ON MODIFIED HERMITE-FEJÉR INTERPOLATION OMITTING DERIVATIVES

SUN XIE-HUA

(Hangzhou University, Hangzhou, China)

§ 1. Introduction

Let us consider the Hermite-Fejér interpolation
\[ H_n(f,x) = \sum_{k=0}^{n} f(x_k) h_{kn}(x), \quad (1.1) \]
on the interval \([-1, 1]\) for a function \(f \in C([-1, 1])\) where
\[-1 < x_{n-1} < \ldots < x_n < 1, \quad n = 1, 2, \ldots,\]
\[ W_n(x) = \prod_{k=1}^{n} (x - x_k), \]
\[ l_{kn}(x) = W_n(x) / [W_n(x_k)(x-x_k)], \quad k = 1, \ldots, n, \]
\[ h_{kn}(x) = [1 - W_n(x_k)(x-x_k) / W_n(x_k)] l_{kn}(x), \quad k = 1, \ldots, n. \]

It is well-known that for zeros of Chebyshev polynomial \(T_n(x)\)
\[ a_{kn} = \cos \theta_{kn} = \cos \left(2k-1\right) \pi / (2n), \quad k = 1, \ldots, n, \quad (1.2) \]
according to a classical result of L. Fejér, \(H_n(f,x)\) converges uniformly to \(f(x)\).

In 1960, P. Turán suggested that perhaps omission of derivatives at a "few" exceptional points would not damage the convergence property of the modified Hermite-Fejér polynomial \(H_{\mu(n)}^*(f,x)\) with the nodes (1.2). In [2], P. Turán proved that \(H_{\mu(n)}^*(f,x)\) does not converge uniformly in general. Later, K. Kumar and K. K. Mathur considered the following question:

Is there any matrix of nodes for which the modified Hermite-Fejér interpolation \(H_{\mu(n)}^*(f,x)\) given by
\[ H_{\mu(n)}^*(f,x) = H_n(f,x) + (x-x_\mu) l_{\mu}^*(x) W_{\mu}^*(x_\mu) \sum_{k=1}^{n} f(x_k) W_{\mu}^*(x_k) \quad (1.3) \]
satisfying the properties
\[ H_{\mu(n)}^*(f,x_\mu) = f(x_\mu), \quad k = 1, \ldots, n, \]
\[ H_{\mu(n)}^*(f,x_k) = 0, \quad 1 \leq k \leq n, \quad k \neq 0, \]
converges uniformly to every \(f \in C([-1, 1])\). They claimed an affirmative answer for the interpolation \(H_{\mu(n)}^*(f,x)\) constructed on the point-systems
\[ \{ \cos(2k-1) \pi / (2n+1) \}_{k=1}^{n}, \quad (1.4) \]
\[ \{ \cos 2k \pi / (2n+1) \}_{k=0}^{n}, \quad (1.4') \]
\[ \{ \cos(k-1) \pi / (n-1) \}_{k=0}^{n}. \quad (1.5) \]

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But, their result was incorrect. In fact, even \( H_{\text{int}}(f_0, 1) \) with the nodes \( \{ \cos((2k-1)\pi/(2n+1)) \}_{k=1}^{n} \) does not converge to \( f_0(1) \), where \( f_0(x) = x \).

On the other hand, P. Turán proved that uniform convergence of \( H_{\mu}(f, x) \) with the nodes (1.2) in \([-1, 1]\) holds if and only if
\[
\int_{-1}^{1} \frac{xf(x)}{\sqrt{1-x^2}} dx = 0. \tag{1.6}
\]
Condition (1.6) is related to \( f \). In the present paper the author considers the following question:

What are the necessary and sufficient conditions which ensure that the uniform convergence of \( H_{\mu}(f, x) \) still holds for every \( f \in C[-1, 1] \) when a derivative out of the points (1.4) and (1.5) is omitted.

\[\section{2. Main Result} \]

\textbf{Theorem 2.1.} For the interpolation \( H_{\mu}(f, x) \) constructed on the pointsystem (1.4) uniform convergence to every \( f \in C[-1, 1] \) holds if and only if
\[
2n - \mu(n) = O(1). \tag{2.1}
\]

\textbf{Proof.} Denote by \( H_{\mu}(f, x) \) the Hermite–Fejér operator based on the nodes \( \{ \cos((2k-1)\pi/(2n+1)) \}_{k=1}^{n} \). From (1.3) we have
\[
H_{\mu}(f, x) = H_{\mu}(f, x) + J_{n}(x), \tag{2.2}
\]
\[
J_{n}(x) = \frac{(1+x)^2 P_{n}(x)}{x-x_{\mu}} \left[ \frac{2}{(2n+1)^2} \sum_{k=1}^{n} f(x_{k}) \right] - \frac{2n(n+1)}{3(2n+1)^2} f(-1),
\]
where \( P_{n}(x) = \cos((2n+1) \theta/2) / \cos(\theta/2) \). To prove (2.1) is necessary.

We suppose that \( H_{\mu}(f, x) \) converges uniformly to every \( f \in C[-1, 1] \). On using Theorem 1 of [4], i.e., \( \lim_{n \to \infty} H_{\mu}(f, x) = f(x) \) uniformly for every \( f \in C[-1, 1] \), we have that for every \( f \in C[-1, 1] \)
\[
\lim_{n \to \infty} J_{n}(x) = 0.
\]
holds uniformly. Particularly, when \( x^* = \cos \theta^* \), \( \theta^* = \theta_\mu - \pi/[2(2n+1)] \) and \( f(x) = \Omega(x) \) where \( \Omega(x) \) satisfies the following conditions:

(1) \( \Omega(x) \in C[0, 2] \) and \( \Omega(x) \) is nondecreasing,

(2) \( \Omega(x) \geq 0 \) \((x \geq 0)\) and \( \Omega(0) = 0 \);

noting that
\[
\sum_{k=1}^{n} 1/(1+x_{k}) = n(n+1)/3,
\]
we have that
\[
\lim_{n \to \infty} J_{n}(x^*) = \lim_{n \to \infty} \left[ \frac{\cos^3 \theta^*/2}{\sin \frac{1}{2} (\theta_\mu - \theta^*) \sin \frac{1}{2} (\theta_\mu + \theta^*)} \cdot \frac{2}{(2n+1)^2} \sum_{k=1}^{n} \Omega(1+x_{k}) \right] = 0
\]
holds. From the monotonicity of \( \Omega(x) \) we obtain
\[
\frac{2}{(2n+1)^2} \sum_{k=1}^{2n} \Omega(1+x_k) = \frac{2}{(2n+1)x^3} \int_{1/2n+1}^{1/2n} \Omega(2x^2) \, dx \\
\leq \frac{2}{(2n+1)x^3} \sum_{k=2}^{2n} \Omega \left( \frac{2}{k+1} \right). \tag{2.5}
\]

On the other hand, it is easy to see that
\[
\left| \sin \left( \frac{1}{2} (\theta_\mu + \theta') \right) \right| \sim \sin \theta_\mu, \quad \cos^3 \theta' / 2 \sim \cos^3 \theta_\mu / 2, \quad \mu \neq n+1
\]
hence
\[
\left| \cos^3 \left( \frac{2}{k+1} \right) \left[ \sin \left( \frac{1}{2} (\theta_\mu - \theta') \right) \sin \left( \frac{1}{2} (\theta_\mu + \theta') \right) \right] \right| \geq c \cdot n \cos \left( \frac{1}{2} \theta_\mu \right) \geq c(n-\mu+2), \quad 1 \leq \mu \leq n+1. \tag{2.6}
\]

From (2.4)–(2.6), we have for every \( \Omega(x) \) satisfying condition (2.3)
\[
\lim_{n \to \infty} \frac{n-\mu}{n} \sum_{k=1}^{2n} \Omega \left( \frac{2}{k+1} \right) = 0. \tag{2.7}
\]

If \( n-\mu(n) \neq O(1) \), since \( n-\mu(n) \) is monotone, \( n-\mu(n) = N(n) \to \infty(n \to \infty) \). Define the following functions \( N(n) \) and \( \Omega_0(x) \):
\[
N(n) := \begin{cases} 
N(n), & x=n \geq 3, \\
N(3), & 0 < x < 3, \\
\text{linear}, & \text{otherwise},
\end{cases}
\]
and
\[
\Omega_0(x) := \begin{cases} 
N^{-1}(\frac{x}{3}), & x > 0, \\
0, & x = 0.
\end{cases}
\]

Obviously, \( \Omega_0(x) \) satisfies (2.3). Then
\[
|J_*(x^*)| \geq c \frac{N(n)}{n} \sum_{k=1}^{2n} \Omega_0 \left( \frac{2}{k+1} \right) \geq c n \Omega_0 (1/n^3) \geq c > 0.
\]

This contradicts (2.7), which completes the proof of necessity.

Now, assume that \( n-\mu(n) = O(1) \). Obviously,
\[
\frac{2}{(2n+1)^2} \sum_{k=1}^{2n} f(x_k) = \frac{2n(n+1)}{3(2n+1)} f(-1) = O \left( \frac{1}{n} \right) \sum_{k=1}^{2n} \omega \left( f, \frac{1}{k^3} \right). \tag{2.8}
\]

Since
\[
\cos \left( \frac{1}{2} \right) \theta = \cos \left( \frac{1}{2} (\theta \pm \theta_\mu) \right) \cos \left( \frac{1}{2} \theta_\mu \right) \pm \sin \left( \frac{1}{2} (\theta \pm \theta_\mu) \right) \sin \left( \frac{1}{2} \theta_\mu \right),
\]
we have
\[
\frac{(1+x)^3 \beta(x)}{|x-x_\mu|} = \frac{2 \cos^3 \left( \frac{\theta}{2} \right) \cos^3 \left( \frac{2n+1}{2} \right)}{\sin \left( \frac{1}{2} (\theta - \theta_\mu) \right) \sin \left( \frac{1}{2} (\theta + \theta_\mu) \right)} = O(1). \tag{2.9}
\]

Combining (2.8) and (2.9) and noting Theorem 1 of [4], we see
\[
\lim_{n \to \infty} H^*_{-\infty, \infty}(f, x) = f(x)
\]
holds uniformly. This completes the proof.

Similarly, we can prove
Theorem 2.2. For the interpolation process \( Q_{\mu(n)}^*(f, x) \) based on the point-system (1.5), uniform convergence for every \( f \in C[-1, 1] \) holds if and only if
\[
\mu(n) = O(1) \quad \text{or} \quad n - \mu(n) = O(1).
\]

References