

A Nonhomogeneous Boundary Value Problem for the Boussinesq Equation on a Bounded Domain

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Abstract. In this paper, we study the well-posedness of an initial-boundary-value problem (IBVP) for the Boussinesq equation on a bounded domain,

$$\begin{cases} u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, & x \in (0,1), t > 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x), \\ u(0,t) = h_1(t), u(1,t) = h_2(t), u_{xx}(0,t) = h_3(t), u_{xx}(1,t) = h_4(t). \end{cases}$$

It is shown that the IBVP is locally well-posed in the space $H^s(0,1)$ for any $s \geq 0$ with the initial data φ, ψ lie in $H^s(0,1)$ and $H^{s-2}(0,1)$, respectively, and the naturally compatible boundary data h_1, h_2 in the space $H_{loc}^{(s+1)/2}(\mathbb{R}^+)$, and h_3, h_4 in the the space of $H_{loc}^{(s-1)/2}(\mathbb{R}^+)$ with optimal regularity.

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Key words: Boussinesq equation, initial-boundary value problem, local well-posedness.

1 Introduction

In this article, we consider a nonhomogeneous boundary value problem for the Boussinesq equation posed on a bounded domain $(0,1)$,

$$\begin{cases} u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, & x \in (0,1), t > 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x), \\ u(0,t) = h_1(t), u(1,t) = h_2(t), u_{xx}(0,t) = h'_3(t), u_{xx}(1,t) = h'_4(t). \end{cases} \quad (1.1)$$

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Here $h'(t)$ denotes the derivative of $h(t)$. Equation of this type, but with negative sign for the fourth derivative term,

$$u_{tt} - u_{xx} + (u^2)_{xx} - u_{xxxx} = 0, \quad (1.2)$$

was originally derived in 1871 by J. Boussinesq in his study [9] on propagation of small amplitude, long waves on the surface of water. The Boussinesq equation (1.2) possesses special traveling wave solutions called the solitary wave, and it is the first equation that gives a mathematical explanation to the phenomenon of solitary waves discovered by Scott Russell reported in 1834. The original Boussinesq equation has been used in a considerable range of applications such as coasts and harbors engineering, simulation of tides and tsunamis.

However, equation (1.2) is ill-posed for its initial-value problem. That is, a slight difference in initial data, might evolve to a large change in solution. For this reason, the equation (1.2) is sometimes called as the “bad” Boussinesq equation. This can be seen, for example, by considering its linear equation as

$$u_{tt} - u_{xxxx} := (\partial_t + \partial_{xx})(\partial_t - \partial_{xx})u = 0.$$

The “ $\partial_t - \partial_{xx}$ ” can be treated as the heat equation and it is well-posed, but “ $\partial_t + \partial_{xx}$ ”, the backward heat equation, is ill-posed. One way to correct this ill-posedness issue is to alter the sign of the fourth derivative term, then the “good” Boussinesq equation

$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, \quad (1.3)$$

is proposed. Similarly, its well-posedness can be seen by considering the linear equation

$$u_{tt} + u_{xxxx} := (\partial_t + i\partial_{xx})(\partial_t - i\partial_{xx})u = 0,$$

because both the Schrödinger and the reversed Schrödinger equation are indeed well-posed.

As the changing of sign to the “bad” Boussinesq equation, the “good” Boussinesq equation cannot be justified as the original Boussinesq’s physical modeling. However, Zakharov [36] has proposed it as a model of nonlinear vibration along a string, and Turitsyn [27] has revealed it when describing electromagnetic waves in nonlinear dielectric materials. Moreover, the “good” Boussinesq equation appeals in the study of Falk et al [11] as a model of shape-memory alloys and it is also raised in a large range of physical phenomena including propagation of ion-sound waves in a plasma and nonlinear lattice waves.

The study of the well-posedness of the initial-value problem (IVP) of the Boussinesq equation,

$$\begin{cases} u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, & x \in \mathbb{R}, t \in \mathbb{R}, \\ u(x, 0) = f(x), & u_t(x, 0) = h(x), \end{cases}$$

in the classical L^2 -based Sobolev spaces $H^s(\mathbb{R})$, was started by Bona and Sach [4]. By using Kato’s abstract semigroup theory, they showed that IVP is well-posed in the space $H^s(\mathbb{R})$ for $s > \frac{5}{2}$ with initial data (f, h) in the space $H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})$. A similar result was obtained by Tsutsumi and Matabashi [26]. Linares [17] obtained Strichartz estimates for solutions of the associated linear problem and showed the IVP is well-posed in the space $H^s(\mathbb{R})$ for any $s \geq 0$ through the contraction mapping principle. With the help of Bourgain spaces, Farah [13] showed that the IVP of the Boussinesq equation is locally well-posed for $s > -\frac{1}{4}$ and ill-posed for $s < -\frac{1}{2}$. Later, using a specially modified Bourgain space, Kishimoto [16] showed that IVP is local well-posed for $s \geq -\frac{1}{2}$.

By contrast, theories for the initial-boundary-value problem (IBVP) of the Boussinesq equation posed on either the right half line, \mathbb{R}^+ ,

$$\begin{cases} u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, & x > 0, t > 0, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \\ u(0, t) = h_1(t), \quad u_{xx}(0, t) = h_2'(t), \end{cases} \tag{1.4}$$

or on a finite domain,

$$\begin{cases} u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, & x \in (0, 1), t > 0, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \\ u(0, t) = h_1(t), \quad u(1, t) = h_2(t), \quad u_{xx}(0, t) = h_3'(t), \quad u_{xx}(1, t) = h_4'(t), \end{cases} \tag{1.5}$$

have remained less developed. The primary goal of our study on these two IBVPs is to develop well-posedness theory in the L^2 -based Sobolev spaces $H^s(\Omega)$ (Here Ω denotes either \mathbb{R}^+ or the finite interval $(0, 1)$). The following two questions arise naturally and will be addressed.

- (1) While the initial data φ and ψ are required to be in the spaces $H^s(\Omega)$ and $H^{s-2}(\Omega)$, respectively, for the well-posedness of the IBVPs (1.4) or (1.5) in the space $H^s(\Omega)$, what are the optimal (minimum) regularity requirements on the boundary data (h_1, h_2) for the IBVP (1.4) and (h_1, h_2, h_3, h_4) for the IBVP (1.5)?
- (2) What is the smallest value of s such that the IBVPs (1.4) and (1.5) are well-posed in the space $H^s(\Omega)$?

Note that it is necessary for $h_j, j = 1, 2, 3, 4$ belonging to the space $H_{loc}^{\frac{2s+1}{4}}(\mathbb{R}^+)$ in order that IBVPs (1.4) and (1.5) are well-posed in the space $H^s(\Omega)$ as the solutions of the pure initial-value problem

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = p(x), \quad u_t(x, 0) = q(x), \end{cases} \tag{1.6}$$

possess the sharp Kato smoothing properties,

$$\sup_{x \in \mathbb{R}} \|\partial_x^j u\|_{H^{\frac{2s-2j+1}{4}}(\mathbb{R}^+)} \leq C \|(p, q)\|_{H^s(\mathbb{R}) \times H^{s-2}(\mathbb{R})},$$

for $j=0,1,2$.

Using the approach developed earlier by Bona, Sun and Zhang in the study of the IBVPs of the KdV equation [5–7], Xue [32] showed that the IBVP,

$$\begin{cases} u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, & x > 0, t > 0, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \\ u(0, t) = h_1(t), \quad u_{xx}(0, t) = h'_2(t), \end{cases}$$

is well-posed in the space $H^s(\mathbb{R}^+)$ for $s > \frac{1}{2}$ with the initial data $(\varphi, \psi) \in H^s(\mathbb{R}^+) \times H^{s-2}(\mathbb{R}^+)$ and the boundary data $(h_1, h_2) \in H^{\frac{s}{2} + \frac{1}{4}}(\mathbb{R}^+) \times H^{\frac{s}{2} + \frac{1}{4} + \epsilon}(\mathbb{R}^+)$ where $\epsilon > 0$. The key step of this approach is to obtain an explicit integral formula for the linear problem,

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0, & x > 0, t > 0, \\ u(x, 0) = 0, \quad u_t(x, 0) = 0, \\ u(0, t) = h_1(t), \quad u_{xx}(0, t) = h'_2(t), \end{cases}$$

using the Laplace transform with respect to t and show that for any $h_1, h_2 \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$,

$$\sup_{t > 0} \|u\|_{H^s(\mathbb{R}^+)} \leq C \|(h_1, h_2)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+) \times H^{\frac{2s+1}{4}}(\mathbb{R}^+)}.$$

In addition, it is worth to point out, as in the study of the well-posedness the IVP or IBVPs of the one-dimensional nonlinear Schrödinger equation [8] in the the space $H^s(\Omega)$, the well-posedness of the IBVPs (1.4) and (1.5) of the Boussinesq equation in the space $H^s(\Omega)$ can be established via classical PDE techniques in the case of $s > \frac{1}{2}$. In contrast, in the case of $s < \frac{1}{2}$, some harmonic analysis based tools are needed and one has to construct some special Banach spaces in order to apply the contraction mapping principle. Indeed, Xue [33,34] was able to show that the IBVP (1.4) is well-posed in the space $H^s(\mathbb{R}^+)$ for $0 \leq s < \frac{1}{2}$ with $(\varphi, \psi) \in H^s(\mathbb{R}^+) \times H^{s-2}(\mathbb{R}^+)$ and boundary data $(h_1, h_2) \in H^{\frac{s}{2} + \frac{1}{4}}(\mathbb{R}^+) \times H^{\frac{s}{2} + \frac{1}{4}}(\mathbb{R}^+)$ only after he succeeded in establishing the $L^p - L^q$ estimates for the linear IBVP associated to (1.4).

For the IBVPs of the Boussinesq equation posed on a finite interval, through the finite element Galerkin method, Pani and Saranga [24] established the local existence and uniqueness for the solution of the following homogeneous IBVP,

$$\begin{cases} u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0, & x \in (0, 1), t > 0, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \\ u(0, t) = 0, \quad u(1, t) = 0, \quad u_{xx}(0, t) = 0, \quad u_{xx}(1, t) = 0. \end{cases} \tag{1.7}$$

Later, Xue [31] showed that the IBVP (1.7) is well-posed in the space $H^s(0,1)$ for $s > \frac{1}{2}$ through deriving an integral formula for the linear problem,

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} = 0, & x \in (0,1), t \in (0,T), \\ u(x,0) = 0, \quad u_t(x,0) = 0, \\ u(0,t) = h_1(t), \quad u(1,t) = h_2(t), \quad u_{xx}(0,t) = h'_3(t), \quad u_{xx}(1,t) = h'_4(t), \end{cases}$$

which enables him to establish the energy estimates needed for the well-posedness of the IBVP (1.7). Note the approach developed in Xue’s work [31] does not work for the nonhomogeneous IBVP (1.1). Neither works for the well-posedness of the homogeneous IBVP (1.7) in the space $H^s(0,1)$ with $0 \leq s < \frac{1}{2}$.

It is our principle purpose to advance the study of the well-posedness of the IBVP (1.1) to the same level as that for the IBVP (1.4) posed on the half line \mathbb{R}^+ . In this paper, we study the well-posedness of the nonhomogeneous boundary-value problem in the space $H^s(0,1)$ not only for the case of $s > \frac{1}{2}$, but also for the case of $0 \leq s < \frac{1}{2}$. Before we state the main results, we denote for given $T > 0$,

$$\mathcal{Q}^s(0,T) := H^{\frac{s+1}{2}}(0,T) \times H^{\frac{s+1}{2}}(0,T) \times H^{\frac{s+1}{2}}(0,T) \times H^{\frac{s+1}{2}}(0,T),$$

and

$$\mathcal{Q}_0^s(0,T) := H_0^{\frac{s+1}{2}}(0,T] \times H_0^{\frac{s+1}{2}}(0,T] \times H_0^{\frac{s+1}{2}}(0,T] \times H^{\frac{s+1}{2}}(0,T].$$

The following two theorems are our main results.

Theorem 1.1. *Let $\frac{1}{2} < s < \frac{9}{2}$, $\gamma > 0$ and $T > 0$ be given. There exist a $T^* \in (0,T]$ depending only on s, γ and T such that for any*

$$(\varphi, \psi, h_1, h_2, h_3, h_4) \in H^s(0,1) \times H^{s-2}(0,1) \times \mathcal{Q}^s(0,T)$$

satisfying

$$\|\varphi\|_{H^s(0,1)} + \|\psi\|_{H^{s-2}(0,1)} + \|(h_1, h_2, h_3, h_4)\|_{\mathcal{Q}^s(0,T)} \leq \gamma,$$

and the compatibility conditions (or “s-compatible condition”)

$$\phi(0) = h_1(0), \quad \phi(1) = h_2(0)$$

if $s < \frac{5}{2}$ and, additionally,

$$\psi(0) = h'_3(0), \quad \psi(1) = h'_4(0)$$

if $\frac{5}{2} < s < \frac{9}{2}$, the IBVP (1.1) admits a unique solution $u \in C([0,T^*]; H^s(0,1))$. Moreover, the corresponding solution map is real analytic.

Theorem 1.2. *Let $0 \leq s < \frac{1}{2}$, $\gamma > 0$ and $T > 0$ be given. There exist a $T^* \in (0, T]$ depending only on s, γ and T such that for any*

$$(\varphi, \psi, h_1, h_2, h_3, h_4) \in H^s(0,1) \times H^{s-2}(0,1) \times \mathcal{Q}^s(0,T)$$

satisfying

$$\|\varphi\|_{H^s(0,1)} + \|\psi\|_{H^{s-2}(0,1)} + \|(h_1, h_2, h_3, h_4)\|_{\mathcal{Q}^s(0,T)} \leq \gamma,$$

the IBVP (1.1) admits a unique solution

$$u \in C([0, T^*]; H^s(0,1)) \cap L^4([0, T^*] \times (0,1)).$$

Moreover, the corresponding solution map is real analytic.

The main idea for this article is based on Bona, Sun and Zhang’s approach for the nonlinear Schrödinger equation [8]:

$$\begin{cases} iu_t + u_{xx} + \lambda|u|^{p-2}u = 0, & x \in (0,1), t \in \mathbb{R}, \\ u(x,0) = \varphi(x), \\ u(0,t) = h_1(t), u(1,t) = h_2(t). \end{cases}$$

One of the keys in their approach is an explicit representation formula for solutions of the linear IBVP:

$$\begin{cases} iu_t + u_{xx} = 0, & x \in (0,1), t \in \mathbb{R}, \\ u(x,0) = 0, \\ u(0,t) = h_1(t), u(1,t) = h_2(t), \end{cases}$$

Instead of deriving an integral representation as a boundary operator using the Laplace transform (c.f. [5–8, 31–34]), they derived an infinite series expression for its solution:

$$u(x,t) = \sum_{n=1}^{+\infty} 2in\pi e^{-i(n\pi)^2t} \int_0^t e^{i(n\pi)^2\tau} (h_1(\tau) - (-1)^n h_2(\tau)) d\tau \sin n\pi x,$$

along with the estimate

$$\|u\|_{L^4([0,T] \times (0,1)) \cap C([0,T]; L^2(0,1))} \leq C_T \left(\|h_1\|_{H^{\frac{1}{2}}(0,T)} + \|h_2\|_{H^{\frac{1}{2}}(0,T)} \right),$$

for given $T > 0$ and $h_1, h_2 \in H^{\frac{1}{2}}(0,T)$. They also showed that the space $H^{\frac{1}{2}}(0,T) \times H^{\frac{1}{2}}(0,T)$ for the boundary data is indeed optimal, even though the index is larger than the one for the IBVP of the Schrödinger equation posed on the half line (e.g. $H^{\frac{1}{4}}(\mathbb{R}^+)$).

The other key in their approach is that when considering the linear equation with homogeneous boundary conditions,

$$\begin{cases} iu_t + u_{xx} = 0, & x \in (0,1), t \in \mathbb{R}, \\ u(x,0) = \varphi(x), \\ u(0,t) = 0, u(1,t) = 0, \end{cases}$$

they treated it as a special case of the Cauchy problem with periodic initial condition:

$$\begin{cases} iu_t + u_{xx} = 0, & x \in (-1,1), t \in \mathbb{R}, \\ u(x,0) = \varphi^*(x), \\ u(-1,t) = u(1,t), \quad u_x(-1,t) = u_x(1,t) \end{cases}$$

where φ^* is a periodic function on \mathbb{R} based on an odd extension of φ from $(0,1)$ to $(-1,1)$. Then, the estimate for the linear IBVP follows from an existing theory of the Schrödinger equation,

$$\|u\|_{L^4([0,T] \times (0,1)) \cap C([0,T]; L^2(0,1))} \leq C_T \|\varphi\|_{L^2(0,1)}.$$

In this paper, we apply the same approach to study the well-posedness of the IBVP (1.1) in the space $H^s(0,1)$. An explicit solution formula will be derived first for solutions of the associated linear IBVP,

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & x \in (0,1), t > 0, \\ u(x,0) = u_t(x,0) = 0, \\ u(0,t) = h_1(t), u(1,t) = h_2(t), u_{xx}(0,t) = h'_3(t), u_{xx}(1,t) = h'_4(t), \end{cases} \tag{1.8}$$

which is quite similar to the solution formula of the relevant linear IBVP of the Schrödinger equation. Based on such an explicit solution formula, we establish the following estimate for solutions of the IBVP (1.8),

$$\|u\|_{C([0,T]; L^2(0,1))} + \|u\|_{L^4([0,T] \times (0,1))} \leq C_T \left(\sum_{j=1}^4 \|h_j\|_{H^{\frac{1}{2}}(0,T)} \right). \tag{1.9}$$

We also show that this estimates is sharp in the sense that the estimate

$$\|u\|_{C([0,T]; L^2(0,1))} + \|u\|_{L^4([0,T] \times (0,1))} \leq C_T \left(\sum_{j=1}^4 \|h_j\|_{H^\alpha(0,T)} \right)$$

fails for any $\alpha < \frac{1}{2}$. Consequently, the regularities imposed on the boundary data $h_j, j = 1,2,3,4$ in Theorem 1.1 and Theorem 1.2 for the well-posedness of the IBVP (1.1) in the space $H^s(0,1)$ are optimal. As in the case of nonlinear Schrödinger equation [8], this

reveals a significant difference of the IBVP of the Boussinesq equation posed on a half line \mathbb{R}^+ and the IBVP of the Boussinesq equation posed on a finite interval $(0,1)$.

The article is organized as follows: In Section 2, we focus on the linear problem. First, we will derive solution formulas for the related linear problems. Secondly, variety of estimates will be given for such formulas. In Section 3, local well-posedness of the nonlinear IBVP (1.1) is established using the contraction mapping principle.

2 Linear problems

This section is devoted to study the IBVP of the linear Boussinesq equation

$$\begin{cases} u_{tt} + u_{xxxx} + g_{xx}(x,t) = 0, & x \in (0,1), t > 0, \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x), \\ u(0,t) = h_1(t), & u(1,t) = h_2(t), u_{xx}(0,t) = h_3'(t), u_{xx}(1,t) = h_4'(t), \end{cases} \quad (2.1)$$

It is divided into two subsections. In Section 2.1, some explicit solution formulas will be derived for the IBVP (2.1). Then various estimates will be established in Section 2.2 for solutions of the IBVP (2.1).

2.1 Boundary operator and solution formulas

First, we consider the following linear IBVP with nonhomogeneous boundary conditions, but zero initial data.

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & x \in (0,1), t > 0, \\ u(x,0) = 0, & u_t(x,0) = 0, \\ u(0,t) = h_1(t), & u(1,t) = h_2(t), u_{xx}(0,t) = h_3'(t), u_{xx}(1,t) = h_4'(t). \end{cases} \quad (2.2)$$

Its solution will be recorded as

$$u(x,t) = [W_{bdr}(\vec{h})](x,t) = [W_{bdr}(h_1, h_2, h_3, h_4)](x,t),$$

here, W_{bdr} will be called as the boundary integral operator. We derive an explicit formula for W_{bdr} . We start with the case that $h_2 = h_3 = h_4 \equiv 0$ and $h_1(0) = h_1'(0) = 0$, Writing the solution u_1 as

$$u_1(x,t) = v(x,t) + (1-x)h_1(t),$$

then $v(x,t)$ solves

$$\begin{cases} v_{tt} + v_{xxxx} = -(1-x)h_1''(t), & x \in (0,1), t > 0, \\ v(x,0) = 0, & v_t(x,0) = 0, \\ v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0. \end{cases}$$

Expanding $v(x, t)$ as a Fourier sine series,

$$v(x, t) = \sum_{k=1}^{\infty} \alpha_k(t) \operatorname{sinc} k\pi x,$$

we have for $k=1, 2, \dots$,

$$\frac{d^2}{dt^2} \alpha_k(t) + k^4 \pi^4 \alpha_k(t) = \beta_k h_1''(t), \quad \alpha_k(0) = 0, \alpha_k'(0) = 0,$$

with

$$\beta_k = -2 \int_0^1 (1-x) \operatorname{sinc} k\pi x dx = -\frac{2}{k\pi}.$$

Thus,

$$\begin{aligned} \alpha_k(t) &= \beta_k h_1(t) + \int_0^t \frac{i\beta_k k^2 \pi^2}{2} \left(e^{ik^2 \pi^2(t-\tau)} - e^{-ik^2 \pi^2(t-\tau)} \right) h_1(\tau) d\tau, \\ v(x, t) &= -(1-x) h_1(x) + \sum_{k=1}^{\infty} \int_0^t \frac{\beta_k}{2} i k^2 \pi^2 \left(e^{ik^2 \pi^2(t-\tau)} - e^{-ik^2 \pi^2(t-\tau)} \right) h_1(\tau) d\tau \operatorname{sinc} k\pi x, \end{aligned}$$

which implies that

$$\begin{aligned} u_1(x, t) &:= [W_{bdr}(h_1, 0, 0, 0)](x, t) \\ &= - \sum_{k=1}^{\infty} \int_0^t i k \pi \left(e^{ik^2 \pi^2(t-\tau)} - e^{-ik^2 \pi^2(t-\tau)} \right) h_1(\tau) d\tau \operatorname{sinc} k\pi x. \end{aligned}$$

Next, we consider the IBVP (2.2) for $h_1 = h_3 = h_4 \equiv 0$ and $h_2(0) = h_2'(0) = 0$. If we set $x' = 1 - x$, it can be reduced to the previous case (as for u_1). Thus, for the IBVP (2.2) with $h_1 = h_3 = h_4 \equiv 0$ and $h_2(0) = h_2'(0) = 0$, we have the solution as

$$\begin{aligned} u_2(x, t) &:= [W_{bdr}(0, h_2, 0, 0)](x, t) \\ &= - \sum_{k=1}^{\infty} \int_0^t (-1)^k i k \pi \left(e^{ik^2 \pi^2(t-\tau)} - e^{-ik^2 \pi^2(t-\tau)} \right) h_2(\tau) d\tau \operatorname{sinc} k\pi x. \end{aligned}$$

Now, attention is turned to the case for $h_1 = h_2 = h_4 \equiv 0$ and $h_3(0) = h_3'(0) = h_3''(0) = 0$. We set the solution u as,

$$u_3(x, t) = v(x, t) + \left(\frac{1}{6} x^3 - \frac{1}{2} x^2 + \frac{1}{3} x \right) h_3'(t).$$

Then v solves,

$$\begin{cases} v_{tt} + v_{xxxx} = - \left(\frac{1}{6} x^3 - \frac{1}{2} x^2 + \frac{1}{3} x \right) h_3'''(t), & x \in (0, 1), t > 0, \\ v(x, 0) = v_t(x, 0) = 0, \\ v(0, t) = v(1, t) = v_{xx}(0, t) = v_{xx}(1, t) = 0. \end{cases}$$

Again, we write

$$v(x,t) = \sum_{k=1}^{\infty} \alpha_k(t) \sin k\pi x.$$

Then, for $k = 1, 2, \dots$,

$$\frac{d^2}{dt^2} \alpha_k(t) + k^4 \pi^4 \alpha_k(t) = \beta_k h_3'''(t), \quad \alpha_k(0) = 0, \alpha_k'(0) = 0,$$

with

$$\beta_k = -2 \int_0^1 \left(\frac{1}{6} x^3 - \frac{1}{2} x^2 + \frac{1}{3} x \right) \sin k\pi x dx = -\frac{2}{k^3 \pi^3}.$$

Solving these equations and integrating by parts, it follows that

$$\alpha_k(t) = \beta_k h_3'(t) - \int_0^t \frac{\beta_k}{2} k^4 \pi^4 \left(e^{ik^2 \pi^2(t-\tau)} + e^{-ik^2 \pi^2(t-\tau)} \right) h_3(\tau) d\tau.$$

Thus, we have the solution of the IBVP (2.2) for $h_1 = h_2 = h_4 \equiv 0$ and $h_3(0) = h_3'(0) = h_3''(0) = 0$ as

$$u_3(x,t) := [W_{bdr}(0,0,h_3,0)](x,t) = \sum_{k=1}^{\infty} \int_0^t k\pi \left(e^{ik^2 \pi^2(t-\tau)} + e^{-ik^2 \pi^2(t-\tau)} \right) h_3(\tau) d\tau \sin k\pi x.$$

Finally, for the case $h_1 = h_2 = h_3 \equiv 0$ and $h_4(0) = h_4'(0) = h_4''(0) = 0$, if we let $x' = 1 - x$ again, this case can be reduced to the case for u_3 . Then, we get

$$\begin{aligned} u_4(x,t) &:= [W_{bdr}(0,0,0,h_4)](x,t) \\ &= \sum_{k=1}^{\infty} \int_0^t (-1)^k k\pi \left(e^{ik^2 \pi^2(t-\tau)} + e^{-ik^2 \pi^2(t-\tau)} \right) h_4(\tau) d\tau \sin k\pi x. \end{aligned}$$

The discussions above leads to the following lemma.

Lemma 2.1. *Given $\vec{h} = (h_1, h_2, h_3, h_4)$, the solution of the IBVP (2.2) can be written in the form*

$$u(x,t) = [W_{bdr}(\vec{h})](x,t) = \sum_{m=1}^4 [W_{bdr}(\vec{h}_m)](x,t)$$

with $\vec{h}_m = \vec{h}$ for $h_j = 0$ if $j \neq m$.

Attention is now turned to the associated linear IBVP with homogeneous boundary conditions, that is,

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & x \in (0,1), t > 0 \\ u(x,0) = \varphi(x), & u_t(x,0) = \psi(x), \\ u(0,t) = 0, u(1,t) = 0, & u_{xx}(0,t) = 0, u_{xx}(1,t) = 0, \end{cases} \quad (2.3)$$

and

$$\begin{cases} u_{tt} + u_{xxxx} + g_{xx}(x,t) = 0, & x \in (0,1), t > 0 \\ u(x,0) = 0, u_t(x,0) = 0, \\ u(0,t) = 0, u(1,t) = 0, u_{xx}(0,t) = 0, u_{xx}(1,t) = 0. \end{cases} \tag{2.4}$$

We first consider the Cauchy problem of the linear equation posed on $(-1,1)$ with the periodic boundary conditions,

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & x \in (-1,1), t > 0, \\ u(x,0) = \tilde{\varphi}(x), u_t(x,0) = \tilde{\psi}(x), \\ \partial_x^j u(-1,t) = \partial_x^j u(1,t) \text{ for } j=0,1,2,3, \end{cases} \tag{2.5}$$

where $\tilde{\varphi}$ and $\tilde{\psi}$ are the odd extension of φ and ψ from $(0,1)$ to $(-1,1)$. We denote the solution of the IVP (2.5) as

$$u(x,t) = [W_{\mathbb{T}}(\tilde{\varphi}, \tilde{\psi})](x,t) \text{ for } x \in (-1,1).$$

One shall notice that the Cauchy problem (2.5) possesses the following invariance.

- If $\tilde{\varphi}$ and $\tilde{\psi}$ are odd functions, then the corresponding solution u of (2.5) is also an odd function with respect to x for any t .
- If $\tilde{\varphi}$ and $\tilde{\psi}$ are even functions, then the corresponding solution u of (2.5) is also an even function with respect to x for any t .

On the other hand, if u is an odd solution with respect to x and solves the IVP (2.5), then its restriction on interval $(0,1)$ will solves the IBVP (2.3) since $u(0,t) = u(1,t) = 0$ and $u_{xx}(0,t) = u_{xx}(1,t) = 0$ will be automatically satisfied. Therefore, we have the solution formula for the IBVP (2.3).

Lemma 2.2. *For any $\varphi \in H^s(0,1)$ and $\psi \in H^{s-2}(0,1)$, let $\tilde{\varphi}$ and $\tilde{\psi}$ be the odd extension of φ and ψ from $(0,1)$ to $(-1,1)$, respectively. Then the solution $u := W_0(\varphi, \psi)$ of the IBVP (2.3) can be written in the form*

$$u(x,t) := [W_0(\varphi, \psi)](x,t) = [W_{\mathbb{T}}(\tilde{\varphi}, \tilde{\psi})](x,t), \quad x \in (0,1).$$

We now turn to the IBVP (2.4). Let g^* be a odd function respect to x on $(-1,1)$ and g be its restriction on $(0,1)$. For the forced Cauchy problem

$$\begin{cases} u_{tt} + u_{xxxx} + g_{xx}^*(x,t) = 0, & x \in (-1,1), t > 0, \\ u(x,0) = 0, u_t(x,0) = 0, \\ \partial_x^j u(-1,t) = \partial_x^j u(1,t) \text{ for } j=0,1,2,3. \end{cases} \tag{2.6}$$

via Duhamel’s principle, we denote its solution as

$$u(x,t) := [W_f(g_{xx}^*)](x,t) = - \int_0^t [W_{\mathbb{T}}(0, g_{xx}^*)](x,t-\tau) d\tau.$$

As a result, one has the following lemma for the formula of the solution for the IBVP (2.4).

Lemma 2.3. For $g \in C^\infty([0, T] \times (0, 1))$, let g^* be its odd extension from $(0, 1)$ to $(-1, 1)$ with respect to x . The corresponding solution of the IBVP (2.4) can be written in the form

$$u(x, t) := - \int_0^t [W_{\mathbb{T}}(0, g_{xx}^*)](x, t - \tau) d\tau \quad x \in (0, 1).$$

2.2 Linear estimates

In this subsection, various linear estimates for the following linear system,

$$\begin{cases} u_{tt} + u_{xxxx} + g_{xx}(x, t) = 0, \\ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \\ u(0, t) = h_1(t), \quad u(1, t) = h_2(t), \quad u_{xx}(0, t) = h_3'(t), \quad u_{xx}(1, t) = h_4'(t) \end{cases}$$

will be derived.

First we consider the case of $\varphi = \psi = g \equiv 0$, i.e., the IBVP (2.2). Recall that for any given $s \geq 0$ and $T > 0$,

$$\begin{aligned} \mathcal{Q}^s(0, T) &:= H^{\frac{s+1}{2}}(0, T) \times H^{\frac{s+1}{2}}(0, T) \times H^{\frac{s+1}{2}}(0, T) \times H^{\frac{s+1}{2}}(0, T), \\ \mathcal{Q}_0^s(0, T) &:= H_0^{\frac{s+1}{2}}(0, T) \times H_0^{\frac{s+1}{2}}(0, T) \times H_0^{\frac{s+1}{2}}(0, T) \times H_0^{\frac{s+1}{2}}(0, T). \end{aligned}$$

Proposition 2.1. Let $s \geq 0$ and $T > 0$ be given. There exists a constant $C > 0$ such that for any $\vec{h} := (h_1, h_2, h_3, h_4) \in \mathcal{Q}_0^s(0, T)$, $u(x, t) = [W_{bdr}(\vec{h})](x, t)$ satisfies

$$\begin{aligned} \partial_x^s u &\in C([0, T]; L^2(0, 1)) \cap L^4([0, T] \times (0, 1)), \\ \sup_{t \in [0, T]} \|\partial_x^s u\|_{L^2(0, 1)} + \|\partial_x^s u\|_{L^4([0, T] \times (0, 1))} &\leq C \|\vec{h}\|_{\mathcal{Q}^s(0, T)}. \end{aligned}$$

Proof. For $\vec{h}_j = (h_1, h_2, h_3, h_4) \in \mathcal{Q}_0^s(0, T)$ with $h_m = 0$ if $m \neq j$ for $m = 1, 2, 3, 4$, let

$$u_1 = W_{bdr}(\vec{h}_1), \quad u_2 = W_{bdr}(\vec{h}_2), \quad u_3 = W_{bdr}(\vec{h}_3) \quad \text{and} \quad u_4 = W_{bdr}(\vec{h}_4).$$

It suffices to show the estimate for u_1 . The proof for others will be similar. Notice that,

$$u_1(x, t) = - \sum_{k=1}^{\infty} \int_0^t ik\pi \left(e^{ik^2\pi^2(t-\tau)} - e^{-ik^2\pi^2(t-\tau)} \right) h_1(\tau) d\tau \operatorname{sinc} \pi x,$$

then the estimate with $s=0$ follows from Proposition 4.6 in [8]. In the case of $s > 0$, we only need to prove it for $s=4$. The case for $s \in (0, 4)$ can be obtained by standard interpolation theory and the proof for $s > 4$ is similar. Set $v = \partial_t^2 u_1$, it then solves

$$\begin{cases} v_{tt} + v_{xxxx} = 0, \\ v(x, 0) = v_t(x, 0) = 0, \\ v(0, t) = h_1''(t), \quad v(1, t) = v_{xx}(0, t) = v_{xx}(1, t) = 0. \end{cases}$$

Thus

$$\|\partial_t^2 u_1\|_{C([0,T];L^2(0,1)) \cap L^4([0,T] \times (0,1))} \leq C \|h_1''\|_{H^{\frac{1}{2}}(0,1)} \leq C \|h_1\|_{H^{\frac{5}{2}}(0,1)},$$

which implies further

$$\|\partial_x^4 u_1\|_{C([0,T];L^2(0,1)) \cap L^4([0,T] \times (0,1))} \leq C \|h_1\|_{H^{\frac{5}{2}}(0,1)}$$

as $u_{xxxx} = -u_{tt}$. The proof is thus complete. \square

Remark 2.1. Note that for the linear Boussinesq equation posed on the half line (cf. [34]), we find the estimate for the boundary integral operator:

$$\|W_{bdr}(t)(h_1, 0)\|_{L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^+)) \cap C([0,T]; L^2(\mathbb{R}^+))} \leq C \|h_1\|_{H^{\frac{1}{4}}(\mathbb{R}^+)},$$

for any $h_1 \in H^{\frac{1}{4}}(\mathbb{R}^+)$. While for the bounded domain problem

$$\|W_{bdr}(t)(h_1, 0, 0, 0)\|_{L^4([0,T] \times (0,1)) \cap C([0,T]; L^2(0,1))} \leq C \|h_1\|_{H^{\frac{1}{2}}(\mathbb{R}^+)},$$

for any $h_1 \in H^{\frac{1}{2}}(\mathbb{R}^+)$. One may wonder whether the estimate can be improved. The example in Appendix shows that if $\|W_{bdr}(t)(h, 0, 0, 0)\|_{L^4([0,T] \times (0,1))} \leq C \|h\|_{H^s(0,T)}$ for any $h(t) \in H^s(0,T)$, then s must be greater than or equal to $\frac{1}{2}$. Thus, the estimate in Proposition 2.1 is indeed optimal.

Next, we turn to consider the Cauchy problem of the linear Boussinesq equation posed on $(-1, 1)$ with the periodic boundary conditions.

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & x \in (-1, 1), \quad t \in \mathbb{R}, \\ u(x, 0) = p(x), \quad u_t(x, 0) = q(x), \\ \partial_x^j u(-1, t) = \partial_x^j u(1, t), \quad j = 0, 1, 2, 3 \end{cases} \quad (2.7)$$

$$\begin{cases} u_{tt} + u_{xxxx} = f_{xx}, & x \in (-1, 1), \quad t \in \mathbb{R}, \\ u(x, 0) = 0, \quad u_t(x, 0) = 0, \\ \partial_x^j u(-1, t) = \partial_x^j u(1, t), \quad j = 0, 1, 2, 3 \end{cases} \quad (2.8)$$

where p, q and f are odd periodic functions with respect to x . The following lemmas comes from Fang and Grillakis's work [12].

Lemma 2.4. Let $0 \leq s < \frac{1}{2}$ and $T > 0$ be given. For any $p \in H^s(-1, 1)$, $q \in H^{s-2}(-1, 1)$, the IVP (2.7) admits a solution u with

$$u(x, t) = [W_{\mathbb{T}}(p, q)](x, t) \in C([0, T], H^s(-1, 1)) \cap L^4([0, T] \times (-1, 1))$$

satisfying

$$\|u\|_{C([0,T], H^s(-1,1)) \cap L^4([0,T] \times (-1,1))} \leq C \left(\|\varphi\|_{H^s(-1,1)} + \|\psi\|_{H^{s-2}(-1,1)} \right),$$

where C depends only on s and T .

Lemma 2.5. Let $0 \leq s < \frac{1}{2}$ and $T > 0$ be given. For any $f \in L^{\frac{4}{3}}([0, T] \times (-1, 1))$, the IVP (2.5) admits a solution u with

$$u(x, t) = - \int_0^t [W_{\mathbb{T}}(0, f_{xx})](x, t) \in C([0, T], H^s(-1, 1)) \cap L^4([0, T] \times (-1, 1))$$

satisfying

$$\|u\|_{C([0, T], H^s(-1, 1)) \cap L^4([0, T] \times (-1, 1))} \leq C \|f\|_{L^{\frac{4}{3}}([0, T] \times (-1, 1))},$$

where C depends only on s and T .

In addition, the following lemma follows from the standard semigroup theory.

Lemma 2.6. Let $s \geq 0$ and $T > 0$ be given. For any $p \in H^s(-1, 1)$, $q \in H^{s-2}(-1, 1)$, and $f \in L^1([0, T]; H^s(-1, 1))$,

$$u(x, t) = W_{\mathbb{T}}(p, q)(x, t) - \int_0^t [W_{\mathbb{T}}(0, f_{xx})](x, t) \in C([0, T], H^s(-1, 1))$$

satisfying

$$\|u\|_{C([0, T], H^s(-1, 1))} \leq C \left(\|\varphi\|_{H^s(-1, 1)} + \|\psi\|_{H^{s-2}(-1, 1)} + \|f\|_{L^1([0, T]; H^s(-1, 1))} \right),$$

where C depends only on s and T .

Now, attention is turned to the IBVP for the linear Boussinesq equation posed on the interval $(0, 1)$ with homogeneous boundary conditions,

$$\begin{cases} u_{tt} + u_{xxxx} = 0, & x \in (0, 1), t \in \mathbb{R}, \\ u(x, 0) = \varphi(x), u_t(x, 0) = \psi(x), \\ u(0, t) = 0, u(1, t) = 0, u_{xx}(0, t) = 0, u_{xx}(1, t) = 0. \end{cases} \tag{2.9}$$

By Lemma 2.2, the solution of IBVP (2.9) can be written in the form

$$u(x, t) = [W_0(\varphi, \psi)](x, t) = W_{\mathbb{T}}(\tilde{\varphi}, \tilde{\psi})(x, t) \quad \text{for } x \in (0, 1).$$

The proposition below follows from Lemma 2.4 and 2.6.

Proposition 2.2. Let $0 \leq s < \frac{9}{2}$ and $T > 0$ be given. For any $\varphi \in H^s(0, 1)$, $\psi \in H^{s-2}(0, 1)$ satisfying the compatibility conditions

$$\varphi(0) = 0 \text{ if } s > \frac{1}{2} \text{ and } \psi(0) = 0 \text{ if } s > \frac{5}{2},$$

$u = [W_0(t)](\varphi, \psi)$ belongs to the space $C([0, T], H^s(0, 1)) \cap L^4([0, T] \times (0, 1))$. Moreover,

$$\|u\|_{C([0, T], H^s(0, 1)) \cap L^4([0, T] \times (0, 1))} \leq C \left(\|\varphi\|_{H^s(0, 1)} + \|\psi\|_{H^{s-2}(0, 1)} \right),$$

where C depends only on s and T .

Furthermore, for the forced linear Boussinesq equation,

$$\begin{cases} u_{tt} + u_{xxxx} = g_{xx}(x,t), & x \in (0,1), t \in \mathbb{R}, \\ u(x,0) = 0, u_t(x,0) = 0, \\ u(0,t) = 0, u(1,t) = 0, u_{xx}(0,t) = 0, u_{xx}(1,t) = 0, \end{cases} \tag{2.10}$$

its solution for $x \in (0,1)$ is given by

$$u(x,t) = [W_f(g_{xx}^*)](x,t) = - \int_0^t [W_T(0, g_{xx}^*)](x,t-\tau) d\tau.$$

where g^* is the odd extension of g from $(0,1)$ to $(-1,1)$. According to Lemma 2.5 and 2.6, we have the following estimate.

Proposition 2.3. Let $0 \leq s \leq 4$ and $T > 0$ be given. For any $g \in L^1([0,T]; H^s(0,1))$,

$$u = - \int_0^t [W_0(0, g_{xx})](x,t-\tau) d\tau \in C([0,T], H^s(0,1))$$

satisfying

$$\|u\|_{C([0,T], H^s(0,1))} \leq C \|g\|_{L^1([0,T]; H^s(0,1))},$$

and, in addition, if $0 \leq s < \frac{1}{2}$ and $g \in L^{\frac{4}{3}}([0,T] \times (0,1))$, then

$$u = - \int_0^t [W_0(0, g_{xx})](x,t-\tau) d\tau \in C([0,T], H^s(0,1)) \cap L^4([0,T] \times (0,1))$$

satisfying

$$\|u\|_{C([0,T], H^s(0,1)) \cap L^4([0,T] \times (0,1))} \leq C \|g\|_{L^4([0,T] \times (0,1))}$$

where C depends only on s and T .

3 Local well-posedness

In this section, we study the nonlinear IBVP.

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (u^2)_{xx} = 0, & x \in (0,1), t > 0, \\ u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \\ u(0,t) = h_1(t), u(1,t) = h_2(t), u_{xx}(0,t) = h_3'(t), u_{xx}(1,t) = h_4'(t) \end{cases} \tag{3.1}$$

for its well-posedness in the space $H^s(0,1)$. Let us define for given $0 \leq s < \frac{9}{2}$ and $T > 0$

$$X_{s,T} = H^s(0,1) \times H^{s-2}(0,1) \times \mathcal{Q}^s(0,T)$$

Theorem 3.1. Let $\frac{1}{2} < s < \frac{9}{2}$, $r > 0$ and $T > 0$ be given. There exists a $T^* \in (0, T]$ such that for any s -compatible $(\phi, \psi, \vec{h}) \in X_{s,T}$ satisfying

$$\|(\phi, \psi, \vec{h})\|_{X_{s,T}} \leq r,$$

the IBVP (3.1) admits a unique solution $u \in C([0, T^*]; H^s(0, 1))$. Moreover, the corresponding solution map is real analytic.

Proof. For $s > \frac{1}{2}$, $H^s(0, 1)$ is a Banach algebra. It follows that there is a constant $C = C_s$ such that

$$\|v^2\|_{H^s(0,1)} \leq C \|v\|_{H^s(0,1)}^2.$$

For any θ with $0 < \theta \leq T$ and $v \in C([0, \theta]; H^s(0, 1))$, Propositions 2.1 and 2.2 imply that the linear IBVP

$$\begin{cases} u_{tt} - u_{xx} + u_{xxxx} + (v^2)_{xx} = 0, & x \in (0, 1), t \in (0, T), \\ u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x) \\ u(0, t) = h_1(t), \quad u(1, t) = h_2(t), \quad u_{xx}(0, t) = h_3'(t), \quad u_{xx}(1, t) = h_4'(t), \end{cases} \quad (3.2)$$

admits a unique solution $u \in C([0, \theta]; H^s(0, 1))$. Moreover, there exists a constant $C > 0$ independent of θ such that

$$\|u\|_{C([0,\theta];H^s(0,1))} \leq C \|(\phi, \psi, \vec{h})\|_{X_{s,T}} + C\theta \|v\|_{C([0,\theta];H^s(0,1))}^2.$$

Thus, for any given $(\phi, \psi, \vec{h}) \in \mathcal{X}_{s,T}$, the IBVP (3.2) defines a nonlinear map Γ from $Y_{s,\theta} := \{w \in C([0, \theta]; H^s(0, 1))\}$ to $Y_{s,\theta}$. If $\theta > 0$ is chosen small enough, there exists an $M > 0$ such that

$$\begin{aligned} \|\Gamma(v_0)\|_{C([0,\theta];H^s(0,1))} &\leq M, \\ \|\Gamma(v_1) - \Gamma(v_2)\|_{C([0,\theta];H^s(0,1))} &\leq \frac{1}{2} \|v_1 - v_2\|_{C([0,\theta];H^s(0,1))} \end{aligned}$$

for any $v_0, v_1, v_2 \in C([0, \theta]; H^s(0, 1))$ with

$$\|v_j\|_{C([0,\theta];H^s(0,1))} \leq M, \quad j = 0, 1, 2.$$

Hence, the map Γ is a contraction whose unique fixed point is the desired solution u of (1.1). The proof is complete. \square

Theorem 3.2. Let $0 \leq s < \frac{1}{2}$ and $T > 0$ be given. There exists a $T^* \in (0, T]$ such that the IBVP (3.1) admits a unique solution $u \in C([0, T^*]; H^s(0, 1)) \cap L^4([0, T^*] \times (0, 1))$ which depends on (ϕ, ψ, \vec{h}) continuously in the corresponding spaces.

Proof. The solution

$$u(x, t) = W_{bdr}(t)\vec{h} + [W_0(t)](\phi, \psi) - \int_0^t [W_0(t-\tau)](0, \partial_x^2(u+u^2))d\tau.$$

can be changed to a problem for finding a fixed point of a nonlinear operator. Consequently, the theorem follows using the same argument as that in the Fang and Grilakis [12] and Proposition 2.1 for the boundary integrals. \square

4 Appendix

In this section, we intend to show that the regularity for the boundary condition presented in Proposition 2.1 is indeed optimal. It does no harm to consider just the case for $\vec{h} = (h_1, 0, 0, 0)$. Here, we note that if the estimate in Proposition 2.1 holds, then,

$$\|u_h\|_{L^2(\Omega_T)} = \|u\|_{L^2([0,T] \times (0,1))} \leq C_T \|h_1\|_{H^{\frac{1}{2}}(0,1)}.$$

For simplicity we drop the footnote for h_1 and consider

$$u_h = \sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \int_0^t (e^{in^2\pi^2(t-\tau)} - e^{-in^2\pi^2(t-\tau)}) h(\tau) d\tau.$$

Assume that $h(t)$ has a Fourier series expansion:

$$h(t) = \sum_{k=-\infty}^{\infty} e^{-i\pi^2 kt} a_k \quad \text{with} \quad a_k = \int_0^{2/\pi} e^{i\pi^2 kt} h(t) dt.$$

Thus,

$$\begin{aligned} u_h &= \sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \int_0^t (e^{in^2\pi^2(t-\tau)} - e^{-in^2\pi^2(t-\tau)}) \sum_{k=-\infty}^{\infty} e^{-i\pi^2 k\tau} a_k d\tau \\ &= \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x + i(n\pi)^2 t} \sum_{k \neq -n^2} \frac{e^{-i\pi^2(n^2+k)t} - 1}{-i\pi^2(n^2+k)} a_k \right) + \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x + i(n\pi)^2 t} t a_{-n^2} \right) \\ &\quad - \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x - i(n\pi)^2 t} \sum_{k \neq n^2} \frac{e^{i\pi^2(n^2-k)t} - 1}{i\pi^2(n^2-k)} a_k \right) - \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x - i(n\pi)^2 t} t a_{n^2} \right) \\ &= \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \sum_{k \neq -n^2} \frac{e^{-i\pi^2 kt} - e^{i(n\pi)^2 t}}{-i\pi^2(n^2+k)} a_k \right) + \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x + i(n\pi)^2 t} t a_{-n^2} \right) \\ &\quad - \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \sum_{k \neq n^2} \frac{e^{-i\pi^2 kt} - e^{-i(n\pi)^2 t}}{i\pi^2(n^2-k)} a_k \right) - \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x - i(n\pi)^2 t} t a_{n^2} \right) \\ &= \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \sum_{k \neq -n^2} \frac{e^{-i\pi^2 kt}}{-i\pi^2(n^2+k)} a_k \right) + \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x + i(n\pi)^2 t} \left(t a_{-n^2} \right. \right. \\ &\quad \left. \left. + \sum_{k \neq -n^2} \frac{a_k}{i\pi^2(n^2+k)} \right) \right) - \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x} \sum_{k \neq n^2} \frac{e^{-i\pi^2 kt}}{i\pi^2(n^2-k)} a_k \right) \\ &\quad - \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x - i(n\pi)^2 t} \left(t a_{n^2} - \sum_{k \neq n^2} \frac{a_k}{i\pi^2(n^2-k)} \right) \right). \end{aligned}$$

Choose $h(t)$ such that

$$a_{n^2} = \int_0^{2/\pi} e^{i\pi^2 n^2 t} h(t) dt = 0, \quad \text{and} \quad a_{-n^2} = \int_0^{2/\pi} e^{-i\pi^2 n^2 t} h(t) dt = 0.$$

Therefore,

$$u_h = \left(\sum_{n=-\infty}^{\infty} \sum_{k \neq \pm n^2} n\pi e^{in\pi x} e^{-i\pi^2 kt} \left(\frac{a_k}{-i\pi^2(n^2+k)} - \frac{a_k}{i\pi^2(n^2-k)} \right) \right) + \left(\sum_{n=-\infty}^{\infty} n\pi \right. \\ \left. \times e^{in\pi x + i(n\pi)^2 t} \left(\sum_{k \neq -n^2} \frac{a_k}{i\pi^2(n^2+k)} \right) \right) - \left(\sum_{n=-\infty}^{\infty} n\pi e^{in\pi x - i(n\pi)^2 t} \left(\sum_{k \neq n^2} \frac{a_k}{i\pi^2(n^2-k)} \right) \right).$$

It is easy to see that all the index in the summation will not equal to each other. Then, it follows,

$$\|u_h\|_{L^2((0,1) \times (0, \frac{2}{\pi}))}^2 = \sum_{n=-\infty}^{\infty} \sum_{k \neq \pm n^2} n^2 \pi^2 \left(\frac{a_k}{-i\pi^2(n^2+k)} - \frac{a_k}{i\pi^2(n^2-k)} \right)^2 \\ + \sum_{n=-\infty}^{\infty} n^2 \pi^2 \left(\sum_{k \neq -n^2} \frac{a_k}{i\pi^2(n^2+k)} \right)^2 + \sum_{n=-\infty}^{\infty} n^2 \pi^2 \left(\sum_{k \neq n^2} \frac{a_k}{i\pi^2(n^2-k)} \right)^2 \\ \geq \sum_{n=-\infty}^{\infty} \frac{n^2}{\pi^2} a_{n^2-1}^2 \quad (\text{picking } k = n^2 - 1).$$

On the other hand, if there is a constant C such that for all $h \in H^\alpha(0, \frac{2}{\pi})$,

$$\|u_h\|_{L^2((0,1) \times (0, \frac{2}{\pi}))}^2 \leq C \|h\|_{H^\alpha(0, \frac{2}{\pi})}^2,$$

then we want to show that $\alpha \geq \frac{1}{2}$. Otherwise, there is a constant C such that for some $\alpha \in (0, \frac{1}{2})$,

$$\|u_h\|_{L^2((0,1) \times (0, \frac{2}{\pi}))}^2 \leq C \|h\|_{H^\alpha(0, \frac{2}{\pi})}^2.$$

Let us choose h defined as

$$h(t) = \sum_{n \neq 0} \frac{1}{|n|^\beta} e^{-i\pi^2(n^2-1)t}.$$

If $h \in H^\alpha(0, \frac{2}{\pi})$, then we need

$$\sum_{n \neq 0} \left| \frac{(n^2-1)^\alpha}{|n|^\beta} \right|^2 < +\infty,$$

that is, $2\beta - 4\alpha > 1$, which follows that $\beta > 2\alpha + \frac{1}{2}$. However, for this h , we have,

$$\|u_h\|_{L^2((0,1) \times (0, \frac{2}{\pi}))}^2 \geq \sum_{n \neq 0, n=-\infty}^{\infty} \frac{n^2}{\pi^2} \frac{1}{|n|^{2\beta}} = \sum_{n \neq 0, n=-\infty}^{\infty} \frac{1}{|n|^{2\beta-2}}.$$

Since $\alpha < \frac{1}{2}$, we can choose β such that $2\alpha + \frac{1}{2} < \beta < \frac{3}{2}$ and it follows that

$$\sum_{n \neq 0, n=-\infty}^{\infty} \frac{1}{|n|^{2\beta-2}} = +\infty.$$

Therefore, we have a contradiction. Hence, we must have $\alpha \geq \frac{1}{2}$.

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References

- [1] J. L. Bona, M. Chen, A Boussinesq system for two-way propagation of nonlinear dispersive waves, *Physica D: Nonlinear Phenomena*, **116**(1998)(1), 191–224.
- [2] J. L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. ii. the nonlinear theory, *Nonlinearity*, **17**(3)(2004), 925–952.
- [3] J. L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. i. derivation and linear theory, *Journal of Nonlinear Science*, **12**(4)(2002), 283–318.
- [4] J. L. Bona, R. L. Sachs, Global existence of smooth solutions and stability of solitary waves for a generalized Boussinesq equation, *Commun. in Mathematical Physics*, **118**(1)(1998), 15–29.
- [5] J. L. Bona, S. M. Sun, B.-Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane, *Transactions of the American Mathematical Society*, **354**(2)(2002), 427–490.
- [6] J. L. Bona, S. M. Sun, B.-Y. Zhang, A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain, *Comm. Partial Differential Equations*, **28**(7-8)(2003), 1391–1436.
- [7] J. L. Bona, S. M. Sun, B.-Y. Zhang, A non-homogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain. ii, *Journal of Differential Equations*, **247**(9)(2009), 2558–2596.
- [8] J. L. Bona, S.-M. Sun, B.-Y. Zhang, Nonhomogeneous boundary value problem for one-dimensional nonlinear Schrodinger equations, in print.
- [9] J. Boussinesq, Theorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal equation, *J.Math.Pures Appl.*, **17**(1872), 55–108.
- [10] J. de Frutos, T. Ortega, J. M. Sanz-Serna, Pseudospectral method for the “good” Boussinesq equation, *Mathematics of Computation*, **57**(195)(1991), 109–122.
- [11] F. Falk, E. Laedke, K. Spatschek, Stability of solitary-wave pulses in shape-memory alloys, *Physical Review B*, **36**(6)(1987), 3031.
- [12] Y.-F. Fang, M. G. Grillakis, Existence and uniqueness for Boussinesq type equations on a circle, *Commun. in Partial Differential Equations*, **21**(7-8)(1996), 1253–1277.
- [13] L. G. Farah, Local solutions in Sobolev spaces with negative indices for the “good” Boussinesq equation, *Commun. in Partial Differential Equations*, **34**(1-3)(2009), 52–73.

- [14] L. G. Farah, M. Scialom, On the periodic “good” Boussinesq equation, *Proceedings of the American Mathematical Society*, **138**(3)(2010), 953–964.
- [15] A. A. Himonas, D. Mantzavinos, The “good” Boussinesq equation on the half-line. *J. Differential Equations*, **258**(9)(2015), 3107–3160.
- [16] N. Kishimoto, Sharp local well-posedness for the “good” Boussinesq equation, *Journal of Differential Equations*, **254**(6)(2013), 2393–2433.
- [17] F. Linares, Global existence of small solutions for a generalized Boussinesq equation, *Journal of Differential Equations*, **106**(2)(1993), 257–293.
- [18] F.-L. Liu, D. L. Russell, Solutions of the Boussinesq equation on a periodic domain, *Journal of Mathematical Analysis and Applications*, **194**(1)(1995), 78–102.
- [19] Y. Liu, Instability of solitary waves for generalized Boussinesq equations, *Journal of Dynamics and Differential Equations*, **5**(3)(1993), 537–558.
- [20] Y. Liu, Instability and blow-up of solutions to a generalized Boussinesq equation, *SIAM Journal on Mathematical Analysis*, **26**(6)(1995), 1527–1546.
- [21] Y. Liu, Decay and scattering of small solutions of a generalized Boussinesq equation, *Journal of Functional Analysis*, **147**(1)(1997), 51–68.
- [22] Y. Liu, Strong instability of solitary-wave solutions of a generalized Boussinesq equation, *Journal of Differential Equations*, **164**(2)(2000), 223–239.
- [23] S. Oh, A. Stefanov. Improved local well-posedness for the periodic “good” Boussinesq equation, *Journal of Differential Equations*, **254**(10)(2013), 4047–4065.
- [24] A. K. Pani, H. Saranga, Finite element galerkin method for the “good” Boussinesq equation, *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal. Series A: Theory and Methods*, **29**(8)(1997), 937–956.
- [25] R. L. Sachs, On the blow-up of certain solutions of the “good” Boussinesq equation, *Applicable Analysis. An International Journal*, **36**(3-4)(1990), 145–152.
- [26] M. Tsutsumi, T. Matabashi. On the Cauchy problem for the Boussinesq type equation, *Mathematica Japonica*, **36**(2)(1991), 371–379.
- [27] S. K. Turitsyn, Nonstable solitons and sharp criteria for wave collapse, *Physical Review E*, **47**(1):R13, 1993.
- [28] H. Wang and A. Esfahani, Well-posedness for the Cauchy problem associated to a periodic Boussinesq equation. *Nonlinear Analysis: Theory, Methods and Applications*, **89**(2013), 267–275.
- [29] H. Wang and A. Esfahani, Well-posedness for the Cauchy problem associated to a periodic Boussinesq equation, *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal. Series A: Theory and Methods*, **89**(2013), 267–275.
- [30] R. Xue, Local and global existence of solutions for the Cauchy problem of a generalized Boussinesq equation, *Journal of Mathematical Analysis and Applications*, **316**(1)(2006), 307–327.
- [31] R. Xue, The initial-boundary value problem for the “good” Boussinesq equation on the bounded domain, *Journal of Mathematical Analysis and Applications*, **343**(2)(2008), 975–995.
- [32] R. Xue, The initial-boundary-value problem for the “good” Boussinesq equation on the half line, *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal. Series A: Theory and Methods*, **69**(2)(2008), 647–682.
- [33] R. Xue, The initial-boundary-value problem of a generalized Boussinesq equation on the half line, *Journal of the Korean Mathematical Society*, **45**(1)(2008), 79–95.
- [34] R. Xue, Low regularity solution of the initial-boundary-value problem for the “good” Boussinesq equation on the half line. *Acta Mathematica Sinica (English Series)*, **26**(12)(2010), 2421–2442.
- [35] Z. Yang, On local existence of solutions of initial boundary value problems for the “bad”

Boussinesq-type equation. *Nonlinear Analysis. Theory, Methods and Applications. An International Multidisciplinary Journal. Series A: Theory and Methods*, **51**(7)(2002), 1259–1271.

- [36] V. Zakharov, On stochastization of one-dimensional chains of nonlinear oscillators, *Sov. Phys.-JETP*, **38**(1974), 108–110.