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Complex Deformation of Critical Kähler Metrics

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Abstract. In this paper, we use Pacard-Xu's methods to discuss the complex deformation of constant scalar curvature metrics in the case of fixed and varying complex structures. Moreover, we also discuss the complex deformation of Kähler-Ricci solitons.

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1 Introduction

In [5, 6], Calabi introduced the extremal Kähler metrics, which is the citical point of the L^2 norm of the scalar curvature in the Kähler class. The existence and uniqueness of the extremal Kähler metrics have been intensively studied during past decades([2,7] and reference therein). By Kodaira-Spencer's work [15], every Kähler manifold admits Kähler metrics under small perturbation of the complex structure. A natural question is whether Kähler-Einstein metrics or extremal Kähler metrics still exist when the complex structures varies. In [17], Koiso showed that the Kähler-Einstein metrics can be perturbed under the complex deformation of the complex structure when the first Chern class is zero or negative. When the first Chern class is positive, Koiso showed this result if the manifold has no nontrivial holomorphic vector fields. In [11, 12], Lebrun-Simanca systematically studied the deformation theory of extremal Kähler metrics and constant scalar curvature metrics and they proved that on a Kähler manifold, the set of Kähler classes which admits extremal metrics is open and the constant scalar curvature metrics can be perturbed under some extra restrictions. Based on Lebrun-Simanca's results, Apostolov-Calderbank-Gauduchon-T. Friedman [1], Rollin-Simanca-Tipler [19, 20] further discussed extremal metrics under the deformation of complex structures.

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The main goal of this paper is to give an alternative proof on the deformation of constant scalar curvature metrics, which was discussed by [11] in the case of fixed complex structure, and later by [1,19] in the case of varying complex structures. Here we use the method of Pacard-Xu in [18] in the context of constant mean curvature problems, which is quite different from [11] in analysis. We will also discuss the deformation of Kähler-Ricci solitons.

First we consider the case of fixed complex structure. The main difficulty of the deformation problems of the Kähler-Einstein metrics or constant scalar curvature metrics is that the linearized equation has nontrivial kernel so that we cannot use the implicit function theorem directly. For this reason, Koiso in [17] assumed that the manifold has no nontrivial holomorphic vector fields, and Lebrun-Simanca in [11] used the surjective version of the implicit function theorem so that the nondegeneracy of the Futaki invariant must be assumed. The same difficulty appears in some other geometrical equations such as the constant mean curvature equation. In [18], Pacard-Xu constructed a new functional to solve the constant mean curvature equation and they removed the nondegeneracy condition of Ye's result in [24]. We observe that Pacard-Xu's method can be applied in our situation and we have the result:

Theorem 1.1. *Let (M,ω_g) be a compact Kähler manifold with a constant scalar curvature metric ω_g. There exists $\epsilon_0>0$ and a smooth function*

$$\Phi:(0,\epsilon_0)\times\mathcal{H}^{1,1}(M)\to\mathbb{R}$$

such that if $\beta\in\mathcal{H}^{1,1}(M)$ has unit norm and satisfies $\Phi(t,\beta)=0$ for some $t\in(0,\epsilon_0)$ then M admits a constant scalar curvature metric in the Kähler class $[\omega_g+t\beta]$. Moreover,

(1) If $\beta\in\mathcal{H}^{1,1}(M)$ is traceless, Φ has the expansion:

$$\Phi(t,\beta)=t^2\int_M(\Pi_g(R_{i\bar{j}}\beta_{j\bar{i}}))^2\omega_g^n+O(t^3).$$

(2) If $\beta\in\mathcal{H}^{1,1}(M)$ is traceless and ω_g is a Kähler-Einstein metric, then Φ has the expansion:

$$\Phi(t,\beta)=t^4\int_M(\Pi_g(\beta_{i\bar{j}}\beta_{j\bar{i}}))^2\omega_g^n+O(t^5).$$

Here the operator Π_g is the projection to the space of Killing potentials with respect to ω_g.

Theorem 1.1 gives us some information in which directions we can find the constant scalar curvature metrics. The function Φ is constructed by the Futaki invariant, and it is automatically zero when the Futaki invariant vanishes. Thus, a direct corollary of Theorem 1.1 is the following result, which was proved by Lebrun-Simanca using the deformation theory of the extremal Kähler metrics and a result of Calabi in [6]:

Corollary 1.1. (Lebrun-Simanca [11]) Let (M,ω_g) be a compact Kähler manifold with a constant scalar curvature metric ω_g. For any $\beta\in\mathcal{H}^{1,1}(M)$, there is a $\epsilon_0>0$ such that if the

Futaki invariant vanishes on the Kähler class $[\omega_g+t\beta]$ for some $t\in(0,\epsilon_0)$, then M admits a constant scalar curvature metric on $[\omega_g+t\beta]$.

In fact, Theorem 1.1 gives us more information on the existence of constant scalar curvature metrics on the class $[\omega_g+t\beta]$. We expand the function $\Phi(t,\beta)$ with respect to t at $t=0$,

$$\Phi(t,\beta)=\sum_{j=1}^{m}a_j(\beta)t^j+O(t^{m+1}),$$

where $a_j(\beta)$ are some functions of β. If we assume some of $a_j(\beta)$ vanish, then we can get "almost constant scalar curvature metrics" in the following sense:

Corollary 1.2. Let ω_g be a constant scalar curvature metric. There are two positive constants ϵ and C such that for any $\beta\in\mathcal{H}^{1,1}(M)$ with

$$a_1(\beta)=a_2(\beta)=\cdots=a_m(\beta)=0,$$

M admits a Kähler metric $\omega_{t,\beta}\in[\omega_g+t\beta]$ for $t\in(0,\epsilon)$ satisfying

$$\|\mathbf{s}(\omega_{t,\beta})-\underline{\mathbf{s}}(t)\|_{C^k(M)}\leq Ct^{\frac{m+1}{2}},$$

where $\underline{\mathbf{s}}(t)$ is the average of the scalar curvature in $[\omega_g+t\beta]$.

The case of varying complex structures is more difficult. In general the extremal metrics may not be perturbed when the complex structure varies ([4]). There are several results on this problem recently. In [1] Apostolov-Calderbank-Gauduchon-T. Friedman showed that the extremal metrics can be perturbed when the deformation of the complex structure is invariant under the action of a maximal compact connected subgroup G of the isometry group of the extremal metrics. Rollin-Simanca-Tipler extend this result in [19] and they allow the group G extends partially to the complex deformation. Here we combine Rollin-Simanca-Tipler and Pacard-Xu's methods to get a similar result as in the case of fixed complex structures.

Before stating the next result, we need to introduce some notations. Let (M,J,g,ω_g) be a compact Kähler manifold with a constant scalar curvature metric (g,ω_g) and G the identity component of the isometry group of (M,g). We assume that a compact connected subgroup G' of G acts holomorphically on a complex deformation (J_t,g_t,ω_t) and we denote by $\mathcal{B}_{G'}$ the space of all such complex deformations. Let $W^{2,k}_{G'}$ be the space of G'-invariant functions in $W^{2,k}$ and $\mathcal{H}^{\mathfrak{z}_0'}_g$ be the space of the space of holomorphic potentials of the elements in the center $\mathfrak{z}_0'$ of $\mathfrak{g}_0'$, where $\mathfrak{g}_0'$ is the ideal of the Killing vector fields with zeroes in the Lie algebra of G'. With these notations, we have

Theorem 1.2. *Let (M,J,g,ω_g) be a compact Kähler manifold with a constant scalar curvature metric ω_g and*

$$\ker\mathbb{L}_g\cap W^{2,k}_{G'}\subset\mathbb{R}\oplus\mathcal{H}^{\mathfrak{z}_0'}_g. \tag{1.1}$$

For any $(J_t,g_t,\omega_t)\in\mathcal{B}_{G'}$*, there is a constant* $\epsilon_0>0$ *and a smooth function* $\Psi:\mathcal{B}_{G'}\to\mathbb{R}$ *such that if*

$$\Psi(J_t,g_t,\omega_t)=0 \tag{1.2}$$

for some $t\in(0,\epsilon_0)$*, then* M *admits a* G'*-invariant constant scalar curvature metric in* $[\omega_t]$ *with respect to* J_t*. In particular, the conclusion holds if the condition* (1.2) *is replaced by the vanishing of the Futaki invariant of* $[\omega_t]$*.*

The condition (1.1) coincides with the non-degeneracy condition of the relative Futaki invariant, which is introduced by Rollin-Simanca-Tipler in [19]. Here we get the same condition from a different point of view. We can get a similar result as Corollary 1 and a similar expansion of the function Ψ as in Theorem 1.1, which are omitted since we will not use them in this paper.

Finally, we will study the deformation of the Kähler-Ricci soliton. A Kähler-Ricci soliton is a Kähler metric ω_g in the first Chern class satisfying

$$Ric(\omega_g)-\omega_g=\sqrt{-1}\partial\bar{\partial}\theta_X,$$

where θ_X is a holomorphic potential of a holomorphic vector field X. As Kähler-Einstein metrics, the existence and uniqueness of Kähler-Ricci soliton are important and has been studied by a series of papers [22, 23] etc. Since Kähler-Ricci solitons must be in the first Chern class, there are no Kähler-Ricci solitons if we deform the Kähler class. However, inspired by the extremal Kähler metrics, we can consider whether there is a metric satisfying the equation

$$\mathbf{s}(\omega_g)-\underline{\mathbf{s}}=\Delta_g\theta_X,$$

where $\underline{\mathbf{s}}$ is the average of the scalar curvature $\mathbf{s}$. This metric is first introduced by Guan in [9] and is called extremal solitons. Using the same idea as in [11,12], we have the result:

Theorem 1.3. *Let* (M,J,g,ω_g) *be a compact Kähler manifold with a Kähler-Ricci soliton* (g,ω_g)*.*

1. *If the complex structure is fixed, for any* $\beta\in\mathcal{H}^{1,1}(M)$ *there is an extremal soliton in the Kähler class* $[\omega_g+t\beta]$ *for small* t*.*

2. *For any* $(J_t,g_t,\omega_t)\in\mathcal{B}_G$ *where* G *is the identity component of the isometry group of* (M,g)*,* M *admits a* G*-invariant extremal soliton in* $[\omega_t]$ *with respect to* J_t*.*

Under the assumption of the second part of Theorem 1.3, if in addition $[\omega_t]$ is the first Chern class of (M,J_t), then $[\omega_t]$ admits a Kähler-Ricci soliton. It is interesting to see whether Theorem 1.3 holds for any extremal soliton. There is a technical difficulty in the proof and we cannot overcome it here.

2 Deformation of cscK metrics

In this section, we will use the method of Pacard-Xu in [18] to solve the constant scalar curvature equation and show that a small perturbation of the Kähler class under some assumptions will admit a constant scalar curvature metric.

2.1 Fixed complex structure

We follow Lebrun-Simanca's notations in [11, 12]. Let (M,J,g,ω_g) be a compact Kähler manifold of complex dimension n with a constant scalar curvature metric ω_g. By Matsushima-Lichnerowicz theorem, the identity component G of the isometry group of (M,g) is a maximal compact subgroup of the identity component $\mathrm{Aut}_0(M,J)$ of the automorphism group $\mathrm{Aut}(M,J)$. Let $W^{2,k}_G(M)$ be the real k-th Sobolev space of G-invariant real-valued functions in $W^{2,k}(M)$. By the Sobolev embedding theorem, the space $W^{2,k}(M)$ is contained in $C^l(M)$ if $k>n+l$. The space of real-valued ω_g-harmonic (1,1) forms on M is denoted by $\mathcal{H}^{1,1}(M)$. Since the metric g is G-invariant, every g-harmonic form $\beta\in\mathcal{H}^{1,1}(M)$ is G-invariant. Let $\mathcal{P}(M,\omega_g)$ be the space of Kähler potentials of ω_g and $\mathcal{U}$ be a small neighborhood of the origin in $W^{2,k}_G(M)$. We can assume that $\mathcal{U}\subset\mathcal{P}(M,\omega_t)$ for small t where $\omega_t=\omega_g+t\beta$. Thus, for any function $\varphi\in\mathcal{U}$ the metric

$$\omega_{t,\varphi}=\omega_g+t\beta+\sqrt{-1}\partial\bar{\partial}\varphi,$$

is G-invariant.

Let $\mathfrak{h}(M,J)$ be the space of holomorphic vector fields on (M,J). By Matsushima-Lichnerowicz theorem, the Lie algebra $\mathfrak{h}(M,J)$ can be decomposed as a direct sum

$$\mathfrak{h}(M,J)=\mathfrak{h}_0(M,J)\oplus\mathfrak{a}(M,J),$$

where $\mathfrak{a}(M,J)$ consists of the autoparallel holomorphic vector fields of (M,J) and $\mathfrak{h}_0(M,J)$ is the space of holomorphic vector fields with zeros. Let $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{g}_0$ the ideal of Killing vector fields with zeros. Any element $\xi\in\mathfrak{g}_0$ corresponds to a holomorphic vector field $X=J\xi+\sqrt{-1}\xi$, and we define a smooth function θ_X satisfying

$$i_X\omega_g=\sqrt{-1}\bar{\partial}\theta_X,\quad \int_M\theta_X\omega_g^n=0.$$

The function θ_X is called holomorphic potential of X with respect to ω_g. Since g is G-invariant, θ_X is a real-valued function. Let $\mathfrak{z}\subset\mathfrak{g}$ denote the center of $\mathfrak{g}$ and $\mathfrak{z}_0=\mathfrak{z}\cap\mathfrak{g}_0$. Then $\mathfrak{z}_0$ corresponds precisely to the Killing vector fields in $\mathfrak{g}_0$ whose holomorphic potentials are G-invariant.

Now we choose a basis $\{\xi_1,\cdots,\xi_d\}$ of $\mathfrak{z}_0$ such that the functions $\{\theta_0,\theta_1,\cdots,\theta_d\}$, where $\theta_0=1$ and θ_i is the holomorphic potential of the holomorphic vector fields $X_i=J\xi_i+\sqrt{-1}\xi_i$, are orthonormal with respect to the L^2 inner product induced by the metric g

$$\langle f,g\rangle_{L^2(\omega_g)}=\frac{1}{V_g}\int_M fg\omega_g^n,\quad f,g\in C^\infty(M,\mathbb{R}),$$

where $V_g=\int_M\omega_g^n$. Using this product, the space $W^{2,k}_G$ has a decomposition

$$W^{2,k}_G=\mathcal{H}_g\oplus\mathcal{H}^\perp_{g,k},$$

where $\mathcal{H}_g$ is spanned by the set $\{\theta_0,\theta_1,\cdots,\theta_d\}$ over $\mathbb{R}$. We define the associate projection operator

$$\begin{aligned}\tilde{\Pi}_g: W^{2,k}_G &\to \mathcal{H}_g\\ f &\to \sum_{i=0}^{d}\langle\theta_i,f\rangle_{L^2(\omega_g)}\theta_i,\end{aligned}$$

and the operator $\tilde{\Pi}_g^{\perp}=I-\tilde{\Pi}_g$.

For any $\varphi\in\mathcal{U}$, we calculate the expansion of the scalar curvature of $\omega_{t,\varphi}$ at $(t,\varphi)=(0,0)$:

$$s(\omega_{t,\varphi})=s(\omega_g)-\Big(\Delta_g^2\varphi+R_{i\bar{j}}\varphi_{j\bar{i}}+t\Delta_g\mathrm{tr}_g\beta+tR_{i\bar{j}}\beta_{j\bar{i}}\Big)+Q_g(\nabla^2\varphi,t\beta),$$

where Q_g collects all the higher order terms. Note that $\mathrm{tr}_g\beta$ is a constant since β is harmonic. The linearized operator of $s(\omega_{t,\varphi})$ at $(t,\varphi)=(0,0)$ is $-\mathbb{L}_g\varphi$, where the operator $\mathbb{L}_g\varphi$ is defined by

$$\mathbb{L}_g\varphi=\Delta_g^2\varphi+R_{i\bar{j}}\varphi_{j\bar{i}},$$

and for any $f\in\ker\mathbb{L}_g$ we can associate a holomorphic vector field $X_f=J\nabla f+\sqrt{-1}\nabla f$ which has nonempty zeros. In general, $\mathbb{L}_g$ has nontrivial kernel and it is difficult to solve the constant scalar curvature equation.

Now we have the following result:

Theorem 2.1. *Let (M,ω_g) be a compact Kähler manifold with a constant scalar curvature metric ω_g. There exists $\epsilon_0>0$ and a smooth function*

$$\Phi:(0,\epsilon_0)\times\mathcal{H}^{1,1}(M)\to\mathbb{R}$$

such that if $\beta\in\mathcal{H}^{1,1}(M)$ has unit norm and satisfies $\Phi(t,\beta)=0$ for some $t\in(0,\epsilon_0)$ then M admits a constant scalar curvature metric in the Kähler class $[\omega_g+t\beta]$.

Proof. Consider the equation for $(\varphi,\tilde{\Xi})\in\mathcal{H}^{\perp}_{g,k}\times\mathbb{R}^{d+1}$:

$$s(\omega_{t,\varphi})=\langle\tilde{\Xi},\tilde{\Theta}\rangle, \tag{2.1}$$

where $\tilde{\Theta}=(\theta_0,\theta_1,\cdots,\theta_d)$ and $\tilde{\Xi}=(c_0,c_1,\cdots,c_d)\in\mathbb{R}^{d+1}$ is a vector with

$$\langle\tilde{\Xi},\tilde{\Theta}\rangle=c_0+\sum_{i=1}^{d}c_i\theta_i.$$

Note that if Eq. (2.1) holds, then c_0 is the average of the scalar curvature and it only depends on the Kähler class $[\omega_t]$. Applying the implicit function theorem, we have

Lemma 2.1. *Fix $\beta\in\mathcal{H}^{1,1}(M)$. Then there exist $\epsilon_0,C>0$ such that for all $t\in(0,\epsilon_0)$ there exists a unique solution $(\varphi_{t,\beta},\tilde{\Xi}_{t,\beta})\in\mathcal{H}^{\perp}_{g,k+4}\times\mathbb{R}^{d+1}$ of Eq.* (2.1) *and satisfying the estimates*

$$\|\varphi_{t,\beta}\|_{W^{2,k+4}(M)}\leq C\epsilon_0,\quad \|\tilde{\Xi}_{t,\beta}\|\leq C\epsilon_0, \tag{2.2}$$

where $\|\tilde{\Xi}\|$ denotes the standard Euclidean norm of $\tilde{\Xi}$ in $\mathbb{R}^{d+1}$.

Proof. We consider the operator

$$\tilde{\Pi}^{\perp}_g s(\omega_{t,\varphi}):(-\epsilon,\epsilon)\times\mathcal{H}^{\perp}_{g,k+4}\to\mathbb{R}.$$

Since the linearized operator at $(t,\varphi)=(0,0)$

$$\begin{aligned}&D_\varphi\tilde{\Pi}^{\perp}s(\omega_{t,\varphi})|_{(0,0)}:\mathcal{H}^{\perp}_{g,k+4}\to\mathcal{H}^{\perp}_{g,k}\\&\psi\to-\mathbb{L}_g\psi\end{aligned}$$

is invertible, for small t there is a solution $\varphi_{t,\beta}\in\mathcal{H}^{\perp}_{g,k+4}$ such that $\tilde{\Pi}^{\perp}_g s(\omega_{t,\varphi_{t,\beta}})=0$ and we can find a vector $\tilde{\Xi}_{t,\beta}\in\mathbb{R}^{d+1}$ such that

$$s(\omega_{t,\varphi_{t,\beta}})=\langle\tilde{\Xi}_{t,\beta},\tilde{\Theta}\rangle. \tag{2.3}$$

The estimates in (2.2) follow directly from the implicit function theorem. □

Now we want to know when the solution $(\varphi_{t,\beta},\tilde{\Xi}_{t,\beta})$ of (2.1) has constant scalar curvature. It suffices to show that the vector $\tilde{\Xi}_{t,\beta}=(c_0,c_1,\cdots,c_d)$ satisfies $c_i=0$ for all $1\leq i\leq d$. Given $\beta\in\mathcal{H}^{1,1}(M)$, the solution $(\varphi_{t,\beta},\tilde{\Xi}_{t,\beta})$ determines a holomorphic vector field

$$X_{t,\beta}=\sum_{k=1}^{d}c_k(t)X_k\in\mathfrak{h}_0(M,J), \tag{2.4}$$

where X_k is the holomorphic vector field defined by θ_k and $c_i(t)$ are the entries of the vector $\tilde{\Xi}_{t,\beta}=(c_0(t),c_1(t),\cdots,c_d(t))$. For simplicity, we write $\omega_{t,\beta}=\omega_{t,\varphi_{t,\beta}}$ for short. Now we define a function on $(0,\epsilon_0)\times\mathcal{H}^{1,1}(M)$ by

$$\Phi(t,\beta)=\int_M X_{t,\beta}h_{\omega_{t,\beta}}\omega^n_{t,\beta},$$

where $h_{\omega_{t,\beta}}$ is determined by $s(\omega_{t,\beta})-c_0(t)=\Delta_{\omega_{t,\beta}}h_{\omega_{t,\beta}}$. Note that the function $\Phi(t,\beta)$ is exactly the Futaki invariant of $(X_{t,\beta},[\omega_t])$, and it is zero if the Futaki invariant of $[\omega_t]$ vanishes. Let Π_g be the L^2-projection from $W^{2,k}_G(M)$ to the subspace which is spanned by the functions $\{\theta_1,\cdots,\theta_d\}$. We denote by $\Xi_{t,\beta}=(c_1,\cdots,c_d)$ the vector in $\mathbb{R}^d$ which removes c_0 from $\tilde{\Xi}_{t,\beta}$ and $\Theta=(\theta_1,\cdots,\theta_d)$. With these notations, we have the lemma:

Lemma 2.2. *There is a $\epsilon_0>0$ such that, if $t\in(0,\epsilon_0)$ and if $\beta\in\mathcal{H}^{1,1}(M)$ with unit norm is a zero of the function $\Phi(t,\beta)$ then $\omega_{t,\beta}$ has constant scalar curvature.*

Proof. Note that

$$\Phi(t,\beta)=\int_M \theta_{t,\beta}(s(\omega_{t,\beta})-c_0(t))\omega_{t,\beta}^n=\int_M \theta_{t,\beta}\langle\Xi_{t,\beta},\Theta\rangle\omega_{t,\beta}^n, \tag{2.5}$$

where $\theta_{t,\beta}$ is the holomorphic potential of $X_{t,\beta}$ with respect to the metric $\omega_{t,\beta}$ under the normalization condition

$$\int_M \theta_{t,\beta}\omega_{t,\beta}^n=0. \tag{2.6}$$

We claim that there is a constant C independent of t and β such that

$$\|\theta_{t,\beta}-\langle\Xi_{t,\beta},\Theta\rangle\|_{L^2(\omega_g)}\le C\epsilon_0\|\Xi_{t,\beta}\|. \tag{2.7}$$

In fact, by definition we have

$$i_{X_{t,\beta}}\omega_g=\sqrt{-1}\bar\partial\langle\Xi_{t,\beta},\Theta\rangle,\quad i_{X_{t,\beta}}\omega_{t,\beta}=\sqrt{-1}\bar\partial\theta_{t,\beta}.$$

This implies that

$$\sqrt{-1}\bar\partial(\theta_{t,\beta}-\langle\Xi_{t,\beta},\Theta\rangle)=i_{X_{t,\beta}}(t\beta+\sqrt{-1}\partial\bar\partial\varphi_{t,\beta})=\sum_{k=1}^d c_k(t)i_{X_k}(t\beta+\sqrt{-1}\partial\bar\partial\varphi_{t,\beta}),$$

where we used the definition (2.4) of $X_{t,\beta}$. Since by Lemma 2.1 $\|\varphi_{t,\beta}\|_{W^{2,k+4}(M)}\le C\epsilon_0$ for any $t\in(0,\epsilon_0)$, we have

$$\begin{aligned}\Big|\Delta_g(\theta_{t,\beta}-\langle\Xi_{t,\beta},\Theta\rangle)\Big|&=\Big|\sum_k c_k(t)\,\mathrm{tr}_g\Big(\partial(i_{X_k}(t\beta+\sqrt{-1}\partial\bar\partial\varphi_{t,\beta}))\Big)\Big|\\&\le C\epsilon_0\|\Xi_{t,\beta}\|,\end{aligned}$$

which implies that

$$\|\theta_{t,\beta}-\langle\Xi_{t,\beta},\Theta\rangle\|_{L^2(\omega_g)}\le C\epsilon_0\|\Xi_{t,\beta}\|$$

by the eigenvalue decomposition of Δ_g and the normalization condition (2.6). Thus, the inequality (2.7) is proved.

Since $\{\theta_0,\cdots,\theta_d\}$ is an orthonormal basis of $\mathcal{H}_g$, we have

$$\|\Xi_{t,\beta}\|^2=\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_g^n\le C\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_{t,\beta}^n, \tag{2.8}$$

where we used the fact that $\|\varphi\|_{W^{2,k+4}(M)}\le C\epsilon_0$ when t small by Lemma 2.1. The assumption $\Phi(t,\beta)=0$ together with (2.8) and (2.7) implies that

$$\begin{aligned}\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_{t,\beta}^n&=\int_M\Big(\langle\Xi_{t,\beta},\Theta\rangle-\theta_{t,\beta}\Big)\langle\Xi_{t,\beta},\Theta\rangle\,\omega_{t,\beta}^n\\&\le C\epsilon_0\|\Xi_{t,\beta}\|\cdot\|\langle\Xi_{t,\beta},\Theta\rangle\|_{L^2(\omega_g)}\\&\le C\epsilon_0\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_{t,\beta}^n.\end{aligned}$$

Thus, if ϵ_0 is small enough we have $\Xi_{t,\beta}=0$. The lemma is proved. □

Thus, the first part of Theorem 1.1 and Corollary 1 follow directly from Lemma 2.2. □

Observe that we can expand the function $\Phi(t,\beta)$ with respect to t at $t=0$:

$$\Phi(t,\beta)=a_1(\beta)t+a_2(\beta)t^2+a_3(\beta)t^3+\cdots+a_m(\beta)t^m+O(t^{m+1}),$$

where $a_j(\beta)$ are the coefficients of t^j. We want to ask what kind of Kähler metric exists if we only assume the first several terms of $a_i(\beta)$ vanish.

Corollary 2.1. *Let ω_g be a constant scalar curvature metric. There are two constants $\epsilon,C>0$ such that for any harmonic form $\beta\in\mathcal{H}^{1,1}(M)$ with unit norm and*

$$a_1(\beta)=a_2(\beta)=\cdots=a_m(\beta)=0, \tag{2.9}$$

M admits a Kähler metric $\omega_{t,\beta}\in[\omega_t+t\beta]$ for $t\in(0,\epsilon_0)$ satisfying

$$\|\mathbf{s}(\omega_{t,\beta})-c_0(t)\|_{C^k(M)}\le Ct^{\frac{m+1}{2}}. \tag{2.10}$$

Proof. We follow the notations in Lemma 2.2. By the assumption (2.9), there are two constants $\epsilon_0,C>0$ such that for any $t\in(0,\epsilon_0)$ we have

$$|\Phi(t,\beta)|\le Ct^{m+1}. \tag{2.11}$$

By equality (2.5) and (2.7) we have

$$\begin{aligned}&\Big|\Phi(t,\beta)-\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_{t,\beta}^n\Big|\\ \le{}& C\epsilon_0\|\Xi_{t,\beta}\|\cdot\|\langle\Xi_{t,\beta},\Theta\rangle\|_{L^2(\omega_g)}\le C\epsilon_0\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_{t,\beta}^n,\end{aligned}$$

where we used (2.8) in the last inequality. Thus, there is a constant $\epsilon_0>0$ such that for any $t\in(0,\epsilon_0)$ we have

$$\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_{t,\beta}^n\le C\cdot\Phi(t,\beta)\le C\cdot t^{m+1},$$

and hence

$$\sum_{i=1}^d c_i(t)^2=\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_g^n\le C\int_M\langle\Xi_{t,\beta},\Theta\rangle^2\omega_{t,\beta}^n\le Ct^{m+1}. \tag{2.12}$$

This implies that for each i when t is small, $|c_i(t)|\le Ct^{\frac{m+1}{2}}$. Since $\omega_{t,\beta}$ is a solution of (2.1), we have

$$\|\mathbf{s}(\omega_{t,\beta})-c_0(t)\|_{C^k(M)}=\|\sum_{i=1}^d c_i(t)\theta_i\|_{C^k(M)}\le Ct^{\frac{m+1}{2}}.$$

The corollary is proved. □

Now we want to compute the coefficients of t in the expansion of the function Φ. Let ω_g be a constant scalar curvature metric on M and $(\varphi_{t,\beta},\tilde{\Xi}_{t,\beta})$ the solution of (2.1). Since the operator

$$\mathbb{L}_g:\mathcal{H}^{\perp}_{g,k+4}\to\mathcal{H}^{\perp}_{g,k}$$

is self-adjoint and invertible, we denote by $\mathbb{G}_g=\mathbb{L}_g^{-1}$ the inverse operator of $\mathbb{L}_g$. Without loss of generality, we can assume that β is traceless with respect to the metric g. Otherwise, we can consider the metric $(1+t\cdot\mathrm{tr}_g\beta)\omega_g$ which still has constant scalar curvature. Let $\mathcal{H}^{1,1}_0(M)$ be the space of traceless harmonic (1,1) form with respect to the metric g on M. Computing the first derivative of $S(t):=s(\omega_{t,\beta})-\langle\tilde{\Xi}_{t,\beta},\tilde{\Theta}\rangle$ with respect to t, we have

Lemma 2.3. *For $\beta\in\mathcal{H}^{(1,1)}_0$, we have the following:*

$$\langle\tilde{\Xi}'(0),\tilde{\Theta}\rangle=-\tilde{\Pi}_g(R_{i\bar{j}}\beta_{j\bar{i}}),$$

$$\varphi'(0)=-\mathbb{G}_g\tilde{\Pi}^{\perp}_g(R_{i\bar{j}}\beta_{j\bar{i}}),$$

$$c_0'(0)=\frac{1}{V_g}\int_M R_{i\bar{j}}\beta_{j\bar{i}}\omega_g^n,$$

where we write $f'(t)=\frac{\partial f}{\partial t}$ for simplicity.

Proof. Since $S(t)=0$ for $t\in(0,\epsilon_0)$, we have

$$0=S'(t)=-\Delta_t^2\varphi'(t)-R_{i\bar{j}}(t)\varphi'_{j\bar{i}}(t)-R_{i\bar{j}}(t)\beta_{j\bar{i}}-\langle\tilde{\Xi}'_{t,\beta}(t),\tilde{\Theta}\rangle. \tag{2.13}$$

Projecting to the space $\mathcal{H}_g$ when $t=0$, we have

$$0=\tilde{\Pi}_g(S'(t))(0)=-\tilde{\Pi}_g(R_{i\bar{j}}\beta_{j\bar{i}})-\langle\tilde{\Xi}'_{t,\beta}(0),\tilde{\Theta}\rangle,$$

which implies that

$$\langle\tilde{\Xi}'_{t,\beta}(0),\tilde{\Theta}\rangle=-\tilde{\Pi}_g(R_{i\bar{j}}\beta_{j\bar{i}}).$$

On the other hand, we project (2.13) to the space $\mathcal{H}^{\perp}_{g,k}$ and we have

$$0=\tilde{\Pi}^{\perp}_g(S'(t))(0)=-\mathbb{L}_g\varphi'(0)-\tilde{\Pi}^{\perp}_g(R_{i\bar{j}}\beta_{j\bar{i}}).$$

This together with $\varphi'(0)\in\mathcal{H}^{\perp}_{g,k+4}$ implies that

$$\varphi'(0)=-\mathbb{G}_g\tilde{\Pi}^{\perp}_g(R_{i\bar{j}}\beta_{j\bar{i}}).$$

Now we calculate $c_0'(t)$. Note that $c_0(t)$ only depends on the Kähler class $[\omega_g+t\beta]$, we compute it using the metric $\omega_t=\omega_g+t\beta$. Since $\frac{\partial}{\partial t}\omega_t^n\Big|_{t=0}=\mathrm{tr}_{\omega_g}\beta\omega_g^n=0$, we have $V_t'=0$ and

$$c_0'(0)=\frac{1}{V_g}\int_M\frac{\partial}{\partial t}s(\omega_t)\Big|_{t=0}\omega_g^n=\frac{1}{V_g}\int_M R_{i\bar{j}}\beta_{j\bar{i}}\omega_g^n.$$

The lemma is proved. $\square$

Corollary 2.2. *If $\beta\in\mathcal{H}_0^{1,1}(M)$, then the function Φ can be expanded as*

$$\Phi(t,\beta)=t^2\int_M(\Pi_g(R_{i\bar{j}}\beta_{j\bar{i}}))^2+O(t^3). \tag{2.14}$$

Proof. Since $\theta_{t,\beta}(0)=0$ and $\Xi_{t,\beta}(0)=0$, we have $\Phi'(0)=0$. Direct calculation shows

$$\Phi''(0)=\int_M 2\theta'_{t,\beta}(0)\langle\Xi'_{t,\beta}(0),\Theta\rangle\omega_g^n. \tag{2.15}$$

Taking the derivative with respect to t, we have

$$\sqrt{-1}\bar{\partial}\theta'_{t,\beta}(0)=\left(i_{X'_{t,\beta}}\omega_{t,\beta}+i_{X_{t,\beta}}\omega'_{t,\beta}\right)\Big|_{t=0}=\sum_{k=1}^d c'_k(0)i_{X_k}\omega_g=\sqrt{-1}\sum_{k=1}^d c'_k(0)\bar{\partial}\theta_k,$$

which implies that

$$\theta'_{t,\beta}(0)=\sum_{k=1}^d c'_k(0)\theta_k=\langle\Xi'_{t,\beta}(0),\Theta\rangle. \tag{2.16}$$

This together with the equality (2.15) and Lemma 2.3 implies that

$$\Phi''(0)=2\int_M(\langle\Xi'_{t,\beta}(0),\Theta\rangle)^2\omega_g^n=2\int_M(\Pi_g(R_{i\bar{j}}\beta_{j\bar{i}}))^2.$$

The corollary is proved. □

If ω_g is a Kähler-Einstein metric, the first term of the right hand side of (2.14) automatically vanishes. In this case, it is not difficult to expand $\Phi(t,\beta)$ for more terms.

Lemma 2.4. *If $\beta\in\mathcal{H}_0^{1,1}(M)$ and satisfies $R_{i\bar{j}}\beta_{j\bar{i}}=0$, then we have*

$$\langle\tilde{\Xi}''_{t,\beta}(0),\tilde{\Theta}\rangle=\tilde{\Pi}_g(2R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}}),$$

$$\varphi''(0)=\mathbb{G}_g\tilde{\Pi}_g^{\perp}(2R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}}),$$

$$c''_0(0)=\frac{1}{V_g}\int_M 2R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}}\omega_g^n.$$

Proof. Following the proof of Lemma 2.3 we have

$$\begin{aligned}S''(t)=&(\beta_{i\bar{j}}+\varphi'_{i\bar{j}})(\Delta_t\varphi')_{j\bar{i}}+\Delta_t((\beta_{i\bar{j}}+\varphi'_{i\bar{j}})\varphi'_{j\bar{i}})-\Delta_t^2\varphi''\\&+(\Delta_t\varphi')_{i\bar{j}}\varphi'_{j\bar{i}}-R_{i\bar{j}}(t)\varphi''_{j\bar{i}}+2R_{i\bar{j}}\varphi'_{j\bar{k}}(\beta+\nabla^2\varphi')_{k\bar{i}}+(\Delta\varphi')_{i\bar{j}}\beta_{j\bar{i}}\\&+R_{i\bar{j}}\beta_{j\bar{k}}(\beta+\nabla^2D_t\varphi)_{k\bar{i}}+R_{i\bar{j}}\beta_{k\bar{i}}(\beta+\nabla^2D_t\varphi)_{j\bar{k}}-\langle\tilde{\Xi}''_{t,\beta},\tilde{\Theta}\rangle.\end{aligned}$$

Thus, projecting to $\mathcal{H}_g$ and $\mathcal{H}_{g,k}^{\perp}$ we have

$$0=\tilde{\Pi}_g(S''(t))(0)=\tilde{\Pi}_g(2R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}})-\langle\tilde{\Xi}''_{t,\beta}(0),\tilde{\Theta}\rangle,$$
$$0=\tilde{\Pi}_g^{\perp}(S''(t))(0)=-\mathbb{L}_g\varphi''+\tilde{\Pi}_g^{\perp}(2R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}}).$$

Moreover, we calculate $c_0''(0)$ as in the proof of Lemma 2.3

$$c_0''(0)=\frac{1}{V_g}\int_M R(\omega_g+t\beta)''\Big|_{t=0}\omega_g^n=\frac{1}{V_g}\int_M 2R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}}\omega_g^n.$$

The lemma is proved. □

Corollary 2.3. *If ω_g is a Kähler-Einstein metric and $\beta\in\mathcal{H}_0^{1,1}(M)$, then we have*

$$\Phi(t,\beta)=t^4\int_M(\Pi_g(R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}}))^2\omega_g^n+O(t^5).$$

Proof. By Lemma 2.3, we have

$$\langle\tilde{\Xi}'_{t,\beta}(0),\tilde{\Theta}\rangle=\varphi'(0)=c_0'(0)=0.$$

Thus, Eq. (2.16) implies that $\theta'_{t,\beta}(0)=0$ and by direct calculation we have

$$\Phi_t'''(0)=3\int_M\left(\theta''_{t,\beta}(0)\langle\Xi'_{t,\beta}(0),\Theta\rangle+\theta'_{t,\beta}(0)\langle\Xi''_{t,\beta}(0),\Theta\rangle\right)\omega_g^n=0.$$

On the other hand, by Lemma 2.4 we have

$$\sqrt{-1}\bar{\partial}\theta''_{t,\beta}(0)=i_{X''_{t,\beta}(0)}\omega_g=\sum_{k=1}^d c_k''(0)i_{X_k}\omega_g=\sqrt{-1}\bar{\partial}\Big(\sum_{k=1}^d c_k''(0)\theta_k\Big), \tag{2.17}$$

which implies that

$$\theta''_{t,\beta}(0)=\langle\Xi''_{t,\beta}(0),\Theta\rangle. \tag{2.18}$$

Thus, by tedious calculation we have

$$\Phi_t^{(4)}(0)=6\int_M\theta''_{t,\beta}(0)\langle\Xi''_{t,\beta}(0),\Theta\rangle\omega_g^n=24\int_M(\Pi_g(R_{i\bar{j}}\beta_{j\bar{k}}\beta_{k\bar{i}}))^2\omega_g^n.$$

The corollary is proved. □

2.2 Varying complex structures

In this section, we will consider the deformation of constant scalar curvature metrics when the complex structure varies. Let (M,J,g,ω_g) be a compact Kähler manifold (M,J) with a Kähler metric g and the associate Kähler form ω_g. Let J_t be a smooth family of complex structures with $J_0=J$. By Kodaira's theorem in [16] there exists a smooth family of Kähler metric g_t with $g_0=g$ which is compatible with the complex structure J_t for small t. Let ω_t be the associate Kähler form of g_t with respect to the complex structure J_t. The triple (J_t,g_t,ω_t) is called a complex deformation of (J,g,ω_g). Given a complex deformation (J_t,g_t,ω_t), we want to know whether there exists a constant scalar curvature metric in the Kähler class $([\omega_t],J_t)$ if we assume that ω_g is a constant scalar curvature metric on (M,J).

Since g is a constant scalar curvature metric, the identity component G of the isometry group of (M,g) is a maximal compact subgroup of $\mathrm{Aut}(M,g)$ by Lichnerowicz-Matsushima theorem. In general the action of the group G may not extend to (M,J_t). We follow the idea of Rollin-Simanca-Tipler in [19] to assume that a compact connected subgroup G' of G can extend to (M,J_t) and G' acts holomorphically on the complex deformation (J_t,g_t,ω_t). We denote by $\mathcal{B}_{G'}$ the space of complex deformations (J_t,g_t,ω_t) which allow the holomorphic action of G'. We denote by $W^{2,k}_{G'}(M)$ the subspace of G'-invariant functions in $W^{2,k}(M)$ and $\mathcal{U}$ a neighborhood of the origin in $W^{2,k}_{G'}(M)$. For any $\varphi\in\mathcal{U}$, we compute the expansion of the scalar curvature of the metric $\omega_{t,\varphi}=\omega_t+\sqrt{-1}\partial_t\bar{\partial}_t\varphi$ at $(t,\varphi)=(0,0)$:

Lemma 2.5. *Suppose that $\partial\omega_t/\partial t=\eta_t$. We have*

$$\mathbf{s}(\omega_{t,\varphi})=\mathbf{s}(\omega_g)-\mathbb{L}_g\varphi-t\Big(\Delta_g\mathrm{tr}_{\omega_g}(\eta+S(\varphi))+R_{i\bar{j}}(\eta+S(\varphi))_{j\bar{i}}+\mathrm{tr}_g(S\log\det g)\Big)+Q,$$

where Q collects all the higher order terms and the operator S is given by $S=\frac{1}{2}dJ_t'(0)df$.

Proof. For any smooth function f, we define the operator

$$S_t(f):=\frac{\partial}{\partial t}\sqrt{-1}\partial_t\bar{\partial}_t(f)=\frac{1}{2}dJ_t'df,$$

where we used the equality $\sqrt{-1}\partial_t\bar{\partial}_t=\frac{1}{2}dJ_td$. Note that

$$\frac{\partial}{\partial t}\omega_{t,\varphi}=\eta_t+S_t(\varphi),\quad D_\varphi\omega_{t,\varphi}(\psi)=\sqrt{-1}\partial_t\bar{\partial}_t\psi.$$

The derivatives of the scalar curvature are given by

$$\frac{\partial}{\partial t}\mathbf{s}(\omega_{t,I,\varphi})=-(\eta_{i\bar{j}}+S_{t,i\bar{j}}(\varphi))R_{j\bar{i}}-g^{i\bar{j}}S_{t,i\bar{j}}(\log\det g)-\Delta_t\mathrm{tr}_{\omega_{t,\varphi}}(\eta+S_t(\varphi)),$$

$$D_\varphi\mathbf{s}(\omega_{t,I,\varphi})(\psi)=-R_{i\bar{j}}\psi_{j\bar{i}}-\Delta_t^2\psi.$$

Thus, the lemma follows directly. □

As in Section 2, we define $\mathfrak{g}$(resp. $\mathfrak{g}'$) the Lie algebra of G(resp. G'), and $\mathfrak{g}_0$ (resp. $\mathfrak{g}_0'$) the ideal of Killing vector fields with zeros in $\mathfrak{g}$ (resp. $\mathfrak{g}'$). The center of $\mathfrak{g}_0$(resp. $\mathfrak{g}_0'$) is denoted by $\mathfrak{z}_0$(resp. $\mathfrak{z}_0'$). Each element of $\mathfrak{z}_0$(resp. $\mathfrak{z}_0'$) is of the form $J\nabla f$ for a G(resp. G')-invariant, real-valued function f. Let $\mathcal{H}_g^{\mathfrak{g}_0'}$ (resp. $\mathcal{H}_g^{\mathfrak{z}_0'}$) the space of holomorphic potentials of the Killing vector fields in $\mathfrak{g}_0'$ (resp. $\mathfrak{z}_0'$) and it is easy to see that the space $\mathcal{H}_g^{\mathfrak{z}_0'}$ is identified to the G'-invariant holomorphic potentials of $\mathcal{H}_g^{\mathfrak{g}_0'}$. Using the L^2 inner product induced by g, the space $W_{G'}^{2,k}(M)$ has the orthogonal decomposition

$$W_{G'}^{2,k}(M)=\mathcal{H}_g\oplus\mathcal{H}_{g,k}^{\perp},$$

where $\mathcal{H}_g=\mathbb{R}\oplus\mathcal{H}_g^{\mathfrak{z}_0'}$ and we assume $\mathcal{H}_g$ is spanned by an orthonormal basis $\{\theta_0,\theta_1,\cdots,\theta_d\}$ where $\theta_0=1$ with respect to the induced L^2 norm of the metric g. Let $\tilde{\Pi}_g$ and $\tilde{\Pi}_g^{\perp}$ be the L^2-orthogonal projection onto $\mathcal{H}_g$ and $\mathcal{H}_{g,k}^{\perp}$ respectively. With these notations, we have the result:

Theorem 2.2. *Let g be a constant scalar curvature metric on M with*

$$\ker\mathbb{L}_g\cap W_{G'}^{2,k}\subset\mathbb{R}\oplus\mathcal{H}_g^{\mathfrak{z}_0'}. \tag{2.19}$$

For any $(J_t,g_t,\omega_t)\in\mathcal{B}_{G'}$, there is a constant $\epsilon_0>0$ and a smooth function $\Psi:\mathcal{B}_{G'}\to\mathbb{R}$ such that if $\Psi(J_t,g_t,\omega_t)=0$ for some $t\in(0,\epsilon_0)$, then M admits a G'-invariant constant scalar curvature metric in $[\omega_t]$ with respect to J_t.

Proof. First, we want to find the solution $(\varphi,\tilde{\Xi})\in\mathcal{H}_{g,k+4}^{\perp}\times\mathbb{R}^{d+1}$ of the equation

$$\mathbf{s}(\omega_{t,\varphi})=\langle\tilde{\Xi},\tilde{\Theta}\rangle, \tag{2.20}$$

where $\tilde{\Theta}=(\theta_0,\theta_1,\cdots,\theta_d)$. As in the proof of Lemma 2.1, we can use the implicit function theorem and Lemma 2.5 to show that

Lemma 2.6. *Suppose that the condition* (2.19) *holds. For any $(J_t,g_t,\omega_t)\in\mathcal{B}_{G'}$, there exist $C,\epsilon_0>0$ such that for all $t\in(0,\epsilon_0)$, there is a solution $(\varphi_t,\tilde{\Xi}_t)\in\mathcal{H}_{g,k+4}^{\perp}\times\mathbb{R}^{d+1}$ which satisfies Eq.* (2.20) *and*

$$\|\varphi_t\|_{W^{2,k+4}(M)}\le C\epsilon_0,\quad \|\tilde{\Xi}_t\|\le C\epsilon_0. \tag{2.21}$$

Proof. The linearization of the operator $\tilde{\Pi}_g^{\perp}\mathbf{s}(\omega_{t,\varphi}):(-\epsilon,\epsilon)\times\mathcal{H}_{g,k+4}^{\perp}\to\mathbb{R}$ at $(t,\varphi)=(0,0)$ is given by

$$D_\varphi\tilde{\Pi}_g^{\perp}\mathbf{s}(\omega_{t,\varphi})|_{(0,0)}(\psi)=-\mathbb{L}_g\psi:\mathcal{H}_{g,k+4}^{\perp}\to W_{G'}^{2,k},$$

which is invertible from $\mathcal{H}_{k+4}^{\perp}$ to $\mathcal{H}_{g,k}^{\perp}$ if and only if the condition (2.19) holds. Thus, the lemma follows directly from the implicit function theorem. □

Let $\xi_i (1 \le i \le d)$ be the Killing vector fields in $\mathfrak{z}_0$ with the holomorphic potentials $\theta_i (1 \le i \le d)$. Since $(J_t, g_t, \omega_t) \in \mathcal{B}_{G'}$, the vector fields $X_i^t := J_t \xi_i + \sqrt{-1}\xi_i$ are holomorphic on (M, J_t) and the holomorphic potential of X_i^t with respect to ω_{t,φ_t} is given by a real-valued function θ_i^t satisfying

$$i_{X_i^t}\omega_{t,\varphi_t} = \sqrt{-1}\bar{\partial}_t \theta_i^t, \quad \int_M \theta_i^t \omega_{t,\varphi_t}^n = 0. \tag{2.22}$$

For the vector $\tilde{\Xi}_t = (c_0(t), c_1(t), \cdots, c_d(t)) \in \mathbb{R}^{d+1}$ obtained in Lemma 2.6, we define the holomorphic vector field

$$X_t = \sum_{i=1}^d c_i(t) X_i^t \in \mathfrak{h}_0(M, J_t).$$

Let θ_t be the holomorphic potential of X_t with respect to ω_{t,φ_t} and

$$\Theta = (\theta_1, \cdots, \theta_d), \quad \Xi_t = (c_1(t), \cdots, c_d(t)),$$

where $c_i(t)$ are the entries of $\tilde{\Xi}_t$.

Lemma 2.7. *If $(J_t, g_t, \omega_t) \in \mathcal{B}_{G'}$ satisfies*

$$\|J_t - J_0\|_{C^1(M)} \le C\epsilon_0, \quad t \in (0, \epsilon_0), \tag{2.23}$$

then there is a constant $C_1 > 0$ such that for all $t \in (0, \epsilon_0)$ we have

$$\|\theta_t - \langle \Xi_t, \Theta \rangle\|_{L^2(\omega_g)} \le C_1 \epsilon_0 \|\Xi_t\|. \tag{2.24}$$

Proof. Define the vector field $\hat{X}_t = \sum_{k=1}^d c_k(t) X_i \in \mathfrak{h}_0(M, J)$ where $c_k(t)$ is given by Lemma 2.6. By definition, we have

$$i_{\hat{X}_t}\omega_g = \sqrt{-1}\bar{\partial}\langle \Xi_t, \Theta \rangle, \quad i_{X_t}\omega_{t,\varphi_t} = \sqrt{-1}\bar{\partial}_t \theta_t,$$

where $\bar{\partial}$ denotes the operator on (M, J). We want to compute the difference of the two functions θ_t and $\langle \Xi_t, \Theta \rangle$:

$$\begin{aligned} \sqrt{-1}\bar{\partial}(\langle \Xi_t, \Theta \rangle - \theta_t) &= i_{\hat{X}_t}\omega_g - i_{X_t}\omega_{t,\varphi_t} + \sqrt{-1}(\bar{\partial}_t - \bar{\partial})\theta_t \\ &= \sum_{k=1}^d c_k(t)(i_{X_k}\omega_g - i_{X_k^t}\omega_{t,\varphi_t}) + \sqrt{-1}(\bar{\partial}_t - \bar{\partial})\theta_t. \end{aligned} \tag{2.25}$$

Note that the estimate $\|\omega_g - \omega_{t,\varphi_t}\|_{W^{2,k+2}(M)} \le C\epsilon_0$ obtained in Lemma 2.6 implies

$$\begin{aligned} &\|\partial\Big(i_{X_k}\omega_g - i_{X_k^t}\omega_t\Big)\|_{C^0} \\ = &\|i_{\partial(X_k - X_k^t)}\omega_g + i_{\partial X_k^t}(\omega_g - \omega_{t,\varphi_t}) + i_{X_k^t}\partial(\omega_g - \omega_{t,\varphi_t})\|_{C^0} \le C\epsilon_0, \end{aligned} \tag{2.26}$$

where we used the estimates

$$\|\partial(X_k^t - X_k)\|_{C^0} = \|\partial(J_t - J_0)\xi_k\|_{C^0} \le C\epsilon_0, \quad t \in (0,\epsilon_0).$$

Now we estimate θ_t. Note that we have

$$\Delta_{\omega_{t,\varphi_t}}\theta_t = \sqrt{-1}\partial_t(i_{X_t}\omega_{t,\varphi_t}) = \sqrt{-1}\sum_{k=1}^{d} c_k(t)\partial_t(i_{X_t}\omega_{t,\varphi_t})$$

and $\|\omega_{t,\varphi_t} - \omega_g\|_{C^{2,\alpha}} \le C\epsilon_0$ if we choose k sufficiently large in Lemma 2.6, there is a constant $C > 0$ independent of t such that

$$\|\theta_t\|_{C^2(M,\omega_g)} \le C\|\Xi_t\|. \tag{2.27}$$

Therefore, we have

$$\left|\partial(\bar\partial_t - \partial)\theta_t\right| = \frac{1}{2}\left|\partial(J_t - J_0)d\theta_t\right| \le C\epsilon_0 \cdot \|\Xi_t\|, \quad t \in (0,\epsilon_0), \tag{2.28}$$

where we used the equality $\bar\partial_t f = \frac{1}{2}(df - \sqrt{-1}J_t df)$ and the inequality (2.27). Combining the estimates (2.25), (2.26) and (2.28), we have

$$\left|\Delta_g(\langle \Xi_t, \Theta\rangle - \theta_t)\right| \le C\epsilon_0 \cdot \|\Xi_t\|.$$

This together the eigenvalue decomposition and the normalization condition (2.22) gives (2.24). The lemma is proved. □

Now we define the function $\Psi: \mathcal{B}_G \to \mathbb{R}$ by

$$\Psi(J_t, g_t, \omega_t) = \int_M X_t h_{t,\varphi_t}\omega_{t,\varphi_t}^n = \int_M \theta_t(\mathbf{s}(\omega_{t,\varphi}) - c_0(t))\omega_{t,\varphi_t}^n,$$

where $c_0(t)$ is the average of $s(\omega_{t,\varphi_t})$ and h_{t,φ_t} is given by $s(\omega_{t,\varphi_t}) - c_0(t) = \Delta_{\omega_{t,\varphi_t}} h_{t,\varphi_t}$. As in Section 2, we have the following result whose proof is omitted.

Lemma 2.8. *There exists $\epsilon_0 > 0$ such that if the complex deformation $(J_t, g_t, \omega_t) \in \mathcal{B}_G$ satisfies $\Psi(J_t, g_t, \omega_t) = 0$ for some $t \in (0,\epsilon_0)$, then ω_{t,φ_t} is a constant scalar curvature metric with respect to the complex structure J_t.*

Theorem 2.2 then follows from the above results. □

3 Deformation of Kähler-Ricci solitons

Let (M,J) be a compact Kähler manifold with a Kähler Ricci soliton g_{KS} with respect to the holomorphic vector field X:

$$Ric(\omega_{KS})-\omega_{KS}=\sqrt{-1}\partial\bar{\partial}\theta_X,$$

where θ_X is the holomorphic potential of X with respect to ω_{KS}. We would like to ask whether we can perturb the Kähler Ricci soliton under complex deformation of the complex structure. Inspired by the discussion before, for any Kähler class $[\omega_g]$we consider the metric $\omega_\varphi\in[\omega_g]$ satisfying the equation of extremal solitons

$$\mathbf{s}(\omega_\varphi)-\underline{\mathbf{s}}=\Delta_\varphi\theta_X(\omega_\varphi). \tag{3.1}$$

By the $\partial\bar{\partial}$-Lemma, we can easily check that

Lemma 3.1. *If $\omega_g\in 2\pi c_1(M)$ satisfies the equation (3.1) with respect to a holomorphic vector field X, then ω_g is a Kähler-Ricci soliton with respect to X.*

By the equation (3.1), if $[\omega_g]$ admits an extremal soliton ω_φ and the Futaki invariant vanishes on $[\omega_g]$, then ω_φ must be a constant scalar curvature metric. In fact,

$$f(X,[\omega_0])=\int_M\theta_X(\varphi)\Delta_\varphi\theta_X(\varphi)\omega_\varphi^n=0$$

implies that $\theta_X(\varphi)$ is a constant.

Theorem 3.1. *If ω_g be a Kähler Ricci soliton with respect to X on M, then for any $\beta\in\mathcal{H}^{1,1}(M)$ there is an extremal soliton in the Kähler class $[\omega_0+t\beta]$ for small t.*

Proof. We follow Lebrun-Simanca's arguments in [11,12]. Let g be a Kähler-Ricci soliton. By Theorem A in the appendix of [22] the identity component G of the isometry group of (M,g) is a maximal compact subgroup of the automorphism group $\mathrm{Aut}(M)$. As in previous sections, we let $W_G^{2,k}$ be the real k-th Sobolev space of G-invariant real-valued functions in $W^{2,k}$. Let $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{z}\subset\mathfrak{g}$ denote the center of $\mathfrak{g}$. We denote by $\mathfrak{g}_0$ the ideal of Killing vector fields with zeros and $\mathfrak{z}_0=\mathfrak{z}\cap\mathfrak{g}_0$. By Lemma A.2 in the appendix of [22], each element of $\mathfrak{z}_0$ is of the form $J\nabla f$, where f is a G-invariant real-valued function satisfying the equation

$$\mathcal{L}_g(f)=f_{\bar{i}\bar{j}}dz^{\bar{i}}\otimes dz^{\bar{j}}=0.$$

We choose a basis $\{\xi_1,\cdots,\xi_d\}$ of $\mathfrak{z}_0$ such that the functionals $\{\theta_0,\theta_1,\cdots,\theta_d\}$, where $\theta_0=1$ and $\theta_i(1\le i\le d)$ is the holomorphic potential of the holomorphic vector fields $X_i=J\xi_i+\sqrt{-1}\xi_i$, are orthonormal with respect to the L^2 inner product

$$\langle f,g\rangle_{L^2(\omega_g)}=\frac{1}{V_g}\int_M fge^{\theta_X}\omega_g^n,\quad f,g\in C^\infty(M,\mathbb{R}),$$

where V_g is the volume of (M,g). Using this product, the space $W^{2,k}_G$ has a decomposition $W^{2,k}_G=\mathcal{H}_g\oplus\mathcal{H}^{\perp}_{g,k}$, where $\mathcal{H}_g$ is spanned by the set $\{\theta_0,\theta_1,\cdots,\theta_d\}$ over $\mathbb{R}$. We define the associate project operator Π_g and $\Pi^{\perp}_g$, and we can assume that $X_1=X$ which defines the Kähler-Ricci soliton ω_g.

Now we consider the equation for $\varphi\in\mathcal{U}$:

$$S(t,\varphi):=\Pi^{\perp}_g\Pi^{\perp}_{\varphi}G_{\varphi}(\mathbf{s}(\omega_{t,\varphi})-\underline{\mathbf{s}}(t))=0,$$

where G_φ is the Green operator with respect to the metric $\omega_{t,\varphi}$. If $\mathcal{U}$ is small enough, $S(t,\varphi)=0$ if and only if $\omega_{t,\varphi}$ is an extremal soliton. We calculate the variation of $S(t,\varphi)$ at $(t,\varphi)=(0,0)$:

$$\begin{aligned}&D_{\varphi}S(t,\varphi)|_{(0,0)}(\psi)\\=&-\Pi^{\perp}_g(D_{\varphi}\Pi_{\varphi})|_{(0,0)}G_g(\mathbf{s}(\omega_g)-\underline{\mathbf{s}})+\Pi^{\perp}_g D_{\varphi}(G_{\varphi}(\mathbf{s}(\omega_{t,\varphi})-\underline{\mathbf{s}}))|_{(0,0)}.\end{aligned}\tag{3.2}$$

Since g is a Kähler Ricci soliton, we have $G_g(\mathbf{s}(\omega_g)-\underline{\mathbf{s}})=\theta_X$. Note that

$$\Pi_{\varphi}\theta_X=\sum_{i=0}^{d}\langle\theta_{i,\varphi},\theta_X\rangle_{L^2(\omega_{t,\varphi})}\theta_{i,\varphi},$$

where $\theta_{i,\varphi}$ is an orthonormal basis of $\mathcal{H}_g$. Now we choose the functions

$$\theta_{0,\varphi}=1,\quad \theta_{i,\varphi}=\frac{\tilde{\theta}_{i,\varphi}}{\|\tilde{\theta}_{i,\varphi}\|_{L^2(\omega_\varphi)}},\quad 1\le i\le d,$$

where $\tilde{\theta}_{i,\varphi}$ are defined by the equalities $i_{X_i}\omega_{t,\varphi}=\sqrt{-1}\bar{\partial}\tilde{\theta}_{i,\varphi}$ such that $\{\theta_{0,\varphi},\cdots,\theta_{d,\varphi}\}$ forms an orthonormal basis of $\mathcal{H}_\varphi$, which is the space defined similar to $\mathcal{H}_g$ using the metric ω_φ. Thus, we have

$$\begin{aligned}&-\Pi^{\perp}_g(D_{\varphi}\Pi_{\varphi})|_{(0,0)}G_g(\mathbf{s}(\omega_g)-\underline{\mathbf{s}})=-\Pi^{\perp}_g(D_{\varphi}\Pi_{\varphi})|_{(0,0)}\theta_X\\=&-\langle\frac{\theta_X}{\|\theta_X\|_{L^2}},\theta_X\rangle_{L^2(\omega_g)}\Pi^{\perp}_g\frac{1}{\|\theta_X\|_{L^2}}D_{\varphi}\tilde{\theta}_{1,\varphi}|_{(0,0)}=-\Pi^{\perp}_g D_{\varphi}\tilde{\theta}_{1,\varphi}|_{(0,0)}.\end{aligned}$$

By the definition of $\tilde{\theta}_{1,\varphi}$, we have

$$i_X D_{\varphi}\omega_{t,\varphi}|_{(0,0)}=\sqrt{-1}\bar{\partial}D_{\varphi}\tilde{\theta}_{1,\varphi},$$

which implies that $X(\psi)=D_{\varphi}\tilde{\theta}_{1,\varphi}|_{(0,0)}(\psi)$. Combining the above equalities, we have

$$-\Pi^{\perp}_g(D_{\varphi}\Pi_{\varphi})|_{(0,0)}G_g(\mathbf{s}(\omega_g)-\underline{\mathbf{s}})=-\Pi^{\perp}_g X(\psi).\tag{3.3}$$

Now we calculate the second term of the right hand side of (3.2). Let $A_\varphi=G_\varphi(\mathbf{s}(\omega_{t,\varphi})-\underline{\mathbf{s}})$, we have

$$\Delta_{\varphi}A_{\varphi}=\mathbf{s}(\omega_{t,\varphi})-\underline{\mathbf{s}}.$$

Differentiating this equation with respect to φ at $(t,\varphi)=(0,0)$, we have

$$-\psi_{i\bar{j}}\theta_{X,j\bar{i}}+\Delta_g D_\varphi A_\varphi|_{(0,0)}=-\Delta_g^2\psi-R_{i\bar{j}}\psi_{j\bar{i}}.$$

Combining this with (3.2) we have

$$\Pi_g^\perp D_\varphi(G_\varphi(\mathbf{s}(\omega_{t,\varphi})-\underline{\mathbf{s}}))|_{(0,0)}=-\Pi_g^\perp G_g\Big(\Delta_g^2\psi+R_{i\bar{j}}\psi_{j\bar{i}}-\psi_{i\bar{j}}\theta_{X,j\bar{i}}\Big). \tag{3.4}$$

Combining the equalities (3.2)-(3.4), we have

$$\begin{aligned}D_\varphi S(t,\varphi)|_{(0,0)}(\psi)&=-\Pi_g^\perp\Big(G_g(\Delta_g^2\psi+R_{i\bar{j}}\psi_{j\bar{i}}-\psi_{i\bar{j}}\theta_{X,j\bar{i}})+X(\psi)\Big)\\&=-\Pi_g^\perp\Big(\Delta_g\psi+\psi+X(\psi)\Big),\end{aligned}$$

here we used the assumption that g is a Kähler-Ricci soliton. Note that by Lemma 2.2 in [22] the function ψ satisfies $\Delta_g\psi+\psi+X(\psi)=0$ if and only if $\Pi_g^\perp\psi=0$ Thus, the operator

$$D_\varphi S(t,\varphi)|_{(0,0)}:\mathcal{H}^\perp_{g,k+2}\to\mathcal{H}^\perp_{g,k}$$

is invertible and by the implicit function theorem there is a solution $\varphi_t\in\mathcal{H}^\perp_{g,k+2}$ satisfies the equation $S(t,\varphi_t)=0$ when t is small. The theorem is proved. □

Remark 3.1. It is interesting to ask whether Theorem 3.1 holds for any extremal soliton g. To prove this, it suffices to show that any function ψ with

$$\psi_{ij\bar{j}\bar{i}}+\theta_{X,\bar{i}}\psi_{ik\bar{k}}=0$$

must satisfy the equation $\psi_{\bar{i}\bar{j}}=0$.

In fact, if g is an extremal soliton, we have

$$\begin{aligned}&G_g(\Delta_g^2\psi+R_{i\bar{j}}\psi_{j\bar{i}}-\psi_{i\bar{j}}\theta_{X,j\bar{i}})+X(\psi)=G_g(\Delta_g^2\psi+R_{i\bar{j}}\psi_{j\bar{i}}-\theta_{i\bar{j}}\psi_{j\bar{i}}+\Delta_g(X\psi))\\&=G_g(\Delta_g^2\psi+R_{i\bar{j}}\psi_{j\bar{i}}+\mathbf{s}_{,\bar{i}}\psi_i+\theta_{X,\bar{i}}\psi_{ik\bar{k}})=G_g(\psi_{ij\bar{j}\bar{i}}+\theta_{X,\bar{i}}\psi_{ik\bar{k}}),\end{aligned}$$

where we used the equality

$$\begin{aligned}\Delta_g(X\psi)&=\frac{1}{2}\Big((\theta_{\bar{i}}\psi_i)_{j\bar{j}}+(\theta_{\bar{i}}\psi_i)_{\bar{j}j}\Big)=\theta_{\bar{i}j}\psi_{i\bar{j}}+\theta_{\bar{i}}(\Delta\psi)_i\\&=\theta_{\bar{i}j}\psi_{i\bar{j}}+\theta_{\bar{i}}(\psi_{ik\bar{k}}-R_{i\bar{j}}\psi_j)=\theta_{\bar{i}j}\psi_{i\bar{j}}+\theta_{\bar{i}}\psi_{ik\bar{k}}+\mathbf{s}_{,\bar{i}}\psi_i.\end{aligned}$$

Here we used the extremal soliton equation in the last equality.

Next, we use the similar method in Section 2 to consider the case when the complex structure varies . Let (g,ω_g) is a Kähler-Ricci soliton on (M,J) and (J_t,g_t,ω_t) a complex deformation of (J,g,ω_g). We assume $(J_t,g_t,\omega_t)\in\mathcal{B}_G$ where G is the identity component of the isometry group of (M,g) and $\mathcal{B}_G$ denotes all the G invariant complex deformation of (J,g,ω_g). With these notations, we have the result:

Theorem 3.2. *Let (M,J,g,ω_g) be a compact Kähler manifold with a Kähler-Ricci soliton (g,ω_g). For any $(J_t,g_t,\omega_t)\in\mathcal{B}_G$, M admits a G-invariant extremal soliton in $[\omega_t]$ with respect to J_t for small t.*

Proof. The proof is more or less the same as in Theorem 3.1, and we only sketch it here. For any $(J_t,g_t,\omega_t)\in\mathcal{B}_G$, we consider the equation

$$S(t,\varphi):=\Pi_g^{\perp}\Pi_\varphi^{\perp}G_\varphi(\mathbf{s}(\omega_{t,\varphi})-\underline{\mathbf{s}}(t))=0, \tag{3.5}$$

where G_φ and $\Pi_\varphi^{\perp}$ are the operators with respect to the metric $\omega_{t,\varphi}=\omega_t+\sqrt{-1}\partial_t\bar\partial_t\varphi$. Let $\{\xi_1,\cdots,\xi_d\}$ be a basis of $\mathfrak{z}_0$. Since $(J_t,g_t,\omega_t)\in\mathcal{B}_G$, the vector fields $\{X_1^t,\cdots,X_d^t\}$ where $X_i^t=J_t\xi_i+\sqrt{-1}\xi_i$ are holomorphic vector fields on (M,J_t) and form a basis of $\mathfrak{h}_0(M,J_t)$. Let $\tilde\theta_i^t(1\le i\le d)$ be the holomorphic potentials of X_i^t with respect to $\omega_{t,\varphi}$ and we assume that the set $\{\tilde\theta_0^t,\tilde\theta_1^t,\cdots,\tilde\theta_d^t\}$ where $\tilde\theta_0^t=1$ are orthonormal and spans the space $\mathcal{H}_\varphi$. Differentiating the equation (3.5) with respect to φ, we have

$$\begin{aligned}&D_\varphi S(t,\varphi)|_{(0,0)}(\psi)\\ =&-\Pi_g^{\perp}(D_\varphi\Pi_\varphi)|_{(0,0)}G_g(\mathbf{s}(\omega_g)-\underline{\mathbf{s}})+\Pi_g^{\perp}D_\varphi(G_\varphi(\mathbf{s}(\omega_{t,\varphi})-\underline{\mathbf{s}}))|_{(0,0)}.\end{aligned}$$

Since $D_\varphi\omega_{t,\varphi}|_{(0,0)}(\psi)=\sqrt{-1}\partial\bar\partial\psi$ and $D_\varphi X^t|_{(0,0)}=0$, we still get the equality (3.3). By the same calculation as in Theorem 3.1, we have the operator

$$D_\varphi S(t,\varphi)|_{(0,0)}(\psi)=-\Pi_g^{\perp}\Big(\Delta_g\psi+\psi+X(\psi)\Big)$$

which is invertible from $\mathcal{H}_{g,k+2}^{\perp}$ to $\mathcal{H}_{g,k}^{\perp}$. The theorem is proved. □

Here we give an easy example on the existence of extremal solitons.

Example 3.1. Let $\pi:\hat M\to M$ be the blowup of $M=\mathbb{CP}^2$ at a point p. Then $\hat M$ has no Kähler-Einstein metrics but admits a Kähler-Ricci soliton in $2\pi c_1(\hat M)$. Thus, $\hat M$ admits extremal solitons in the Kähler class $2\pi c_1(\hat M)-t[E]$ for $t\in(0,\epsilon)$ where $E=\pi^{-1}(p)$ is the exceptional divisor and $\epsilon>0$ is small.

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