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Overdetermined Boundary Value Problems in $\mathbb{S}^n$

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Abstract. In this paper we use the maximum principle and the Hopf lemma to prove symmetry results to some overdetermined boundary value problems for domains in the hemisphere or star-shaped domains in $\mathbb{S}^n$. Our method is based on finding suitable P-functions as Weinberger ([26]).

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1 Introduction

In a seminal paper [21], Serrin proved that for a bounded open connected domain $\Omega\subset\mathbb{R}^n$ with sufficient regular boundary $\partial\Omega$, if there exists a solution of the following overdetermined boundary value problem

$$\begin{cases} \Delta u = n & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu}=c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where c is a constant, then Ω must be a ball and u is radially symmetric. Here ν denotes the outward unit normal of $\partial\Omega$.

The main tool of Serrin's proof is well-known as the method of moving planes, which is due to Alexandrov. Immediately after Serrin's paper, Weinberger [26] give an alternative proof of the same result, based on a Rellich-Pohozaev type identity and an interior maximum principle for a subharmonic function (In literatures, it is often referred to as P-function). Each of their proofs has its own merits. Serrin's argument applies to very

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general partial differential equations if an additional assumption $u>0$ is added, while Weinberger's argument is more elementary.

Since the works of Serrin and Weinberger, there have been numerous generalizations to overdetermined problems for general elliptic operators in $\mathbb{R}^n$, the interested readers may refer to [4–6, 8, 11–14, 17, 25] and references therein.

On the other hand, Serrin's result has been extended to the hemisphere $\mathbb{S}^n_+$ and the hyperbolic space $\mathbb{H}^n$. Precisely, Molzon [16] considered equation $\Delta u = f(x)$ where $f(x) = \cos r$ ($\cosh r$ resp.) in the case $\mathbb{S}^n_+$ ($\mathbb{H}^n$ resp.) and r is the distance function from a fixed point or $f(x)=n$. Kumaresan and Prajapat [15] considered equation $\Delta u + f(u) = 0$ in $\Omega\subset\mathbb{S}^n_+$ or $\mathbb{H}^n$, where f is a C^1 function. They proved that if $\Delta u + f(u) = 0$ with the boundary condition $u=0$ and $\frac{\partial u}{\partial \nu} = constant$ admits a *positive* solution, then Ω is a geodesic ball and u is radially symmetric. They used Serrin's method of moving planes to achieve this, where the positivity of u is an unremovable assumption.

In this paper, we will study an overdetermined problem corresponding to a particular equation on $\mathbb{S}^n$:

$$\begin{cases} \Delta u + nu = n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Equation (1.2) is related to Schiffer's problem (See [28] problem 80) on $\mathbb{S}^n$. See for instance [1–3, 7, 9, 10, 23, 24, 27] for recent developments of Schiffer's problem. Previously, Souam [24] showed that for $n=2$ and $\Omega\subset\mathbb{S}^n$ *simply connected*, if (1.2) admits a solution, then Ω must be a geodesic ball.

Our first result is the following.

Theorem 1.1. *Let $\Omega\subset\mathbb{S}^n$ be a bounded open connected domain such that $\overline{\Omega}$ is contained in a hemisphere $\mathbb{S}^n_+$. If the overdetermined problem* (1.2) *admits a solution u, then Ω must be a geodesic ball and u is radially symmetric.*

We remark that since the first Dirichlet eigenvalue for a domain $\overline{\Omega}\subset\mathbb{S}^n_+$ is strictly larger than n, there exists a unique solution for the Dirichlet problem $\Delta u + nu = n$ in Ω and $u=0$ on $\partial\Omega$. However, it is not a priori known whether the solution has a definite sign. Therefore, Theorem 1.1 does not follow from the result of Kumaresan and Prajapat [15].

Our approach to Theorem 1.1 is parellel to Weinberger's, namely, we use a maximum principle for a subharmonic function P and a Rellich-Pohozaev type identity. We remark that our method also applies to equation $\Delta u - nu = n$ in $\Omega\subset\mathbb{H}^n$. In this case, u is negative in Ω by the maximum principle. Hence the conclusion also follows from the result of Kumaresan and Prajapat.

Our next result concerns the same overdetermined problem (1.2) in $\Omega\subset\mathbb{S}^n$ without the assumption that $\overline{\Omega}$ is contained in a hemisphere $\mathbb{S}^n_+$. Instead, we shall add a star-shapedness assumption on Ω. A domain $\Omega\subset\mathbb{S}^n$ is called star-shaped with respect to $p\in\mathbb{S}^n$ if Ω can be written as a graph over a geodesic sphere centered at p. It is clear that a domain Ω which is star-shaped w.r.t. p does not contain the antipodal point $-p$ and the unique geodesic connecting p and $q\in\Omega$ is contained in Ω.

Theorem 1.2. *Let Ω be a bounded open connected domain in $\mathbb{S}^n$. Assume that Ω is star-shaped with respect to some fixed point $p\in\mathbb{S}^n$. If the overdetermined problem* (1.2) *admits a solution u, then Ω must be a geodesic ball and u is radially symmetric.*

For the proof of Theorem 1.2, we find another harmonic function $\tilde{P}$. By examining the boundary behavior of P and $\tilde{P}$ and using the Hopf lemma, we see either P or $\tilde{P}$ is a constant function, which implies our conclusion.

We remark that, in general one can construct round symmetric annuli on $\mathbb{S}^n$ such that (1.2) has solutions.† That means, the condition that Ω is contained in a hemisphere or Ω is star-shaped cannot be totally removed.

Notation: In the following sections, we denote by Δ and ∇ the Laplacian and the gradient on $\mathbb{S}^n$ respectively. We denote by $d\Omega$ and dA the volume measure and the surface measure of Ω and $\partial\Omega$ respectively. For simplicity, we will use $u_i, u_{ij}, \cdots$ and u_ν to denote covariant derivatives and normal derivative of a function u with respect to the round metric on $\mathbb{S}^n$ respectively. We will also follow Einstein's summation convention.

2 Two P-functions

In this section we find two P-functions P and $\tilde{P}$ for u satisfying equation $\Delta u+nu=n$ on $\mathbb{S}^n$. The first P-function in Lemma 2.1 is analog to the one in $\mathbb{R}^n$ used in [26, 18].

Lemma 2.1. *Let u satisfies the equation $\Delta u+nu=n$ in $\Omega\subset\mathbb{S}^n$. Then the following P-function*

$$P:=|\nabla u|^2+u^2-2u, \tag{2.1}$$

satisfies subharmonic property

$$\Delta P\geq 0. \tag{2.2}$$

Proof. Let g be the round metric on $\mathbb{S}^n$. Since $Ric_g=(n-1)g$, the Bochner formula gives

$$\begin{aligned}
\Delta|\nabla u|^2 &= 2|\nabla^2 u|^2+2\langle\nabla\Delta u,\nabla u\rangle+2Ric(\nabla u,\nabla u)\\
&= 2|\nabla^2 u+ug|^2-4u\Delta u-2nu^2+2\langle\nabla(n-nu),\nabla u\rangle+2(n-1)|\nabla u|^2\\
&\geq \frac{2}{n}(\Delta u+nu)^2-2|\nabla u|^2+2nu^2-4nu\\
&= 2n-2|\nabla u|^2+2nu^2-4nu.
\end{aligned} \tag{2.3}$$

†We thank Dr. Guanghao Hong for pointing out this to us.

We have used the Schwarz's inequality $|\nabla^2 u+ug|^2\geq\frac{1}{n}(\Delta u+nu)^2$. Direct computation gives

$$\Delta u^2 \;=\; 2|\nabla u|^2+2u\Delta u=2|\nabla u|^2+2nu-2nu^2. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$\Delta P=\Delta|\nabla u|^2+\Delta u^2-2\Delta u\geq 0.$$

The completes is the proof. □

The second P-function is a harmonic function.

Lemma 2.2. *Let u satisfies the equation $\Delta u+nu=n$ in $\Omega\subset\mathbb{S}^n$. Let $p\in\Omega$ and its antipodal point $-p\notin\Omega$. Denote $V(x)=\cos(r(x))$, where r is the distance function from p. Then V is differentiable in Ω and the following P-function*

$$\widetilde{P}:=\langle\nabla u,\nabla V\rangle+uV-V, \tag{2.5}$$

satisfies

$$\Delta\widetilde{P}=0. \tag{2.6}$$

Proof. Since $\{-p\}$ is the cut locus of p and $-p\notin\Omega$, V is differentiable in Ω. It is well known that $\nabla^2 V=-Vg$ and $\Delta V=-nV$ in $\Omega\subset\mathbb{S}^n$. Direct computation yields:

$$\begin{aligned}\Delta\langle\nabla V,\nabla u\rangle&=\langle\Delta\nabla V,\nabla u\rangle+\langle\nabla V,\Delta\nabla u\rangle+2\langle\nabla^2 V,\nabla^2 u\rangle\\&=\langle\nabla\Delta V,\nabla u\rangle+Ric(\nabla V,\nabla u)+\langle\nabla V,\nabla\Delta u\rangle+Ric(\nabla V,\nabla u)-2V\Delta u\\&=-n\langle\nabla V,\nabla u\rangle+2(n-1)\langle\nabla V,\nabla u\rangle+\langle\nabla V,\nabla(n-nu)\rangle-2V(n-nu)\\&=-2\langle\nabla V,\nabla u\rangle-2nV+2nVu,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\Delta(Vu)&=\Delta Vu+V\Delta u+2\langle\nabla V,\nabla u\rangle\\&=-2nVu+2\langle\nabla V,\nabla u\rangle+nV.\end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8), we conclude $\Delta\widetilde{P}=-Vn-\Delta V=0$. □

3 A Rellich-Pohozaev type identity

In order to prove Theorem 1.1, we need the following Rellich-Pohozaev type identity.

Lemma 3.1. *Let u be a solution of problem* (1.2) *and $V(x)$ defined as in Lemma* 2.2. *Then the following integral identity holds:*

$$\int_\Omega \left(-\frac{n(n+2)}{2}Vu^2+(n+2)nVu\right)d\Omega=\int_{\partial\Omega} c^2 V_\nu dA. \tag{3.1}$$

Proof. Multiplying $\Delta u+nu$ by $\langle\nabla u,\nabla V\rangle$ and integrating among Ω, we have from integration by parts that

$$\begin{aligned}
&\int_\Omega (\Delta u+nu)\langle\nabla u,\nabla V\rangle d\Omega\\
=&\int_{\partial\Omega} u_j\nu^j u_i V_i dA+\int_\Omega\left(-\frac{1}{2}|\nabla u|_i^2 V_i - V_{ij}u_iu_j+\frac{n}{2}(u^2)_iV_i\right)d\Omega\\
=&\int_{\partial\Omega}\left(u_j\nu^ju_iV_i-\frac{|\nabla u|^2}{2}V_\nu+\frac{n}{2}u^2V_\nu\right)dA+\int_\Omega\left(\frac{|\nabla u|^2}{2}\Delta V-V_{ij}u_iu_j-\frac{n}{2}u^2\Delta V\right)d\Omega\\
=&\int_{\partial\Omega}\frac{1}{2}c^2V_\nu dA+\int_\Omega\left(-\frac{n-2}{2}V|\nabla u|^2+\frac{n^2}{2}u^2V\right)d\Omega\\
=&\int_{\partial\Omega}\frac{1}{2}c^2V_\nu dA+\int_\Omega\left(-\frac{n-2}{4}V(\Delta u^2-2u\Delta u)+\frac{n^2}{2}u^2V\right)d\Omega\\
=&\int_{\partial\Omega}\frac{1}{2}c^2V_\nu dA+\int_\Omega\left(-\frac{n(n-2)}{4}Vu^2+\frac{n(n-2)}{2}Vu+\frac{n^2}{2}u^2V\right)d\Omega. \qquad (3.2)
\end{aligned}$$

Here we have used $V_{ij}=-Vg_{ij}$ and $u=0$ on $\partial\Omega$. On the other hand, since $\Delta u+nu=n$,

$$\int_\Omega(\Delta u+nu)\langle\nabla u,\nabla V\rangle d\Omega=\int_\Omega n\langle\nabla u,\nabla V\rangle d\Omega=\int_\Omega n^2uVd\Omega. \tag{3.3}$$

Combine (3.2) and (3.3), we obtain (3.1). □

4 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let $V(x)=\cos(r(x))$, where r is the distance function from the origin of $\mathbb{S}^n_+$. It follows from Lemma 2.1 and the strong maximum principle that $P\le c^2$ in Ω and only two cases happen:

Case 1: $P\equiv constant$;

Case 2: $\exists$ a point $x\in\Omega$, such that $P<c^2$.

In the first case, $\Delta P\equiv 0$, then equality must hold in (2.3). It follows that $\nabla^2u+ug\equiv\lambda(x)g$ for some function λ. Taking into account of $\Delta u+nu=n$, we see $\nabla^2u+ug\equiv g$.

Hence $\nabla^2(u-1)=-(u-1)g$ and $u-1|_{\partial\Omega}=-1$. It follows from an Obata type result (see Reilly [20]) that Ω must be a geodesic ball and u must be radial symmetric, precisely, $u(x)=-\sqrt{1+c^2}\cos r(x)+1$, and Ω is a geodesic ball of radial $\arccos(\frac{1}{\sqrt{1+c^2}})$.

We will show that the second case cannot happen. Suppose it happens. Then thanks to the smoothness of P, by integrating VP over Ω, we obtain

$$\int_\Omega [V|\nabla u|^2+Vu^2-2Vu]d\Omega < c^2\int_\Omega V d\Omega \tag{4.1}$$

note here we use $V>0$ on $\mathbb{S}^n_+$.

Integrating by part, we have

$$\begin{aligned}\int_\Omega |\nabla u|^2 V d\Omega &= \int_\Omega (-uu_iV_i - u\Delta u V)d\Omega + \int_{\partial\Omega} uu_\nu V dA\\ &= \int_\Omega \left(-\frac{u^2nV}{2} - u\Delta u V\right)d\Omega\\ &= \int_\Omega \left(-nuV + \frac{nu^2V}{2}\right)d\Omega.\end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$(n+2)\int_\Omega \left(-uV+\frac{1}{2}u^2V\right)d\Omega < c^2\int_\Omega V d\Omega. \tag{4.3}$$

On the other hand, Lemma 3.1 tells us that

$$\int_\Omega \left(-\frac{(n+2)n}{2}Vu^2+(n+2)nVu\right)d\Omega = \int_{\partial\Omega} c^2 V_\nu dA = -\int_\Omega c^2 nV d\Omega. \tag{4.4}$$

We see (4.4) contradicts with (4.3). We complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let $\Omega\subset\mathbb{S}^n$ be a star-shaped domain with respect to p and $V(x)=\cos(r(x))$, where r is the distance function from p. We claim that $P\equiv constant$ or $\widetilde{P}\equiv constant$ in Ω.

Suppose not, i.e., neither P nor $\tilde{P}$ is a constant. On one hand, by applying the Hopf lemma to P, we know $P_\nu>0$ on the whole boundary $\partial\Omega$ because $P=c^2$ on $\partial\Omega$. Thus for every point on $\partial\Omega$,

$$0<P_\nu = 2\sum_{i=1}^n u_iu_{i\nu} + 2uu_\nu - 2u_\nu = 2u_\nu(u_{\nu\nu}-1). \tag{4.5}$$

Here we used $u=0$ and $u_\nu=c$ on $\partial\Omega$. Hence

$$c(u_{\nu\nu}-1)>0 \quad \text{on } \partial\Omega. \tag{4.6}$$

On the other hand, by applying the Hopf lemma to $\tilde{P}$, at the maximum point of $\widetilde{P}$, say $y_1\in\partial\Omega$, we have $\widetilde{P}_\nu(y_1)>0$. Thus at y_1,

$$\begin{aligned}0<\widetilde{P}_\nu(y_1)=&\sum_{i=1}^n u_{i\nu}V_i+u_iV_{i\nu}+u_\nu V+uV_\nu-V_\nu\\=&u_{\nu\nu}V_\nu-u_\nu V+u_\nu V-V_\nu\\=&u_{\nu\nu}V_\nu-V_\nu.\end{aligned}\tag{4.7}$$

If Ω is star-shaped, then $V_\nu=-\sin r\langle\partial_r,\nu\rangle<0$ on $\partial\Omega$. So we deduce from (4.7) that

$$u_{\nu\nu}(y_1)<1.\tag{4.8}$$

Similarly, at the minimum point of $\widetilde{P}$, say $y_2\in\partial\Omega$, we have

$$0>\widetilde{P}_\nu(y_2)=u_{\nu\nu}V_\nu-V_\nu.$$

So we get

$$u_{\nu\nu}(y_2)>1.\tag{4.9}$$

One of (4.8) and (4.9) must contradict with (4.6), because c is a constant. Therefore, we have shown that only two cases happen:

Case 1: $P\equiv constant$,

Case 2: $\widetilde{P}\equiv constant$.

In the first case, $\Delta P\equiv 0$, then the conclusion follows from the same argument as the proof of Theorem 1.1 . In the second case, $\widetilde{P}_\nu=0$ on the boundary which implies that $u_{\nu\nu}=1$. Then $P_\nu=c(u_{\nu\nu}-1)=0$ and by the Hopf lemma, P must be a constant and we reduce to the first case. We complete the proof of Theorem 1.2. □

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