Journal of Computational Mathematics
Vol.33, No.1, 2015, 86–99.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1406-m4309

ANISOTROPIC CROUZEIX-RAVIART TYPE NONCONFORMING FINITE ELEMENT METHODS TO VARIATIONAL INEQUALITY PROBLEM WITH DISPLACEMENT OBSTACLE*

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Abstract

In this paper, anisotropic Crouzeix-Raviart type nonconforming finite element methods are considered for solving the second order variational inequality with displacement obstacle. The convergence analysis is presented and the optimal order error estimates are obtained under the hypothesis of the finite length of the free boundary. Numerical results are provided to illustrate the correctness of theoretical analysis.

Mathematics subject classification: 65N15, 65N30.
Key words: Crouzeix-Raviart type nonconforming finite elements, Anisotropy, Variational inequality, Displacement obstacle, Optimal order error estimates.

1. Introduction

The variational inequality problem with displacement obstacle has been a very interesting subject in many fields, see, e.g., [1,2]. As usual, it reads as: to find $u \in K$, such that

$$a(u, v-u) \geq f(v-u), \quad \forall\ v \in K, \tag{1.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dxdy, \qquad f(v) = \int_\Omega f v dxdy, \tag{1.2a}$$

$$K = \Big\{\ v \in H_0^1(\Omega): \quad v \geq \chi \ \ a.e. \text{ in } \Omega; \quad \chi \leq 0 \text{ on } \partial\Omega\ \Big\}, \tag{1.2b}$$

$\Omega \subset R^2$ is bounded convex domain. $f \in L^\infty(\Omega)$ and $\chi \in H^2(\Omega)$ are given functions.

The variational inequality theory was first introduced by Hartman and Stampacchia [3] to study the partial differential equations, and has been playing more and more important role in the contact problem, obstacle problem, elasticity problem, traffic problem, and so on.

* Received November 8, 2012 / Revised version received May 23, 2014 / Accepted June 26, 2014 / Published online December 1, 2014 /

As to the problem shown in (1.1), there have been numerous studies with different finite elements, such as conforming linear triangular element [1,4-6], quadratic element [7], nonconforming Crouzeix-Raviart type linear triangular element and rectangular Wilson element [8-10]. Based on the detailed analysis, the error bound of order $O(h^{3/2-\varepsilon})$, for any $\varepsilon > 0$, was obtained in [4] for the quadratic finite element under the hypothesis of that the free boundary has finite length. Further, [7] derived the same error bound as [4] for the same quadratic element without the above hypothesis. In [8], the Crouzeix-Raviart type nonconforming linear triangular element was used to problem (1.1) and the error bound was estimated with order $O(h)$.

However, to the best knowledge of the authors, all of the above studies on error estimates depend on the essential condition of the discrete meshes, i.e., regular assumption $\frac{h_K}{\rho_K} \le C$ or quasi-uniform assumption $\frac{h}{h_K} \le C$, $\forall\ K \in J_h$, where $h_K,\ \rho_K$ denote the diameters of element K and biggest circle contained in K, respectively, $h = \max\limits_{K\in J_h} h_K$, J_h is a subdivision of Ω, C is a positive constant which is independent of h and the function under consideration.

As we know, the domain considered may be narrow or irregular, and the cost of calculation will be very expensive if we employ the regular subdivision on the domain. Naturally, it is an obvious idea to use an anisotropic partition with fewer degrees of freedom for simplicity in the application. But, in this case, some difficulties will arise in the convergence analysis and error estimates of interpolation and consistency errors for nonconforming finite element methods. For example, the Bramble-Hilbert lemma, the traditional interpolation theory in Sobolev spaces, can not be directly applied to the interpolation error estimates for the meshes are characterized by $\frac{h_K}{\rho_K} \to \infty$, where the limit can be considered as $h \to 0$. On the other hand, when we deal with the consistency error estimate on the longer or the longest edge F of the element K, there will appear a factor $\frac{|F|}{|K|}$, which may tend to infinity and makes the estimate in vain.

In order to overcome the above difficulties, some researches have been devoted to the investigation on the narrow and anisotropic finite elements for the practical problems [11-14]. But there are only a few of articles considering the variational inequality problem with nonconforming finite elements. For example, anisotropic Carey element and Wilson element approximations to the second order obstacle problem were investigated in [15], in which the proofs of the main results are simplified greatly comparing with [8] and [9]. But the techniques used in [15] are only valid to the finite elements when their interpolations can be separated into the conforming part and nonconforming part. Moreover, a class of Crouzeix-Raviart type finite elements were applied to the Signorini variational problem in [16], and [17] extended them to the parabolic variational inequality problem with moving grids.

In [18], a nonconforming rotated Q_1 element was proposed, of which the degrees of freedom are function values of the midpoints of four edges of element K, and the shape function space is spanned by $\{1, x, y, x^2 - y^2\}$. However, it has been proved in [14] that this element can not be applied to anisotropic meshes directly by a counter example. At the same time, [14] also proposed a kind of modified nonconforming finite element with the degrees of freedom of mean-values on the four edges of element K , and the shape function space is spanned by $\{1, x, y, x^2\}$ or $\{1, x, y, y^2\}$, and proved its convergence for the second order problem on a special anisotropic meshes, i.e., the longer edges of all the elements should parallel to x-axis or y-axis, respectively. Obviously, the shape function space of this modification is asymmetrical and the requirement on meshes is too strong.

Recently, there have appeared a lot of studies focusing on the analysis of convergence, supercloseness and supercongvergence for some anisotropic finite element methods (cf. [19-23]). However, the applications of Crouzeix-Raviart type anisotropic nonconforming linear triangular element [14] and EQ_1^{rot} rectangular element [24] to the variational inequality problem (1.1) have never been seen, although [25] applied the latter one to a class of nonlinear Sobolev equations and obtained the optimal error estimates and the supercloseness for both semi-discrete and fully-discrete approximate schemes, and [26] discussed the supercloseness and superconvergence of nonconforming rectangular elements for the second order elliptic problems.

In this paper, we will have a try to consider the approximations of Crouzeix-Raviart type nonconforming linear triangular element [14] and EQ_1^{rot} rectangular element [24] to problem (1.1) on anisotropic meshes. With the same hypothesis of finite length of the free boundary as [1,4], the optimal error estimate of order $O(h)$ is obtained and some numerical results are provided to illustrate the correctness of our theoretical analysis. Thus the gap on this aspect is filled.

2. Construction of the Finite Elements and Some Lemmas

Suppose Ω is a bounded convex domain with the boundary $\partial\Omega$ parallel to x-axis and y-axis respectively, J_h is a family of triangular or rectangular mesh grading of Ω satisfying the maximum angle and coordinate system conditions [13]. But it is not required to satisfy the regular assumption or quasi-uniform assumption.

For a given rectangle K, by dividing each rectangle diagonally, we obtain triangular mesh. Without lose of generality, we assume the center point of K is (x_K, y_K), the sides of K parallel to x-axis and y-axis are of lengths $2h_{Kx}$ and $2h_{Ky}$ respectively. In addition, we assume that

$$\begin{aligned} & h_{Kx} \gg h_{Ky}, h_x = \max_{K\in J_h} h_{Kx}, h_y = \max_{K\in J_h} h_{Ky}, \\ & \tilde{h}_x = \min_{K\in J_h} h_{Kx}, \tilde{h}_y = \min_{K\in J_h} h_{Ky}, \frac{h_y}{\tilde{h}_y} \le C,\ \frac{h}{\tilde{h}_x} \le C, \end{aligned}$$

where C is a positive constant independent of h and $\frac{h_K}{\rho_K}$, $h_K = \mathrm{diam}(K)$ be the diameter of the element K, $h = \max\limits_{K\in J_h} h_K$. For convenience, h_{Kx}, h_{Ky} are simply denoted by h_L and h_S respectively. Obviously, $h_S \le h_L \le h_K \le h$.

On the other hand, let the vertices of K be $d_i(x_i, y_i)$ for the rectangular or triangular element, and the corresponding edges be $l_i = \overline{d_i d_{i+1}} (i = 1, 2, 3, 4 \bmod (4)$ or $i = 1, 2, 3 \bmod (3))$.

Let $\hat{K} = [-1, 1; -1, 1]$ be the reference element, the middle points of the four edges $\hat{l}_1$, $\hat{l}_2$, $\hat{l}_3$ and $\hat{l}_4$ are denoted by $\hat{a}_1(0, -1)$, $\hat{a}_2(1, 0)$, $\hat{a}_3(0, 1)$ and $\hat{a}_4(-1, 0)$ respectively. Let $\hat{K}$ be the reference element on (λ_1, λ_2)-plane with vertices

$$\hat{d}_1 = (1, 0),\ \hat{d}_2 = (0, 1),\ \hat{d}_3 = (0, 0),\ \hat{l}_1 = \overline{\hat{d}_2\hat{d}_3},\ \hat{l}_2 = \overline{\hat{d}_3\hat{d}_1},\ \hat{l}_3 = \overline{\hat{d}_1\hat{d}_2}.$$

The corresponding affine mapping $F_K : \hat{K} \to K$ of the rectangular or triangular element is defined by

$$\left\{ \begin{array}{l} x = x_K + h_L\xi, \\ y = y_K + h_S\eta, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} x = (x_1 - x_3)\lambda_1 + (x_2 - x_3)\lambda_2 + x_3, \\ y = (y_1 - y_3)\lambda_1 + (y_2 - y_3)\lambda_2 + y_3. \end{array} \right.$$

Now, we define the finite element space $(\hat{K}, \hat{P}, \hat{\Sigma})$ as

$$\hat{\Sigma} = \{\hat{v}_i\} \quad (i = 1, 2, 3, 4, 5 \text{ or } i = 1, 2, 3),$$

where

$$\hat{v}_i = \frac{1}{|\hat{l}_i|}\int_{\hat{l}_i} \hat{v} d\hat{s} \ \ (i=1,2,3,4 \text{ or } i=1,2,3) \text{ and } \hat{v}_5 = \frac{1}{|\hat{K}|}\int_{\hat{K}} \hat{v} d\xi d\eta.$$

Moreover, let $\hat{P} = \text{span}\{1,\ \xi,\ \eta,\ \varphi(\xi),\ \varphi(\eta)\}$, $\varphi(t) = \frac{1}{2}(3t^2-1)$, for EQ_1^{rot} rectangular element([24]), or $\hat{P} = \text{span}\{1,\ \lambda_1,\ \lambda_2\}$, for the linear triangular element([14]).

It can be easily checked that the corresponding interpolation can be expressed as

$$\begin{aligned}\hat{\Pi}\hat{v} = \hat{v}_5 &+ \frac{1}{2}(\hat{v}_2 - \hat{v}_4)\xi + \frac{1}{2}(\hat{v}_3 - \hat{v}_1)\eta + \frac{1}{2}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5)\varphi(\xi) \\ &+ \frac{1}{2}(\hat{v}_3 + \hat{v}_1 - 2\hat{v}_5)\varphi(\eta),\end{aligned} \tag{2.1}$$

or

$$\hat{\Pi}\hat{v} = \hat{v}_1 + \hat{v}_2 - \hat{v}_3 + 2(\hat{v}_3 - \hat{v}_1)\lambda_1 + 2(\hat{v}_3 - \hat{v}_2)\lambda_2. \tag{2.2}$$

Then the finite element space is defined as

$$\begin{aligned}V_h = \Big\{ v_h : \hat{v}_h = v_h|_K \circ F_K \in \hat{P} \ \text{ satisfying } \ (2.1) \text{ or } (2.2),\ \forall K \in J_h; \\ \int_F [v_h] ds = 0,\ \forall F \subset \partial K \Big\},\end{aligned} \tag{2.3}$$

where $[v_h]$ denotes the jump value of v_h crossing the edge F if F is an interior edge, and it is equal to v_h if F is a boundary edge of $\partial\Omega$.

Let the associated interpolation operator $\Pi_h :\ H^2(\Omega) \longrightarrow V_h$, be defined by

$$\Pi_h|_K = \Pi_K,\ \ \Pi_K v = \hat{\Pi}\hat{v} \circ F_K^{-1},\ \ \forall v \in H^2(K).$$

Now, we introduce the following two important lemmas which can be found in [14,19].

Lemma 2.1. $\hat{\Pi}$ *has the anisotropic interpolation property, i.e.,* $\forall\ \hat{v} \in H^2(\hat{K})$, $\alpha = (\alpha_1, \alpha_2)$ *with* $|\alpha| = 1$, *there holds*

$$\|\hat{D}^\alpha(\hat{v} - \hat{\Pi}\hat{v})\|_{0,\hat{K}} \le C|\hat{D}^\alpha \hat{v}|_{1,\hat{K}}.$$

Consequently, we have

$$|u - \Pi_h u|_{1,K} \le C \sum_{q\in\{L,S\}} h_{q,K} |\partial_q u|_{1,K}, \tag{2.4}$$

$$|u - \Pi_h u|_{0,K} \le C \sum_{q\in\{L,S\}} h_{q,K}^2 |\partial_{qq}^2 u|_{0,K}. \tag{2.5}$$

Lemma 2.2. *Let* F *be any edge of element* K, $v \in H^1(K)$, $v_h \in V_h$. *Then*

$$\begin{aligned}&\left|\int_F (v - M_F v)(v_h - M_F v_h) ds\right| \\ &\le C\frac{|F|}{|K|}\Big(\sum_{q\in\{L,S\}} h_{q,K}^2 \|\partial_q v\|_{0,K}^2\Big)^{\frac{1}{2}} \Big(\sum_{q\in\{L,S\}} h_{q,K}^2 \|\partial_q v_h\|_{0,K}^2\Big)^{\frac{1}{2}},\end{aligned} \tag{2.6}$$

where $M_F v = \dfrac{1}{|F|}\displaystyle\int_F v ds$.

As we know, Lemma 2.2 plays a very important role in estimating the consistency error for nonconforming elements on regular meshes. But for anisotropic meshes, if F is the longer or the longest edge of K, the factor $\frac{1}{h_S}$ will appear in (2.6), which may tend to infinite when h_S is small enough.

In order to overcome this difficulty, we introduce the following auxiliary space $\tilde{V}_h$ which is sufficiently close to V_h as: for EQ_1^{rot} element,

$$\tilde{V}_h = \left\{ \tilde{u}_h \in L^2(\Omega) : \ \tilde{u}_h|_K \in \text{span}\{1, y, \varphi(y)\}, \ \forall K \in J_h; \ \int_{F_L} [\tilde{u}_h] ds = 0 \right\}, \tag{2.7}$$

or for linear triangular element

$$\tilde{V}_h = \left\{ \tilde{u}_h \in L^2(\Omega) : \ \tilde{u}_h|_K \in \text{span}\{1, y\}, \ \forall K \in J_h; \ \int_{F_L} [\tilde{u}_h] ds = 0 \right\}, \tag{2.8}$$

where F_L are the two longer sides of K.

For any $v_h \in V_h$, let $\tilde{v}_h \in \tilde{V}_h$, and satisfy: for EQ_1^{rot} element,

$$\int_{F_L} v_h ds = \int_{F_L} \tilde{v}_h ds, \quad \int_K v_h dxdy = \int_K \tilde{v}_h dxdy,$$

or for linear triangular element,

$$\int_{F_L} v_h ds = \int_{F_L} \tilde{v}_h ds, \qquad \forall F_L \in \partial K.$$

Obviously the above definitions are meaningful since each element K has two longer edges F_L. It is not hard to check that

$$\partial_y v_h = \partial_y \tilde{v}_h, \quad \partial_x \tilde{v}_h = 0. \tag{2.9}$$

Then applying Poincaré inequality yields

$$\|w_h - \tilde{w}_h\|_{0,K} \leq C h_L \|\partial_x w_h\|_{0,K}, \ \forall w_h \in V_h. \tag{2.10}$$

Lemma 2.3. *For each $v_h \in V_h$, if there exists a point $(x^K, y^K) \in K$ such that $v_h(x^K, y^K) = 0$, then there holds*

$$\|v_h\|_{0,K} \leq C h_L |v_h|_{1,K}. \tag{2.11}$$

Proof. If there exists a point $(x^K, y^K) \in K$ such that $v_h(x^K, y^K) = 0$, then for the linear triangular element defined above,

$$\begin{aligned}
\|v_h\|_{0,K}^2 &= \int_K |v_h(x,y) - v_h(x^K, y^K)|^2 dxdy \\
&= \int_K |\partial_x v_h(x^K, y^K)(x - x^K) + \partial_y v_h(x^K, y^K)(y - y^K)|^2 dxdy \\
&= \int_K |\partial_x v_h(x,y)(x - x^K) + \partial_y v_h(x,y)(y - y^K)|^2 dxdy \\
&\leq 2h_L^2 \|\partial_x v_h\|_{0,K}^2 + 2h_S^2 \|\partial_y v_h\|_{0,K}^2 \leq 4h_L^2 |v_h|_{1,K}^2.
\end{aligned}$$

While for EQ_1^{rot} rectangular element,

$$\begin{aligned}
&v_h(x,y)-v_h(x^K,y^K)\\
&=\partial_x v_h(x^K,y^K)(x-x^K)+\partial_y v_h(x^K,y^K)(y-y^K)\\
&\qquad+\frac{1}{2}\partial^2_{xx}v_h(x^K,y^K)(x-x^K)^2+\frac{1}{2}\partial^2_{yy}v_h(x^K,y^K)(y-y^K)^2\\
&=[\partial_x v_h(x^K,y^K)-\partial^2_{xx}v_h(x^K,y^K)(x-x^K)](x-x^K)\\
&\quad+[\partial_y v_h(x,y)-\partial^2_{yy}v_h(x^K,y^K)(y-y^K)](y-y^K)\\
&\qquad+\frac{1}{2}\partial^2_{xx}v_h(x^K,y^K)(x-x^K)^2+\frac{1}{2}\partial^2_{yy}v_h(x^K,y^K)(y-y^K)^2\\
&=\partial_x v_h(x,y)(x-x^K)+\partial_y v_h(x,y)(y-y^K)-\frac{1}{2}\partial^2_{xx}v_h(x,y)(x-x^K)^2\\
&\qquad-\frac{1}{2}\partial^2_{yy}v_h(x,y)(y-y^K)^2.
\end{aligned}$$

Note that

$$\|\partial^2_{xx}v_h(x,y)\|_{0,K}\le h_L^{-1}\|\partial_x v_h(x,y)\|_{0,K},\quad \|\partial^2_{yy}v_h(x,y)\|_{0,K}\le h_S^{-1}\|\partial_y v_h(x,y)\|_{0,K},$$

we have

$$\begin{aligned}
\|v_h\|_{0,K}&=\left(\int_K |v_h(x,y)-v_h(x^K,y^K)|^2dxdy\right)^{\frac{1}{2}}\\
&\le h_L\|\partial_x v_h(x,y)\|_{0,K}+\frac{1}{2}h_L^2\|\partial^2_{xx}v_h(x,y)\|_{0,K}+h_S\|\partial_y v_h(x,y)\|_{0,K}\\
&\qquad+\frac{1}{2}h_S^2\|\partial^2_{yy}v_h(x,y)\|_{0,K}\le Ch_L|v_h|_{1,K}.
\end{aligned}$$

The proof is completed. $\square$

3. Convergence Analysis and Error Estimates

According to [5], the variational problem of (1.1) is equivalent to:

$$\begin{cases}
-\Delta u=f & \text{in}\quad \Omega^+=\{\,x\in\ \Omega:\ u(x)>\chi(x)\},\\
-\Delta u\ge f & \text{in}\quad \Omega^0=\{\,x\in\ \Omega:\ u(x)=\chi(x)\},\\
u\ge\chi & \text{in}\quad \Omega,\\
u=0 & \text{on}\quad \partial\Omega.
\end{cases}\tag{3.1}$$

The closed convex nonempty set K_h is defined by

$$K_h=\left\{v_h\in V_h:\ \int_{l_i}v_h ds\ge\int_{l_i}\chi ds,\ l_i\subset\partial K, i=1,2,3,4 \text{ or } i=1,2,3\right\}.$$

Then the approximation problem of (1.1) reads as: to find $u_h\in K_h$, such that

$$a_h(u_h,v_h-u_h)\ge f(v_h-u_h),\quad \forall\ v_h\in K_h,\tag{3.2}$$

where $a_h(u,v)=\sum\limits_{K\in J_h}\int_K\nabla u\cdot\nabla v dxdy.$

Now, we will prove the main result of this paper.

Theorem 3.1. *Let u,u_h be the solutions of* (1.1) *and* (3.2) *respectively, $u-\chi\in W^{2,\infty}(\Omega)$, $f\in L^\infty(\Omega)$. With the hypothesis of finite length of the free boundary, we have for anisotropic meshes that*

$$\|u-u_h\|_h=\tilde{C}(u,f,\chi)h,\tag{3.3}$$

where $\|\cdot\|_h = (\sum\limits_K |\cdot|_{1,K}^2)^{\frac{1}{2}}$, $\tilde{C}$ *denotes a positive constant which depends on* u, f *and* χ, *but is independent of* h *and* h_K/ρ_K.

Proof. Here, we only give the proof for the linear triangular element, and the EQ_1^{rot} rectangular element can be treated similarly. Because

$$\|u-u_h\|_h \leq \|u-\Pi_h u\|_h + \|\Pi_h u - u_h\|_h, \tag{3.4}$$

by Lemma 2.1 and (2.4), the first term on the right hand of (3.4) can be estimated as

$$\|u-\Pi_h u\|_h \leq Ch|u|_{2,\Omega}. \tag{3.5}$$

The main difficulty is the estimate of $\|\Pi_h u - u_h\|_h$, the second term on the right hand of (3.4). For the method employed in [9] is not valid for anisotropic meshes, we should develop new techniques to deal with this term. Notice that

$$\begin{aligned}\|\Pi_h u - u_h\|_h^2 =& a_h(\Pi_h u - u_h, \Pi_h u - u_h)\\ =& a_h(\Pi_h u - u, \Pi_h u - u_h) + a_h(u, \Pi_h u - u_h) - a_h(u_h, \Pi_h u - u_h)\\ \leq& C\|\Pi_h u - u\|_h \|\Pi_h u - u_h\|_h + a_h(u, \Pi_h u - u_h) - a_h(u, \Pi_h u - u_h)\\ \leq& Ch|u|_{2,\Omega}\|\Pi_h u - u_h\|_h + E_h(u, \Pi_h u - u_h),\end{aligned} \tag{3.6}$$

where $E_h(u,w_h) = a_h(u,w_h) - f(w_h)$, $w_h = \Pi_h u - u_h$.

Then by Green's formula and (2.9), we have

$$\begin{aligned}E_h(u,w_h) =& a_h(u,w_h) - f(w_h) = \sum_{K\in J_h}\int_K (\nabla u\cdot\nabla w_h - f w_h)dxdy\\ =& \sum_{K\in J_h}\int_K\Big(\partial_x u\partial_x w_h + \partial_y u \partial_y w_h - f w_h\Big)dxdy\\ =& \sum_{K\in J_h}\int_K\Big(\partial_x u\partial_x w_h + \partial_y u \partial_y \tilde{w}_h - f w_h\Big)dxdy\\ =& -\sum_{K\in J_h}\int_K\Big(\partial_{xx}^2 u w_h + \partial_{yy}^2 u\tilde{w}_h + f w_h\Big)dxdy + \sum_{K\in J_h}\int_{\partial K}\Big(\partial_x u w_h n_x + \partial_y u \tilde{w}_h n_y\Big)ds\\ =& -\sum_{K\in J_h}\int_K (\partial_{xx}^2 u + f)(w_h - \tilde{w}_h)dxdy - \sum_{K\in J_h}\int_K (\Delta u + f)\tilde{w}_h dxdy\\ &+ \sum_{K\in J_h}\int_{\partial K}\Big(\partial_x u w_h n_x + \partial_y u \tilde{w}_h n_y\Big)ds\\ =:& I_1 + I_2 + I_3,\end{aligned} \tag{3.7}$$

where

$$I_1 = -\sum_{K\in J_h}\int_K (\partial_{xx}^2 u + f)(w_h - \tilde{w}_h)dxdy, \qquad I_2 = -\sum_{K\in J_h}\int_K (\Delta u + f)\tilde{w}_h dxdy,$$

$$I_3 = \sum_{K\in J_h}\int_{\partial K}\Big(\partial_x u w_h n_x + \partial_y u\tilde{w}_h n_y\Big)ds.$$

Now, we start to estimate each term I_i $(i = 1, 2, 3)$. Firstly, I_1 can be estimated by (2.10) that

$$\begin{aligned}|I_1| &\le C\sum_{K\in J_h}\|\partial^2_{xx}u + f\|_{0,K}\|w_h - \tilde{w}_h\|_{0,K}\\ &\le C\sum_{K\in J_h} h_L\|\partial^2_{xx}u + f\|_{0,K}\|\partial_x w_h\|_{0,K}\\ &\le Ch(|u|_{2,\Omega} + \|f\|_{0,\Omega})\|w_h\|_h. \end{aligned}\tag{3.8}$$

Secondly, thanks to Lemma 2.3 and the fact that $n_x\frac{|F|}{|K|}$ is for all faces of order h_L^{-1} or even zero, we have

$$\begin{aligned}&\sum_{K\in J_h}\int_{\partial K}(\partial_x u)w_h n_x ds = \sum_{K\in J_h}\sum_{F\subset\partial K}\int_F(\partial_x u)w_h n_x ds\\ &= \sum_{K\in J_h}\sum_{F\subset\partial K} n_x\int_F(\partial_x u - M_F\partial_x u)(w_h - M_F w_h)ds\\ &\le C\sum_{K\in J_h}\sum_{F\subset\partial K} n_x\frac{|F|}{|K|}\Big(\sum_{q\in\{L,S\}} h^2_{q,K}\|\partial^2_{xq}u\|^2_{0,K}\Big)^{\frac12}\Big(\sum_{q\in\{L,S\}} h^2_{q,K}\|\partial_q w_h\|^2_{0,K}\Big)^{\frac12}\\ &\le C\sum_{K\in J_h}\Big(\sum_{q\in\{L,S\}} h^2_{q,K}\|\partial^2_{xq}u\|^2_{0,K}\Big)^{\frac12}\|w_h\|_h. \end{aligned}\tag{3.9}$$

Similarly, by (2.9) we have

$$\begin{aligned}&\sum_{K\in J_h}\int_{\partial K}\frac{\partial u}{\partial y}\tilde{w}_h n_y ds\\ &\le C\sum_{K\in J_h}\sum_{F\subset\partial K} n_y\frac{|F|}{|K|}\Big(\sum_{q\in\{L,S\}} h^2_{q,K}\|\partial^2_{yq}u\|^2_{0,K}\Big)^{\frac12}\Big(\sum_{q\in\{L,S\}} h^2_{q,K}\|\partial_q \tilde{w}_h\|^2_{0,K}\Big)^{\frac12}\\ &\le C\sum_{K\in J_h}\sum_{F\subset\partial K} h_L^{-1}\Big(\sum_{q\in\{L,S\}} h^2_{q,K}\|\partial^2_{yq}u\|^2_{0,K}\Big)^{\frac12} h_S\|\partial_y\tilde{w}_h\|_{0,K}\\ &\le C\sum_{K\in J_h}\Big(\sum_{q\in\{L,S\}} h^2_{q,K}\|\partial^2_{yq}u\|^2_{0,K}\Big)^{\frac12}\|w_h\|_h. \end{aligned}\tag{3.10}$$

Thus combining (3.9) and (3.10) yields

$$|I_3| \le Ch|u|_{2,\Omega}\|w_h\|_h. \tag{3.11}$$

In order to estimate the term I_2, let

$$\begin{aligned}&w = -(\Delta u + f),\quad J_h^1 = \{K\in J_h;\ K\cap\Omega^0 = \emptyset\},\\ &J_h^2 = \{K\in J_h;\ K\cap\Omega^+ = \emptyset\},\ J_h^3 = \{K\in J_h;\ K\in J_h - J_h^1 - J_h^2\}.\end{aligned}$$

Then I_2 can be rewritten as

$$\begin{aligned}I_2 &= (w, \widetilde{w}_h) = (w, \Pi_h\widetilde{u - u_h})\\ &= (w, \Pi_h\widetilde{(u-\chi)} - (u-\chi)) + (w, u-\chi) + (w, \Pi_h\widetilde{\chi - u_h})\\ &=: I_{21} + I_{22} + I_{23}, \end{aligned}\tag{3.12}$$

where

$$I_{21} = (w, \widetilde{\Pi_h(u-\chi)} - (u-\chi)), \quad I_{22} = (w, u-\chi), \quad I_{23} = (w, \widetilde{\Pi_h\chi - u_h}).$$

By virtue of the boundness of $h_y/\tilde{h}_y$ and $h_x/\tilde{h}_x$, I_{21} can be estimated as

$$\begin{aligned}
I_{21} &\le \sum_{K\in J_h} \|w\|_{0,K}\|\widetilde{\Pi_h(u-\chi)} - (u-\chi)\|_{0,K} \\
&\le C\sum_{K\in J_h} \|w\|_{0,K}\Big(\|\Pi_h(u-\chi) - \widetilde{\Pi_h(u-\chi)}\|_{0,K} + \|\Pi_h(u-\chi)-(u-\chi)\|_{0,K}\Big) \\
&\le C\sum_{K\in J_h} \|w\|_{0,K}\Big(h_L|\Pi_h(u-\chi)|_{1,K} + \sum_{q\in\{L,S\}} h_{q,K}^2 |\frac{\partial^2(u-\chi)}{\partial q^2}|_{0,K}\Big) \\
&\le C\sum_{K\in J_h} \|w\|_{0,K}\Big(h_L|\Pi_h(u-\chi)-(u-\chi)|_{1,K} + h_L|u-\chi|_{1,K} + h_L^2|u-\chi|_{2,K}\Big) \\
&\le C\sum_{K\in J_h} h_L h_L h_S \|w\|_{0,\infty,K}|u-\chi|_{1,\infty,K} + Ch^2\|w\|_{0,\Omega}|u-\chi|_{2,\Omega} \\
&\le Ch^2\|w\|_{0,\infty,\Omega}|u-\chi|_{1,\infty,\Omega} + Ch^2\|w\|_{0,\Omega}|u-\chi|_{2,\Omega}.
\end{aligned} \tag{3.13}$$

From [5] we know that

$$I_{22} = (w, u-\chi) = 0. \tag{3.14}$$

As to I_{23}, we have

$$\begin{aligned}
I_{23} &= (w, \widetilde{\Pi_h\chi - u_h} - (\Pi_h\chi - u_h)) + (w, \Pi_h\chi - u_h) \\
&\le C\sum_{K\in J_h} h_L\|w\|_{0,K}|\Pi_h\chi - u_h|_{1,K} + (w, \Pi_h\chi - u_h) \\
&\le C\sum_{K\in J_h} h_L\|w\|_{0,K}|\Pi_h\chi - u_h|_{1,K} + (w, \Pi_h\chi - u_h) \\
&\le C\sum_{K\in J_h} h_L\|w\|_{0,K}\Big(|\Pi_h\chi - \chi|_{1,K} + |\chi - u_h|_{1,K}\Big) + (w, \Pi_h\chi - u_h) \\
&=: CI_{23}^1 + I_{23}^2,
\end{aligned} \tag{3.15}$$

where

$$I_{23}^1 = \sum_{K\in J_h} h_L\|w\|_{0,K}\Big(|\Pi_h\chi - \chi|_{1,K} + |\chi - u_h|_{1,K}\Big), \quad I_{23}^2 = (w, \Pi_h\chi - u_h).$$

When the element K lies in the contact part Ω^0, we have $u = \chi,\ w \ge 0$. In this situation $(\Pi_h\chi - u_h)|_K \le 0$, then $I_{23}^2 \le 0$. Otherwise, there exists a point $X^K = (x^K, y^K) \in K$ satisfying $(\Pi_h\chi - u_h)(X^K) = 0$. Therefore, by Lemma 2.3

$$\|\Pi_h\chi - u_h\|_{0,K} \le Ch_L|\Pi_h\chi - u_h|_{1,K}. \tag{3.16}$$

Thus, we have

$$\begin{aligned}
I_{23}^2 &\le C\sum_{K\in J_h} \|w\|_{0,K}\|\Pi_h\chi - u_h\|_{0,K} \le \sum_{K\in J_h} Ch_L\|w\|_{0,K}|\Pi_h u - u_h|_{1,K} \\
&\le Ch^2\|w\|_{0,\Omega}^2 + \frac{1}{8}\|\Pi_h u - u_h\|_h^2.
\end{aligned} \tag{3.17}$$

On the other hand, we can see from (3.1) that for $K \in J_h^1$, $w = 0$, and for $K \in J_h^2$, $u = \chi$. So,

$$\begin{aligned} I_{23}^1 = & \sum_{K\in J_h^2} h_L \|w\|_{0,K}(|\Pi_h\chi-\chi|_{1,K}+|u-u_h|_{1,K}) + \sum_{K\in J_h^3} h_L\|w\|_{0,K}(|\Pi_h\chi-\chi|_{1,K}+|\chi-u_h|_{1,K}|) \\ \le & \sum_{K\in J_h^2} h_L\|w\|_{0,K}\Big(|\Pi_h\chi-\chi|_{1,K}+|u-\Pi_h u|_{1,K}+|\Pi_h u-u_h|_{1,K}\Big) \\ & + \sum_{K\in J_h^3} h_L\|w\|_{0,K}\Big(|\Pi_h\chi-\chi|_{1,K}+|\chi-u|_{1,K}+|u-\Pi_h u|_{1,K}+|\Pi_h u-u_h|_{1,K}\Big) \\ \le & Ch^2\|w\|_{0,\Omega}\Big(|\chi|_{2,\Omega}+\|f\|_{0,\Omega}+\|w\|_{0,\Omega}\Big)+\frac{1}{8}\|\Pi_h u-u_h\|_h^2+C\sum_{K\in J_h^3} h_L\|w\|_{0,K}|\chi-u|_{1,K}. \end{aligned} \tag{3.18}$$

Notice that the hypothesis of the free boundary length is finite implies the total number of $K \in J_h^3$ is no more than $O(\tilde{h}_y^{-1})$. Hence

$$\begin{aligned} &\sum_{K\in J_h^3} h_L\|w\|_{0,K}|\chi-u|_{1,K} \le \sum_{K\in J_h^3} h_L^2 h_{Ky}\|w\|_{0,\infty,\Omega}|\chi-u|_{1,\infty,\Omega} \\ \le & \frac{Ch_x^2 h_y}{\tilde{h}_y}\|w\|_{0,\infty,\Omega}|\chi-u|_{1,\infty,\Omega} \le Ch^2\|w\|_{0,\infty,\Omega}|\chi-u|_{1,\infty,\Omega} \end{aligned}$$

and

$$\begin{aligned} I_{23} &\le Ch^2(|u|_{2,\Omega}+\|f\|_{0,\Omega})(\|w\|_{0,\Omega}+|\chi|_{2,\Omega})+\frac{1}{4}\|\Pi_h u-u_h\|_h^2+Ch^2\|w\|_{0,\infty,\Omega}|\chi-u|_{1,\infty,\Omega} \\ &\le \tilde{C}(u,f,\chi)h^2+\frac{1}{4}\|\Pi_h u-u_h\|_h^2. \end{aligned} \tag{3.19}$$

Collecting (3.7), (3.8), and (3.11)-(3.19), yields

$$|E_h(u,w_h)| \le \tilde{C}(u,f,\chi)h^2+\frac{1}{4}\|\Pi_h u-u_h\|_h^2. \tag{3.20}$$

Substituting (3.20) into (3.6), we have

$$\|u_h-\Pi_h u\|_h^2 \le \tilde{C}(u,f,\chi)h^2. \tag{3.21}$$

Then the desired result follows from (3.5) and (3.21). □

Remark 3.1. We can also use the auxiliary finite element space

$$\tilde{V}_h = \{\tilde{u}_h \in L^2(\Omega);\ \tilde{u}_h|_K \in \mathrm{span}\{1,x,\varphi(x)\}\ \text{ or }\ \tilde{u}_h|_K \in \mathrm{span}\{1,x\},\ \forall K, \int_{F_L}[\tilde{u}_h]ds=0\},$$

if the long edge of rectangular element K or long right-angle edge of triangle element K parallel to $y-$axis.

Remark 3.2. How to get the optimal order estimates remains open for the above two anisotropic finite elements when the length of free boundary is infinite. Furthermore, how to extend the results obtained herein to the quadrilateral cases of [28, 29] is also a very interesting topic in the future studying.

4. Numerical Example

In order to investigate the numerical behavior of the above two finite elements, we consider the following example [27]:

$$\begin{cases} -\triangle u \geq -50, & \text{in } \Omega, \\ u \geq -0.5, & \text{in } \Omega, \\ (-\triangle u + 50)(u+0.5) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega = [0,1] \times [0,1]$.

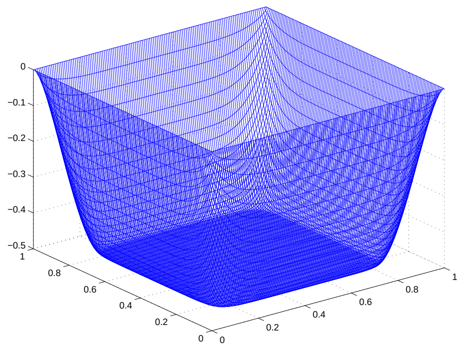

Fig. 4.1 The reference solution of the obstacle problem

We use the following projection SOR algorithm to solve this problem.

Algorithm:

Step 1. Set $u_h^{(0)}$.

Step 2. For $i = 1, ..., N$

$$\tilde{u}_{hi}^{(k+1)} = \frac{1}{a_{ii}}\Big(b_i - \sum_{j=1}^{i-1} a_{ij} u_{hj}^{(k+1)} - \sum_{j=i+1}^{N} a_{ij} u_{hj}^{(k)}\Big).$$

Step 3. Set $u_{hi} = \max\{\chi_i,\ u_{hi}^{(k)} + \omega(\tilde{u}_{hi}^{(k+1)} - u_{hi}^{(k)})\}$, $0 < \omega < 2$ is relaxation factor.

Step 4. If $\|u_h^{(k+1)} - u_h^{(k)}\| \leq \varepsilon$, then go to the next step, otherwise go back to step 2.

Step 5. Output u_h.

Because there is no exact solution to the above problem (4.1), the numerical solution about bilinear finite element on a sufficient refined mesh ($h = 1/256$) is used as the reference solution (see Fig. 4.1).

We subdivide Ω in two ways: Mesh 1 and Mesh 2 (see Fig. 4.2). Mesh 1 is rectangular mesh and Mesh 2 is right triangular mesh, on which $\max\limits_{K\in J_h} h_K/\rho_K \approx 14$. The numerical results about the two nonconforming finite elements are listed in Table 1 and pictured in Fig. 4.3 and Fig. 4.4, where m and n are the subdivision numbers along x-direction and y-direction, respectively.

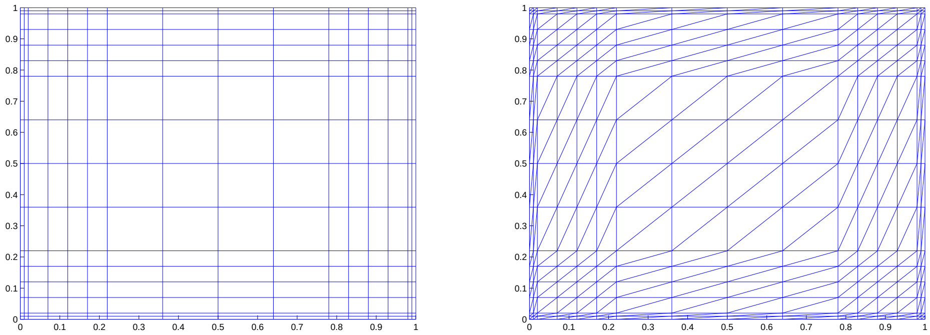

Fig. 4.2 Mesh 1 (left) and Mesh 2 (right) with $m \times n = 16 \times 16$

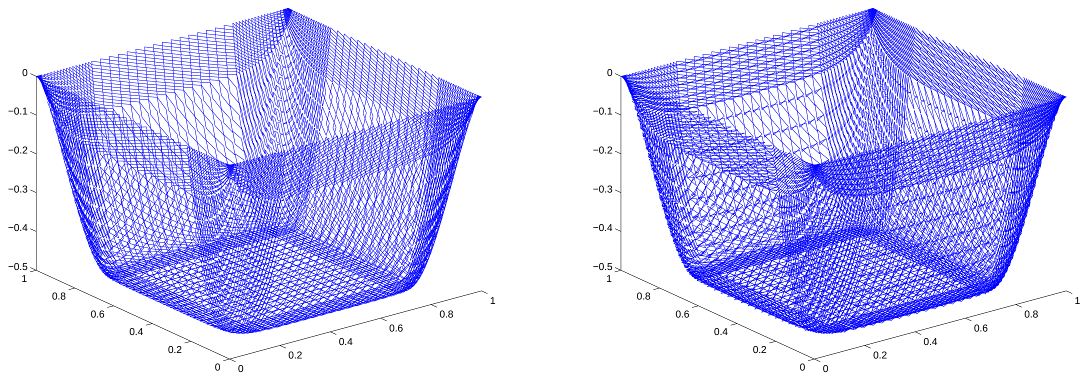

Fig. 4.3 The finite element solutions on Mesh 1 (left) and Mesh 2 (right) with $m \times n = 64 \times 64$

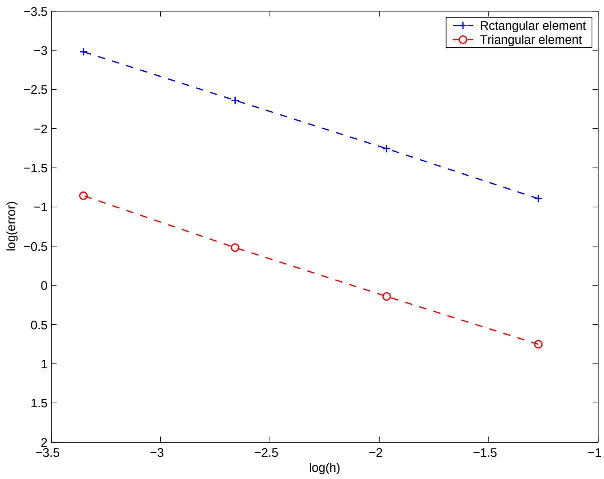

Fig. 4.4 The error graphes of the two finite elements

Table 1 Error estimates in broken energy norm

$m \times n$	8×8	16×16	32×32	64×64
Rectangular element	0.33078256	0.17471666	0.09437756	0.05078487
Order	/	0.9209	0.8885	0.8940
Triangular element	2.11976070	1.15148513	0.61768332	0.31866036
Order	/	0.8804	0.8986	0.9548

From Fig. 4.1 we see that the solution presents an anisotropic phenomenon, i.e., it changes rapidly near the boundary. Thus in order to get more accurate approximation solution and improve the computing efficiency (see Fig. 4.3), we subdivide the domain with anisotropic meshes in local regions (see Fig. 4.2). On the other hand, the numerical results listed in Table 1 and the error graphes pictured in Fig. 4.4 indicate that the convergence rates with order $O(h)$ in broken energy norm are obtained for two nonconforming finite elements considered, which confirm our theoretical analysis.

Acknowledgments. The work of the first author was supported by the NSF of China (No.10971203, 11271340). At the same time, the authors gratefully acknowledge the referees for their great efforts and valuable suggestions or questions on our manuscript.

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