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On the behavior of the four order iteration in Euler's family near a zero

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Abstract. The aim of this paper is to study the local convergence of the four order iteration of Euler's family for solving nonlinear operator equations. We get the optimal radius of the local convergence ball of the method for operators satisfying the weak third order generalized Lipschitz condition with $L-$average. We also show that the local convergence of the method is determined by a period 2 orbit of the method itself applied to a real function.

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Key words: local convergence; convergence ball; period 2 orbit; generalized Lipschitz condition.

1 Introduction

Let

$$f(x)=0, \tag{1.1}$$

where $f:D\subset X\to Y$ is a nonlinear operator defined on a convex set D of a real or complex Banach space X and valued in a same type space Y.

The method used often to solve a solution of (1.1) is Newton's method

$$\begin{cases} x_{n+1}=N_f^n(x), \quad x\in D,\, n\geq 1,\\ N_f(x)=x-f'(x)^{-1}f(x) \end{cases} \tag{1.2}$$

and its modifications with an approximation to f', or the modifications using higher derivatives(such as Euler's family, Halley's family and etc.).

The analysis of the convergence of an iteration is always the one of fields mostly interested in numerical nonlinear algebra. There are many papers concerning the semi-local and the local convergence of iterations for solving nonlinear equations. Parts of the latest papers are [1–6, 11].

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It is proved that for operators satisfying some kinds of Lipschitz condition with $L-$average, the criteria to guarantee the semi-local convergence of each stationary iteration of Euler's family and of Halley's family are the same, or, in other words, the criteria is an invariant independent on iterations [12–14]. The Lipschitz condition with $L-$average takes the classical Kantorovich-type condition [7] and Smale-type condition [8] as two special cases.

What interesting is that status for the local convergence is different. For Newton's method (1.2) and Euler's method

$$\begin{cases} x_{n+1}=E_f^n(x), \quad x\in D, \;\; n\geq 1, \\ E_f(x)=x-[1-\dfrac{1}{2}f'(x)^{-1}f''(x)f'(x)^{-1}f(x)]f'(x)^{-1}f(x). \end{cases} \tag{1.3}$$

[10] proved that the criteria to guarantee the local convergence are linked closely to the constructions of iterations and the dynamical properties of themselves applied to a real or complex function. In details, under the second order Lipschitz condition with $L-$average, [10] proved that the local convergence behavior of Newton's method is determined by a period 2 orbit[‡] of itself applied to a real or complex function, and the behavior of Euler's method is determined by an repelling additional fixed point (we call $p\in X$ is an additional fixed point of $E_f(x)$, if $E_f(p)=p$ and $f(p)\neq 0$) of itself applied to the same function. If we write R_N,R_E as the optimal radii of Newton's method and Euler's method, respectively, then $R_E<R_N$.

It is natural to ask how the local convergence behavior of iterations changes along with the convergent order becomes higher. Based on that Newton's method and Euler's method are the first two members of Euler's family, in this paper, we consider the local convergence behavior of the third member of Euler's family, which is defined by

$$x_{n+1}=G_f^n(x), \quad x\in D, \;\; n\geq 1, \tag{1.4}$$

where

$$G_f(x)=x+\sum_{i=1}^{3}\frac{1}{i!}[f_x^{-1}(f(x))]^{(i)}(-f(x))^i,$$

and f_x^{-1} is the local inverse of f at x. [9] shows us that (1.4) can be written in the following formula:

$$\begin{aligned} G_f(x)=x+\Delta_1-\frac{1}{2}f'(x)^{-1}f''(x)\Delta_1^2-f'(x)^{-1}f''(x)\Delta_1\Delta_2 \\ -\frac{1}{3!}f'(x)^{-1}f'''(x)\Delta_1^3, \end{aligned} \tag{1.5}$$

where

$$\begin{cases} \Delta_1=\Delta_{f,1}=-f'(x)^{-1}f(x), \\ \Delta_2=\Delta_{f,2}=E_f(x)-N_f(x). \end{cases} \tag{1.6}$$

‡ $\{t_1,t_2\}$, a subset of real numbers, is called a period 2 orbit of a real function $Iter(t)$ if $t_2=Iter(t_1)$ and $t_1=Iter(t_2)$. Any period 2 orbit of the iterative function is called the period 2 orbit of the corresponding iterative method.

We will prove that if f satisfies the weak third order Lipschtiz condition with $L-$ average described in section 2, then the local convergence of (1.4) near a zero of (1.1) is determined by a period 2 orbit of (1.4) applied to a real function defined in section 2. Furthermore, if we denotes R_G as the optimal radius of the local convergence ball of (1.4), then $R_G < R_E < R_N$.

The paper is organized as follows. In section 2, we will state the Basic Assumptions and main results, and in section 3, we want to display some useful formulas that are important for our analysis. In section 4 and section 5, we are going to study the local convergence of (1.4) in the real field and in Banach spaces, respectively.

2 Assumptions and Results

Denote $\mathcal{L}(X,Y)$ as the space of linear operators defined on a convex set D of the Banach space X and valued in the Banach space Y. Let $L(t)$ be a real function with following properties: $L(t)$ is $C^2[0,r]$ with $L^{(i)}(0)>0$ for $i=1,2$ and $\frac{L''(t)}{t}>0$ increases in $(0,r)$, where r is a positive number satisfying

$$\int_0^r L(t)dt=1. \tag{2.1}$$

Basic Assumptions:

- $f(\zeta)=0$, where $\zeta\in O(\zeta,r)\subset D$ and $O(\zeta,r)=\{x|\|x-\zeta\|<r\}$.

- $f(x)$ has the third Fréchet derivatives in $\mathring{O}(\zeta,r)=\{x|0<\|x-\zeta\|<r\}$.

- $f'(\zeta)$ exists and has inverse, and $f(x)$ satisfies the weak third order Lipschitz condition with $L-$average defined by

$$\begin{cases} \limsup\limits_{x\to\zeta}\|f'(\zeta)^{-1}f^{(i)}(x)\|\le L^{(i-1)}(0), \quad i=2,3, \\ \|f'(\zeta)^{-1}[f'''(x)-f'''(\zeta+\theta(x-\zeta))]\|\le \displaystyle\int_{\theta\|x-\zeta\|}^{\|x-\zeta\|}L''(u)du, \\ \qquad\qquad \forall\theta\in(0,1],\ \forall x\in\mathring{O}(\zeta,r). \end{cases} \tag{2.2}$$

We say f satisfies the weak third order Lipschitz condition with $L-$average due to that here we do not assume f has the second and the third Fréchet derivatives at the zero point ζ.

Let

$$h(t)=\begin{cases} h_r(t) & t\ge 0, \\ -h_r(-t) & t<0. \end{cases} \tag{2.3}$$

where

$$h_r(t) = -t + \int_0^t (t-u)L(u)du. \tag{2.4}$$

We have

Theorem 2.1. *If f satisfies Basic Assumptions and R_G is the smallest positive zero of $G_h(t)+t$, then*

1) *$\forall x \in \mathring{O}(\zeta, R_G)$, the iteration sequence $\{G_f^n(x)\}_{n\geq 1}$, generated by* (1.4) *starting from x, converges to ζ with the error estimates*

$$\|G_f^n(x) - \zeta\| \leq q(x)^{4^n - 1}\|x - \zeta\|, \quad n \geq 1, \tag{2.5}$$

if $G_f^n(x) \neq \zeta$ for all $n \geq 1$, where

$$q(x) = \sqrt[3]{\frac{G_h(\|x-\zeta\|)}{-\|x-\zeta\|}} \in (0,1). \tag{2.6}$$

2) *R_G is the optimal radius of the local convergence ball of* (1.4) *with*

$$G_h(R_G) = -R_G, \quad G_h(-R_G) = R_G, \tag{2.7a}$$

or

$$|G_{h_r}(R_G)| = R_G, \tag{2.7b}$$

and

$$R_G < R_E < R_N. \tag{2.8}$$

Here R_E and R_N are the optimal radii of local convergence balls of Newton's method and Euler's method for nonlinear operators satisfying the Basic Assumptions, respectively.

Remark 2.1. (2.7a) shows that $\{R_G, -R_G\}$ is a period 2 orbit of iteration G_h. It determines the local convergence of G_f in Banach spaces for f satisfying the Basic Assumptions. (2.5) and (2.7b) shows that R_G is an non-attractive fixed point of $|G_h|$.

Remark 2.2. If f is asked to have the third Fréchet derivatives in $O(\zeta,r)$, then

$$\begin{cases} \|f'(\zeta)^{-1}f^{(i)}(\zeta)\| \leq L^{(i-2)}(0), \qquad i=2,3, \\ \|f'(\zeta)^{-1}[f'''(x) - f'''(\zeta+\theta(x-\zeta))]\| \leq \int_{\theta\|x-\zeta\|}^{\|x-\zeta\|} L''(u)du, \\ \qquad\qquad \forall \theta \in [0,1], \ \forall x \in O(\zeta,r). \end{cases}$$

is used instead of (2.2) , and Theorem 2.1 is still true except that R_G is optimal .

3 Basic formulas

In this section, we want to display three basic formulas that are important for the proof of Theorem 2.1.

Suppose $N_f(x), E_f(x)$ and $G_f(x)$ exist for some $x \neq \zeta$, where $x \in D$, then by Taylor's expansion, we have for any $\tau \in (0,1]$ and $\tau_x \triangleq \zeta + \tau(x-\zeta) \in \mathring{O}(\zeta, r)$

$$
\begin{aligned}
\zeta - N_f(x) &= \zeta - x - \Delta_1 \\
&= \zeta - \tau_x + f'(x)^{-1} f(\tau_x) - f'(x)^{-1} \int_0^1 \theta f''(\tau_x + \theta(x-\tau_x) d\theta (x-\tau_x)^2, \qquad (3.1) \\
\zeta - E_f(x) &= \zeta - x - \Delta_1 - \Delta_2 \\
&= \zeta - \tau_x + f'(x)^{-1} f(\tau_x) + f'(x)^{-1}[-f(\tau_x) + f(x) + f'(x)(\tau_x - x) + \frac{1}{2} f''(x)(\tau_x - x)^2] \\
&\quad - \frac{1}{2} f'(x)^{-1} f''(x) \left[\tau_x - x + f'(x)^{-1} f(x)\right] \left[\tau_x - x - f'(x)^{-1} f(x)\right] \\
&= \zeta - \tau_x + f'(x)^{-1} f(\tau_x) + \frac{1}{2} f'(x)^{-1} f''(x) f'(x)^{-1} [2f(x) - f(\tau_x)] f'(x)^{-1} f(\tau_x) \\
&\quad + f'(x)^{-1} \int_0^1 \theta [f''(x) - f''(\tau_x + \theta(x-\tau_x))] d\theta (x-\tau_x)^2 \\
&\quad + \frac{1}{2} f'(x)^{-1} f''(x) f'(x)^{-1} \int_0^1 \theta f''(\tau_x + \theta(x-\tau_x) d\theta (x-\tau_x)^2 \\
&\quad \cdot \left[2 - f'(x)^{-1} \int_0^1 \theta f''(\tau_x + \theta(x-\tau_x) d\theta (x-\tau_x)\right] (\tau_x - x), \qquad (3.2) \\
G_f(x) - \zeta &= \tau_x - \zeta - f'(x)^{-1} f(\tau_x) \\
&\quad + f'(x)^{-1}[-f(x) - f'(x)(\tau_x - x) - \frac{1}{2} f''(x)(x-\tau_x)^2] \\
&\quad + f'(x)^{-1} f''(x)[(\tau_x - x)^2 - \Delta_1^2 - 2\Delta_1 \Delta_2] - \frac{1}{3!} f'(x)^{-1} f'''(x) \Delta_1^3 \\
&= \zeta - \tau_x + f'(x)^{-1} f(\tau_x) + I_f(x,\tau) + II_f(x,\tau) + III_f(x,\tau), \qquad (3.3)
\end{aligned}
$$

where

$$
\begin{aligned}
I_f(x,\tau) &= f'(x)^{-1} \left[\int_0^1 \frac{\theta^2}{2} f'''(\tau_x + \theta(x-\tau_x)) d\theta (\tau_x - x)^3 + \frac{1}{3!} f'''(x)(x-\tau_x)^3\right] \\
&= f'(x)^{-1} \int_0^1 \frac{\theta^2}{2} \left[f'''(x) - f'''(\tau_x + \theta(x-\tau_x))\right] d\theta (x-\tau_x)^3, \qquad (3.4a) \\
II_f(x,\tau) &= \frac{1}{2} f'(x)^{-1} f''(x)[(\tau_x - x)^2 - \Delta_1^2 - 2\Delta_1\Delta_2] \\
&= \frac{1}{2} f'(x)^{-1} f''(x)[(\tau_x - N_f(x))^2 + 2(\tau_x - E_f(x))\Delta_1], \qquad (3.4b) \\
III_f(x,\tau) &= \frac{1}{3!} f'''(x) \left[(\tau_x - x)^3 - \Delta_1^3\right]
\end{aligned}
$$

$$=\frac{1}{3!}f'''(x)(\tau_x-N_f(x))\Big[(\tau_x-N_f(x))^4+3(\tau_x-x)\Delta_1\Big]. \tag{3.4c}$$

Here, in (3.2) and (3.3), operations of vectors linked with higher order derivatives used in [9] are adopted.

4 The convergence of $G_h^n(t)(n\geq 1)$

It can be verified that

Lemma 4.1. *All the* $-\frac{1}{h'(t)}>0, h''(t)>0$ *and* $h'''(t)>0$ *increase on* $[0,r)$, *and*

$$h^{(i)}(t)=(-1)^{i+1}h^{(i)}(-t),\quad -r\leq t<0,\quad i=0,1,2,3,4. \tag{4.1}$$

We need following lemmas.

Lemma 4.2. *Let* $R_G=\min\{u>0|G_h(u)+u=0\}$, *and* R_E *and* R_N *be the optimal radii of the local convergence ball of* E_h *and of* N_h, *respectively, then* R_E *and* R_N *are the smallest positive reals satisfying*†

$$-\frac{1}{2}\frac{h''(t)h(t)}{h'(t)}\Big|_{t=R_E}=1,\qquad \frac{h(t)}{th'(t)}\Big|_{t=R_N}=2, \tag{4.2}$$

and $R_G<R_E<R_N$.

Proof. By almost the same deduction procedure used in [10], we have that R_E and R_N, the optimal radii of the local convergence balls of $E_h^n(t)$ and $N_h^n(t)$, respectively, are the smallest positive reals satisfying (4.2), $R_E<R_N$ and

$$-t<t-\frac{h(t)}{h'(t)}<0,\quad \forall t\in(0,R_E).$$

Following this inequality and

$$-\frac{h''(t)}{h'(t)}\Delta_{h,2}\Big|_{t=R_G}=2\Big(\frac{h''(t)h(t)}{2h^2(t)}\Big)^2\Big|_{t=R_G}=2,$$

we have

$$\begin{aligned}(G_h(t)+t)\Big|_{t=R_E}&=\Big(2t-\frac{h(t)}{h'(t)}\Delta_{h,1}\Delta_{h,2}-\frac{h'''(t)}{3!h'(t)\Delta_{h,1}^3}\Big)\Big|_{t=R_E}\\&=\Big[2\Big(t-\frac{h(t)}{h'(t)}\Big)-\frac{h'''(t)}{3!h'(t)}\Delta_{h,1}^3\Big]\Big|_{t=R_E}<0,\end{aligned}$$

which, by combining with $[G_h(t)+t]'_+|_{t=0}=1$, and $[G_h(t)+t]|_{t=0^+}=0$, deduces that there is a positive zero of $G_h(t)+t$ in $(0,R_E)$. Consequently R_G exists with $R_G<R_E$ and the proof is completed. □

† R_N and R_E satisfying (4.2) were first defined in [10].

Lemma 4.3. *We have*

i) R_G*, defined in Lemma 4.2, is the unique zero of* $G_h(t)+t$ *in* $(0,R_E)$*.*

ii) $\forall |t|\in(0,R_G)$*, hold*

$$sgn(t)G_h(t)<0, \quad and \quad |G_h(t)|<|t|. \tag{4.3}$$

iii) Let $q(t)=\sqrt[3]{\frac{G_h(t)}{-t}}$ *for* $0<t<R_G$*. Then*

$$\begin{cases} q(t)=q(|t|)\in(0,1), \\ |G_h(t)|=|G_h(|t|)|=|G_{h_r}(|t|)|, \end{cases} \qquad \forall |t|\in(0,R_G). \tag{4.4}$$

iv) Moreover,

$$\begin{cases} G_h(R_G)=-R_G, \quad G_h(-R_G)=R_G, \\ |G_{h_r}(R_G)|=R_G \end{cases} \tag{4.5}$$

Proof. If $h'(t)\neq 0$ for some $t\in(0,r)$, denote

$$\begin{cases} D_{N,\,h}=-\dfrac{1}{h'(t)}\displaystyle\int_0^1 \theta h''(\theta t)d\theta, \\ D_{E,\,h}=-\dfrac{1}{h'(t)}\displaystyle\int_0^1 \dfrac{\theta^2}{2}h'''(\theta t)d\theta \\ \qquad\qquad +\dfrac{1}{2}\dfrac{h''(t)}{h'(t)^2}\displaystyle\int_0^1 \theta h''(\theta t)d\theta\Big[2-\int_0^1\theta\dfrac{h''(\theta t)}{h'(t)}d\theta t\Big]. \end{cases} \tag{4.6}$$

By Lemma 4.1, we have

$$\begin{cases} D_{N,\,h}(t)=-D_{N,\,h}(-t)>0, \\ D_{E,\,h}(t)=D_{E,\,h}(-t)>0, \end{cases} \qquad \forall t\in(0,R_E). \tag{4.7}$$

It can be verified that $D_{N,\,h}(t)$ and $D_{E,\,h}(t)$ increase in $(0,R_E)$. Let

$$\begin{cases} Q_1(t)=\dfrac{1}{h'(t)}\displaystyle\int_0^1 \dfrac{\theta^2}{2}\cdot\dfrac{h'''(t)-h'''(0)}{t}d\theta, \\ Q_2(t)=\dfrac{h''(t)}{2h'(t)}\Big[D^2_{N,\,h}(t)+2D_{E,\,h}\Big(1-\dfrac{\int_0^1\theta h''(\theta t)d\theta}{h'(t)}t\Big)\Big], \\ Q_3(t)=\dfrac{h'''(t)}{3!h'(t)}D_{N,\,h}(t)\Big[D^4_{N,\,h}(t)t^2+3\Big(1-\dfrac{\int_0^1\theta h''(\theta t)d\theta}{h'(t)}t\Big)\Big]. \end{cases} \tag{4.8}$$

Then $-Q_i(t)$ is positive and increases in $(0,R_G)$ for each $i=1,2,3$ from the properties of the function L definition in section 2, Lemma 4.1 and (4.7). Since $h_+^{(i)}(0)$ and $h_-^{(i)}(0)$ exist for $i=2,3$, it can be checked easily that (3.1)−(3.4) still hold when $X=Y=D$ are the real field and $f=h$. In this case, $h_+^{(i)}(0)$ and $h_-^{(i)}(0)$ are used when $\tau=0$. In other words,

$$\begin{aligned} &G_h(t)=I_h(t,0)+II_h(t,0)+III_h(t,0),\\ &I_h(t,0)=Q_1(t)t^4, \quad II_h(,0)=Q_2(t)t^4, \quad III_h(t,0)=Q_3(t)t^4. \end{aligned} \tag{4.9}$$

It follows that

$$\frac{G_h(t)}{-t}=-[Q_1(t)+Q_2(t)+Q_3(t)]t^3>0 \tag{4.10}$$

increases in $(0,R_E)$. Therefore, by Lemma 4.2, R_G is the unique zero of $G_h(t)+t$ in $(0,R_E)$, or equivalently, is the unique solution of $\frac{G_h(t)}{-t}=1$ in $(0,R_E)$, which proves i) and deduces

$$\begin{cases} 0<q(t)=\sqrt[3]{\dfrac{G_h(t)}{-t}}<1, \quad \forall t\in(0,R_G),\\ G_h(R_G)=-R_G. \end{cases} \tag{4.11}$$

Since

$$Q_1(t)=-Q_1(-t),\ \ Q_2(t)=-Q_2(-t),\ \ Q_3(t)=-Q_3(-t),\ \ \forall t\in(0,R_G) \tag{4.12}$$

is true by Lemma 4.1, holds

$$G_h(t)=-G_h(-t), \quad \text{and} \quad q(t)=q(-t), \quad \forall t\in(0,R_G). \tag{4.13}$$

Then, ii) and iii), or (4.3) and (4.4), follow from (4.10)–(4.13), and iv), or (4.5), follows from ii) and iii), respectively. The proof is completed. □

By the lemmas above, we can get

Proposition 4.1. For any $|t|\in(0,R_G)$, $\{G_h^n(t)\}_{n\geq 1}$ converges to 0 with error estimates

$$\left|G_h^n(t)\right|<q(t)^{4^n-1}|t|, \qquad n\geq 1, \tag{4.14}$$

where R_G and $q(t)$ are defined in Lemma 4.3. Also, R_G is the radius of the local convergence ball of $G_h(t)$ (or equivalently, a non-attractive fixed point of $|G_{h_r}(t)|$), and $\{R_G,-R_G\}$ is a period 2 orbit of $G_h(t)$.

Proof. It is obviously that (4.14) is true for $n=0$. Suppose (4.14) is true for $n=k$, then by (4.12), for any $0<|t|<R_G$, hold $|G_h^k(t)|<|t|$ and

$$\begin{aligned} |G_h^{k+1}(t)|&=|Q_1(G_h^k(t))+Q_2(G_h^k(t))+Q_3(G_h^k(t))|(G_h^k(t))^4\\ &=-\Big(Q_1(|G_h^k(t)|)+Q_2(|G_h^k(t)|)+Q_3(|G_h^k(t)|)\Big)(G_h^k(t))^4\\ &\leq|Q_1(t)+Q_2(t)+Q_3(t)|q(t)^{4^{k+1}-4}t^4\\ &=q(t)^{4^{k+1}-1}|t|, \end{aligned}$$

that is to say, (4.14) is still true for $n=k+1$. (4.14) is then proved by induction method.

Following (4.5) in Lemma 4.3, R_G, a fixed point of $|G_{h_r}(t)|$, is the radius of the local convergence ball of $G_h(t)$, and $\{R_G,-R_G\}$ is a period 2 orbit of $G_h(t)$. (4.4) in lemma 4.3 and (4.14) show us also that R_G is a non-attractive fixed point of $|G_{h_r}(t)|$, which completes the proof. $\square$

Remark 4.1. Proposition 4.1 shows that the local convergence of $G_h(t)$ is determined by its period 2 orbit nearest to 0.

Remark 4.2. By Lemma 4.3, (4.14) is equivalent to

$$\left|G_h^n(|t|)\right| < q(|t|)^{4^n-1}|t|, \qquad \forall |t|\in(0,R_G), \quad n\geq 0.$$

Remark 4.3. Let $t_n = G_h^n(t_0)(n\geq 1)$, then for all $|t_0|\in(0,R_G)$,

$$|t_{n+1}|<|t_n|, t_n t_{n+1}<0. \qquad n\geq 0,$$

or

$$\begin{cases} t_1<t_3<\cdots<t_{2k+1}<\cdots<0<\cdots<t_{2k+2}<\cdots<t_2<t_0, & 0<t_0<R_G, \\ t_1>t_3>\cdots>t_{2k+1}>\cdots>0>\cdots>t_{2k+2}>\cdots>t_2>t_0, & -R_G<t_0<0 \end{cases}$$

follows from (4.3).

5 The convergence of $G_f^n(x)(n\geq 1)$

We need the following lemmas.

Lemma 5.1. *If f satisfies* (2.2) *, then* $\forall x\in\mathring{O}(\zeta,r)$,

$$\|f'(\zeta)^{-1}\left[f''(x)-f''(\zeta+\theta(x-\zeta))\right]\|\leq\int_{\theta\|x-\zeta\|}^{\|x-\zeta\|}L'(u)du \tag{5.1a}$$

holds for all $0<\theta\leq 1$ *and*

$$\|f'(\zeta)^{-1}\left[f'(x)-f'(\zeta+\theta(x-\zeta))\right]\|\leq\int_{\theta\|x-\zeta\|}^{\|x-\zeta\|}L(u)du \tag{5.1b}$$

holds for all $0\leq\theta\leq 1$

Proof. Since for all $x\in\mathring{O}(\zeta,r)$ and all $0<\theta\leq 1$, there is

$$f''(x)-f''(\zeta+\theta(x-\zeta))=\int_0^1 f'''(\zeta+\theta(x-\zeta)+\mu(1-\theta)(x-\zeta))d\mu(1-\theta)(x-\zeta).$$

By (2.2) in Basic Assumptions,

$$\|f'(\zeta)^{-1}\left[f''(x)-f''(\zeta+\theta(x-\zeta))\right]\|$$

$$
\begin{aligned}
&\le \int_0^1 \|f'(\zeta)^{-1}\Big[f'''(\zeta+\theta(x-\zeta)+\mu(1-\theta)(x-\zeta))-f'''(\tau_x)\Big]\|d\mu(1-\theta)\|x-\zeta\| \\
&\quad +(1-\theta)\|f'''(\tau_x)\|\|x-\zeta\| \\
&\le \int_0^1 \int_{\tau\|x-\zeta\|}^{(\theta+\mu(1-\theta))\|x-\zeta\|} L''(u)dud\mu(1-\theta)\|x-\zeta\|+(1-\theta)\|f'''(\tau_x)\|\|x-\zeta\| \\
&\le \int_0^1 \int_0^{(\theta+\mu(1-\theta))\|x-\zeta\|} L''(u)dud\mu(1-\theta)\|x-\zeta\|+(1-\theta)\|f'''(\tau_x)\|\|x-\zeta\|
\end{aligned}
$$

holds for any $0<\tau<1$. Letting $\tau\to 0^+$, we have

$$
\begin{aligned}
&\|f'(\zeta)^{-1}\Big[f''(x)-f''(\zeta+\theta(x-\zeta))\Big]\| \\
\le& \int_0^1 \int_0^{(\theta+\mu(1-\theta))\|x-\zeta\|} L''(u)dud\mu(1-\theta)\|x-\zeta\|+(1-\theta)\|L'(0)\|\|x-\zeta\| \\
=& \int_0^1 L'(\theta+\mu(1-\theta))\|x-\zeta\|)d\mu(1-\theta)\|x-\zeta\| \\
=& \int_{\theta\|x-\zeta\|}^{\|x-\zeta\|} L'(u)du.
\end{aligned}
$$

(5.1a) is proved. (5.1b) follows from (5.1a) by the similar method, which completes the proof. □

Lemma 5.2. *If f satisfies* (2.2) *, then $\forall x\in \mathring{O}(\zeta,R_E)$, where R_E is defined in Lemma 4.2, we have*

$$\|f'(\zeta)^{-1}f''(x)\|\le h''(\|x-\zeta\|), \tag{5.2}$$

and $f'(x)$ is conversable with

$$\|f'(x)^{-1}f'(\zeta)\|\le -\frac{1}{h'(\|x-\zeta\|)}, \text{ and } \|f'(x)^{-1}f(x)\|\le \frac{h(\|x-\zeta\|)}{h'(\|x-\zeta\|)}. \tag{5.3}$$

Proof. For any $x\in \mathring{O}(\zeta,R_E)$ and $0<\theta<1$, we have by (5.1a)

$$
\begin{aligned}
\|f'(\zeta)^{-1}f''(x)\| &\le \|f'(\zeta)^{-1}[f''(x)-f''(\zeta+\theta(x-\zeta))]\|+\|f'(\zeta)^{-1}f''(\zeta+\theta(x-\zeta))\| \\
&\le \int_{\theta\|x-\zeta\|}^{\|x-\zeta\|} L'(u)du+\|f'(\zeta)^{-1}f''(\zeta+\theta(x-\zeta))\| \\
&\le \int_0^{\|x-\zeta\|} L'(u)du+\|f'(\zeta)^{-1}f''(\zeta+\theta(x-\zeta))\|.
\end{aligned}
$$

Letting $\theta\to 0^+$, holds

$$\|f'(\zeta)^{-1}f''(x)\|\le \int_0^{\|x-\zeta\|} L'(u)du+L(0)=L(\|x-\zeta\|)=h''(\|x-\zeta\|)$$

from the definition of the function h. That is to say (5.2) is true.

For any $x\in O(\zeta,R_E)$ and $\theta_x=\zeta+\theta(x-\zeta)$ with $0<\theta<1$, we have by (5.1b) and (5.2)

$$\begin{aligned}
&\|f'(\zeta)^{-1}f'(x)-I\| \\
\le&\|f'(\zeta)^{-1}[f'(x)-f'(\theta_x)]\|+\|f'(\zeta)^{-1}[f'(\theta_x)-f'(\zeta)]\| \\
\le&\|\int_0^1 f'(\zeta)^{-1}f''(\theta_x+\tau(x-\theta_x))d\tau(x-\zeta)\|+\|f'(\zeta)^{-1}[f'(\theta_x)-f'(\zeta)]\| \\
\le&\int_0^1 h''((\theta+\tau-\theta\tau)\|x-\zeta\|)d\tau\|x-\zeta\|+\|f'(\zeta)^{-1}[f'(\theta_x)-f'(\zeta)]\|.
\end{aligned}$$

Letting $\theta\to 0^+$, it follows

$$\|f'(\zeta)^{-1}f'(x)-I\|\le\int_0^1 h''(\tau\|x-\zeta\|)d\tau\|x-\zeta\|=h'(\tau\|x-\zeta\|)+1.$$

The inequality above and Banach Lemma derive that $f'(\zeta)^{-1}f'(x)$ or $f'(x)$ inverses and the first part of (5.3) holds. Since for any $\tau_x=\zeta+\tau(x-\zeta)$ with $0<\tau<1$

$$\begin{aligned}
&\|f'(x)^{-1}[f(\tau_x)-f(x)]\| \\
\le&\|I\|+\int_0^1(1-\theta)\|f'(x)^{-1}f''(x+\theta(\tau_x-x))\|d\theta\|\tau_x-x\|^2 \\
\le&1+\int_0^1(1-\theta)\|f'(x)^{-1}f'(\zeta)\|\|f'(\zeta)^{-1}f''(x+\theta(\tau_x-x))\|d\theta\|\tau_x-x\|^2 \\
\le&1-\int_0^1(1-\theta)\frac{h''((1-\theta+\theta\tau)\|x-\zeta\|)}{h'(\|x-\zeta\|)}d\theta\|\tau_x-x\|^2
\end{aligned}$$

follows from (5.2) and the first part of (5.3), letting $\tau\to 0^+$ above we have

$$\|f'(x)^{-1}[f(\zeta)-f(x)]\|\le 1-\int_0^1(1-\theta)\frac{h''((1-\theta)\|x-\zeta\|)}{h'(\|x-\zeta\|)}d\theta\|\zeta-x\|^2.$$

Therefore,

$$\begin{aligned}
\|f'(x)^{-1}f(x)\|&=\|f'(x)^{-1}[f(\zeta)-f(x)]\| \\
&\le 1-\int_0^1(1-\theta)\frac{h''((1-\theta)\|x-\zeta\|)}{h'(\|x-\zeta\|)}d\theta\|\zeta-x\|^2 \\
&=-\frac{h(\zeta)-h(\|\zeta-x\|)}{h'(\|\zeta-x\|)}=\frac{h(\|\zeta-x\|)}{h'(\|\zeta-x\|)}
\end{aligned}$$

deduces the second part of (5.3). The proof is completed. □

Proof of Theorem 2.1. By (3.1), (3.2) and Lemma 5.1, using almost the same procedure in [10], it is very easy for us to know that R_N,R_E defined in Lemma 4.2, or (4.2), are still the optimal radii of local convergence balls of Newton's method and Euler's method for operators f satisfying (2.2) , respectively.

Following Lemma 4.2 and Lemma 4.3, R_G exits and (2.7a) and (2.8) holds. From (3.1) and (4.6), we have for each $x \in \mathring{O}(\zeta, R_G)$

$$\begin{aligned}\|\zeta - N_f(x)\| \le & \|\zeta - \tau_x + f'(x)^{-1} f(\tau_x)\| \\ & + \int_0^1 \theta \|f'(x)^{-1} f'(\zeta)\| \|f'(\zeta)^{-1} f''(\tau_x + \theta(x-\tau_x)\| d\theta \|x - \tau_x\|^2 \\ & \le \|\zeta - \tau_x + f'(x)^{-1} f(\tau_x)\| - \int_0^1 \theta \frac{h''((\theta + \tau - \theta\tau)\|x-\zeta\|)}{h'(\|x-\zeta\|)} d\theta \|x - \tau_x\|^2\end{aligned}$$

for any $0 < \tau < 1$ by Lemma 5.2. Letting $\tau \to 0^+$ above, it follows

$$\begin{aligned}\|\zeta - N_f(x)\| \le & -\int_0^1 \theta \frac{h''((\theta)\|x-\zeta\|)}{h'(\|x-\zeta\|)} d\theta \|x-\zeta\|^2 \\ & = D_{E,\,h}(\|x-\zeta\|)\|x-\zeta\|^3, \qquad \forall x \in \mathring{O}(\zeta, R_G).\end{aligned} \tag{5.4}$$

Similarly, we have

$$\|\zeta - E_f(x)\| \le D_{E,\,h}(\|x-\zeta\|)\|x-\zeta\|^3, \qquad \forall x \in \mathring{O}(\zeta, R_G) \tag{5.5}$$

from (3.2) and (4.6), and

$$\begin{cases}\limsup\limits_{\tau\to 0^+} \|I_f(x,\tau)\| \le |I_h(\|x-\zeta\|)|, \\ \limsup\limits_{\tau\to 0^+} \|II_f(x,\tau)\| \le |II_h(\|x-\zeta\|)|, \\ \limsup\limits_{\tau\to 0^+} \|III_f(x,\tau)\| \le |III_h(\|x-\zeta\|)|,\end{cases} \qquad \forall x \in \mathring{O}(\zeta, R_G) \tag{5.6}$$

from (3.4) and (4.8), (4.9) and (4.12), respectively.

Since from (3.3), for any $x \in \mathring{O}(\zeta, R_G)$ and any $0 < \tau < 1$ holds

$$\|G_f(x) - \zeta\| \le \|\zeta - \tau_x + f'(x)^{-1} f(\tau_x)\| + \|I_f(x,\tau)\| + \|II_f(x,\tau)\| + \|III_f(x,\tau)\|.$$

Letting $\tau \to 0^+$, we have

$$\begin{aligned}\|G_f(x) - \zeta\| \le & |I_h(\|x-\zeta\|,0)| + |II_h(\|x-\zeta\|,0)| + |III_h(\|x-\zeta\|,0)| \\ & = -\Big(Q_1(\|x-\zeta\|) + Q_2(\|x-\zeta\|) + Q_3(\|x-\zeta\|)\Big)\|x-\zeta\|^4 \\ & = -G_h(\|x-\zeta\|).\end{aligned} \tag{5.7}$$

from (4.12) and (5.4), (5.5) and (5.6).

Then induction method and Lemma 4.3 derive

$$\|G_f^n(x) - \zeta\| \le (-1)^n G_h^n((-1)^{n-1}\|x-\zeta\|) = |G_h^n(\|x-\zeta\|)|, \quad n \ge 1 \tag{5.8}$$

can be proved for all $x\in \mathring{O}(\zeta,R_G)$ if $G_f^n(x)\neq\zeta$ for all $n\geq 1$. Thus, following Proposition 4.1, $\{G_f^n(x)\}_{n\geq 1}$ initialed from any $x\in\mathring{O}(\zeta,R_G)$ converges to ζ and (2.5) holds if $G_f^n(x)\neq\zeta$ for all $n\geq 1$.

Since $h(t)$ satisfies the Basic Assumptions when $\zeta=0$ and $X=Y$ are real fields, R_G, as the radius of the local convergence ball by Proposition 4.1, is the optimal value of radii of the local convergence ball of $G_f(x)$ for any f satisfies the Basic Assumptions. The proof is completed. $\square$

6 Conclusion

In this paper, the optimal local convergence radius of the four order iteration (1.4) of Euler's family is determined for nonlinear operators satisfying (2.2), the weak third order Lipschitz condition with $L-$average. Since this optimal radius is the absolute value of real numbers appeared in the period 2 orbit nearest to 0 of the method itself applied to the function defined by (2.3), we can say the local convergence of the method, which is essentially a fixed point problem, is determined by one of the period 2 orbits of the method itself applied to the function. It is just like the result for Newton's method obtained in [10]. It is expected that the local convergence of each method in the family with even number order is determined by one of period 2 orbits of the method itself applied to a similar function under mild conditions.

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