opt}	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$1.472e-5$	2.931	$1.347e-5$	2.949
2^6	$1.931e-6$	2.966	$1.744e-6$	2.978
2^7	$2.472e-7$	2.983	$2.214e-7$	2.990
2^8	$3.126e-8$	2.992	$2.786e-8$	2.996
2^9	$3.931e-9$	2.995	$3.493e-9$	2.997
2^{10}	$4.930e-10$	2.593	$4.375e-10$	2.434
2^{11}	$8.168e-11$	–	$8.095e-11$	–

In Tables 5.9–5.20 we compare errors obtained using the Bakhvalov-Shishkin mesh and two recursively graded S-type meshes defined in Section 4 for those two values of ε. For the Gartland-Shishkin and the Duran-Shishkin mesh, N_1 denotes the total number of mesh points, while M is the number of mesh points in the interval $[0, \tilde{\tau}]$.

Table 5.7: Galerkin FEM with cubic elements, $\varepsilon = 2^{-25}$

	mesh with rational ψ		Shishkin mesh		Vulanović mesh	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$7.567e-4$	1.434	$6.089e-4$	2.140	$2.134e-5$	2.835
2^6	$2.800e-4$	1.452	$1.381e-4$	2.306	$2.991e-6$	2.878
2^7	$1.024e-4$	1.475	$2.793e-5$	2.412	$4.068e-7$	2.907
2^8	$3.683e-5$	1.487	$5.247e-6$	2.487	$5.425e-8$	2.926
2^9	$1.314e-5$	1.494	$9.359e-7$	2.543	$7.139e-9$	2.940
2^{10}	$4.666e-6$	1.497	$1.606e-7$	2.587	$9.304e-10$	2.894
2^{11}	$1.653e-6$	–	$2.672e-8$	–	$1.252e-10$	–

Table 5.8: Galerkin FEM with cubic elements, $\varepsilon = 2^{-25}$

	Bakhvalov-Shishkin mesh		mesh with ψ_{opt}	
N	$E(N)$	$Ord(N)$	$E(N)$	$Ord(N)$
2^5	$2.270e-5$	2.930	$1.871e-5$	2.950
2^6	$2.979e-6$	2.966	$2.422e-6$	2.977
2^7	$3.814e-7$	2.983	$3.076e-7$	2.989
2^8	$4.824e-8$	2.992	$3.873e-8$	2.995
2^9	$6.066e-9$	2.996	$4.858e-9$	2.997
2^{10}	$7.605e-10$	2.840	$6.083e-10$	2.752
2^{11}	$1.062e-10$	–	$9.027e-11$	–

Table 5.9: Galerkin FEM with linear elements, $\varepsilon = 2^{-12}$

B-S mesh	G-S mesh				
$E(N_1)$	$E(N_1)$	N	M	N_1	M/N
$1.663e-2$	$1.093e-2$	8	16	20	2.000
$8.516e-3$	$4.588e-3$	16	33	41	2.063
$4.310e-3$	$2.160e-3$	32	65	81	2.031
$2.142e-3$	$1.066e-3$	64	130	162	2.031
$1.081e-3$	$5.321e-4$	128	258	322	2.016
$5.430e-4$	$2.662e-4$	256	514	642	2.008
$2.721e-4$	$1.332e-4$	512	1027	1283	2.006
$1.362e-4$	$6.660e-5$	1024	2051	2563	2.003
$6.815e-5$	$3.331e-5$	2048	4099	5123	2.001

Table 5.10: Galerkin FEM with linear elements, $\varepsilon = 2^{-12}$

B-S mesh	D-S mesh				
$E(N_1)$	$E(N_1)$	N	M	N_1	$M/(N\ln(2\ln N))$
$1.519e-2$	$1.105e-2$	8	18	22	1.579
$7.125e-3$	$4.591e-3$	16	41	49	1.496
$3.203e-3$	$2.160e-3$	32	92	108	1.485
$1.461e-3$	$1.066e-3$	64	206	238	1.519
$6.625e-4$	$5.321e-4$	128	462	526	1.588
$3.028e-4$	$2.661e-4$	256	1024	1152	1.662
$1.393e-4$	$1.331e-4$	512	2250	2506	1.741
$6.447e-5$	$6.658e-5$	1024	4902	5414	1.821
$3.002e-5$	$3.330e-5$	2048	10605	11629	1.901

All integrals are calculated using three-point Gauss-Legendre quadrature formula for linear elements, and four-point Gauss-Legendre quadrature formula for quadratic, and cubic elements.

Numerical experiments completely confirm our theoretical results. Namely, comparing er-

Table 5.11: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-12}$

B-S mesh $E(N_1)$	G-S mesh $E(N_1)$	N	M	N_1	M/N
$5.387e-4$	$1.239e-3$	8	23	27	2.875
$1.205e-4$	$3.420e-4$	16	48	56	3.000
$3.032e-5$	$8.603e-5$	32	96	112	3.000
$7.584e-6$	$1.970e-5$	64	193	225	3.016
$1.899e-6$	$3.429e-6$	128	385	449	3.008
$4.753e-7$	$5.045e-7$	256	769	897	3.004
$1.187e-7$	$7.771e-8$	512	1538	1794	3.004
$2.972e-8$	$1.456e-8$	1024	3074	3586	3.002
$7.435e-9$	$3.258e-9$	2048	6146	7170	3.000

Table 5.12: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-12}$

B-S mesh $E(N_1)$	D-S mesh $E(N_1)$	N	M	N_1	$M/(N\ln(3\ln N))$
$6.283e-4$	$1.366e-3$	8	20	24	1.366
$1.395e-4$	$3.535e-4$	16	44	52	1.298
$2.927e-5$	$8.952e-5$	32	98	114	1.308
$5.993e-6$	$2.074e-5$	64	220	252	1.362
$1.251e-6$	$3.819e-6$	128	488	552	1.424
$2.634e-7$	$6.595e-7$	256	1076	1204	1.495
$5.609e-8$	$1.320e-7$	512	2354	2610	1.570
$1.209e-8$	$3.043e-8$	1024	5110	5622	1.644
$2.635e-9$	$7.439e-9$	2048	11021	12045	1.719

Table 5.13: Galerkin FEM with cubic elements, $\varepsilon = 2^{-12}$

B-S mesh $E(N_1)$	G-S mesh $E(N_1)$	N	M	N_1	M/N
$1.235e-5$	$1.456e-5$	8	31	35	3.875
$1.482e-6$	$1.523e-6$	16	63	71	3.937
$2.199e-8$	$1.708e-7$	32	127	143	3.969
$1.227e-8$	$1.477e-8$	64	256	288	4.000
$2.763e-9$	$1.739e-9$	128	512	576	4.000
$3.467e-10$	$2.179e-10$	256	1024	1152	4.000
$7.954e-11$	$2.124e-10$	512	2049	2305	4.002
$2.602e-10$	$8.438e-10$	1024	4097	4609	4.001

rors on the Vulanović and the Bakhvalov-Shishkin meshes with errors on the Shishkin and with rational ψ meshes (see Tables 5.1-5.8), we see that the Bakhvalov-Shishkin mesh and the Vulanović meshes become significantly better as the order of elements increases.

As well as predicted by theoretical results, the mesh with ψ_{opt} gives the best results, i.e. gives the smallest errors of all tested meshes for all values of ε and N.

The last columns in tables containing numerical results for the Gartland-Shishkin and the Duran-Shishkin meshes confirm the estimates (4.10) and (4.11) for the minimum of mesh points M^* in the fine mesh regions of those meshes.

Table 5.14: Galerkin FEM with cubic elements, $\varepsilon = 2^{-12}$

B-S mesh $E(N_1)$	D-S mesh $E(N_1)$	N	M	N_1	$M/(N \ln(4 \ln N))$
$3.379e-5$	$2.685e-5$	8	21	25	1.239
$3.187e-6$	$3.539e-6$	16	46	54	1.195
$3.149e-7$	$4.187e-7$	32	103	119	1.224
$2.985e-8$	$5.303e-8$	64	229	261	1.273
$2.851e-9$	$6.694e-9$	128	507	571	1.336
$2.783e-10$	$8.436e-10$	256	1113	1241	1.403
$9.435e-11$	$1.357e-10$	512	2428	2684	1.474
$4.158e-10$	$3.504e-10$	1024	5258	5770	1.545

Table 5.15: Galerkin FEM with linear elements, $\varepsilon = 2^{-25}$

B-S mesh $E(N_1)$	G-S mesh $E(N_1)$	N	M	N_1	M/N
$1.662e-2$	$1.083e-2$	8	16	20	2.000
$8.508e-3$	$4.523e-3$	16	33	41	2.063
$4.306e-3$	$2.123e-3$	32	65	81	2.031
$2.140e-3$	$1.048e-3$	64	130	162	2.031
$1.080e-3$	$5.234e-4$	128	258	322	2.016
$5.425e-4$	$2.618e-4$	256	514	642	2.008
$2.719e-4$	$1.310e-4$	512	1027	1283	2.006
$1.361e-4$	$6.553e-5$	1024	2051	2563	2.003
$6.809e-5$	$3.277e-5$	2048	4099	5123	2.001

Table 5.16: Galerkin FEM with linear elements, $\varepsilon = 2^{-25}$

B-S mesh $E(N_1)$	D-S mesh $E(N_1)$	N	M	N_1	$M/(N \ln(2 \ln N))$
$1.517e-2$	$1.095e-2$	8	18	22	1.579
$7.119e-3$	$4.526e-3$	16	41	49	1.496
$3.200e-3$	$2.124e-3$	32	92	108	1.485
$1.459e-3$	$1.048e-3$	64	206	238	1.519
$6.619e-4$	$5.233e-4$	128	462	526	1.588
$3.025e-4$	$2.618e-4$	256	1024	1152	1.662
$1.391e-4$	$1.310e-4$	512	2250	2506	1.741
$6.442e-5$	$6.553e-5$	1024	4902	5414	1.821
$3.000e-5$	$3.277e-5$	2048	10605	11629	1.901

Comparing results for the same N in all numerical experiments it can be seen that the Gartland-Shishkin mesh takes small precedence over the Duran-Shishkin mesh.

Comparison of the results obtained on the Gartland-Shishkin and the Bakhvalov-Shishkin meshes shows that for linear elements the Gartland-Shishkin mesh is better than the Bakhvalov-Shishkin mesh, while for quadratic elements the Bakhvalov-Shishkin mesh has the advantage for most values of ε and N_1. For cubic elements the Gartland-Shishkin mesh becomes better than the Bakhvalov-Shishkin mesh for smaller ε.

The results obtained on the Bakhvalov-Shishkin mesh are in general better then those obtained on the Duran-Shishkin mesh. The advantage of the Duran-Shishkin mesh is it's much

Table 5.17: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-25}$

B-S mesh $E(N_1)$	G-S mesh $E(N_1)$	N	M	N_1	M/N
$5.382e-4$	$1.235e-3$	8	23	27	2.875
$1.208e-4$	$3.411e-4$	16	48	56	3.000
$3.074e-5$	$8.621e-5$	32	96	112	3.000
$7.754e-6$	$2.196e-5$	64	193	225	3.016
$1.947e-6$	$5.499e-6$	128	385	449	3.008
$4.879e-7$	$1.377e-6$	256	769	897	3.004
$1.219e-7$	$3.452e-7$	512	1538	1794	3.004
$3.048e-8$	$8.643e-8$	1024	3074	3586	3.002
$7.568e-9$	$2.164e-8$	2048	6146	7170	3.000

Table 5.18: Galerkin FEM with quadratic elements, $\varepsilon = 2^{-25}$

B-S mesh $E(N_1)$	D-S mesh $E(N_1)$	N	M	N_1	$M/(N\ln(3\ln N))$
$6.277e-4$	$1.365e-3$	8	20	24	1.366
$1.398e-4$	$3.526e-4$	16	44	52	1.298
$2.968e-5$	$8.962e-5$	32	98	114	1.308
$6.133e-6$	$2.282e-5$	64	220	252	1.362
$1.284e-6$	$5.736e-6$	128	488	552	1.424
$2.704e-7$	$1.440e-6$	256	1076	1204	1.495
$5.758e-8$	$3.609e-7$	512	2354	2610	1.570
$1.235e-8$	$9.044e-8$	1024	5110	5622	1.644
$2.660e-9$	$2.264e-8$	2048	11021	12045	1.719

Table 5.19: Galerkin FEM with cubic elements, $\varepsilon = 2^{-25}$

B-S mesh $E(N_1)$	G-S mesh $E(N_1)$	N	M	N_1	M/N
$1.234e-5$	$1.119e-5$	8	31	35	3.875
$1.481e-6$	$7.976e-7$	16	63	71	3.937
$1.813e-7$	$6.803e-8$	32	127	143	3.969
$2.197e-8$	$4.822e-9$	64	256	288	4.000
$2.761e-9$	$5.280e-10$	128	512	576	4.000
$3.464e-10$	$8.326e-11$	256	1024	1152	4.000

Table 5.20: Galerkin FEM with cubic elements, $\varepsilon = 2^{-25}$

B-S mesh $E(N_1)$	D-S mesh $E(N_1)$	N	M	N_1	$M/(N\ln(4\ln N))$
$3.376e-5$	$2.392e-5$	8	21	25	1.239
$3.184e-6$	$3.095e-6$	16	46	54	1.195
$3.147e-7$	$3.908e-7$	32	103	119	1.224
$2.983e-8$	$5.076e-8$	64	229	261	1.273
$2.849e-9$	$6.479e-9$	128	507	571	1.336
$2.781e-10$	$8.193e-10$	256	1113	1241	1.403
$9.485e-11$	$1.351e-10$	512	2428	2684	1.474

simpler recursion formula. Taking this into account, we observe that the results obtained on Duran-Shishkin mesh are quite comparable to the results obtained on Gartland-Shishkin and Bakhvalov-Shishkin meshes.

6. Conclusions and Open Problems

The numerical results in Section 5 confirm that it is extremely useful to use graded meshes instead of piecewise equidistant meshes for higher order polynomials. For instance, if $k = 3$, $\varepsilon = 2^{-12}$ and $N = 128$, the error on the Shiskin mesh for our example is $1.4 * 10^{-5}$, while on the Bakhvalov-Shishkin mesh is $2.5 * 10^{-7}$ which is significantly smaller! We also observed that recursively generated meshes, especially the Gartland-Shishkin mesh, are an interesting alternative to the use of mesh generating functions.

In principle, it is possible to extend those results to Q_k-elements in 2D (or 3D). But that analysis would require a solution decomposition analogous to (2.1) in the more-dimensional case. Such decomposition is so far verified only under unrealistic compatibility conditions, see [10]. A more realistic approach at least requires an analysis of the weak corner singularities which appear in the two-dimensional case in polygonal domains.

For *linear* elements on S-type meshes one can prove a supercloseness result, see Theorem 5.10 in [6] (for recursively generated meshes, see [3]). Supercloseness implies an improved error estimate in the L_2-norm, which is important because Nitsche's trick cannot be used. However, supercloseness for higher order elements is an open problem, even in 1D. There are some results for SDFEM [11] and moreover a numerical study for Q_k-elements [4].

S-type meshes do have the property that the mesh is not locally uniform, especially where the mesh changes from graded to equidistant and the mesh size changes abruptly. It is open to investigate a smoothing of the mesh generating function in the sense of [13] or to generalize the results on locally uniform modifications of Bakhvalov-Shishkin meshes for linear elements [8] to higher order finite elements.

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