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Finite Difference/Collocation Method for Two-Dimensional Sub-Diffusion Equation with Generalized Time Fractional Derivative

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Abstract. In this paper, we propose a finite difference/collocation method for two-dimensional time fractional diffusion equation with generalized fractional operator. The main purpose of this paper is to design a high order numerical scheme for the new generalized time fractional diffusion equation. First, a finite difference approximation formula is derived for the generalized time fractional derivative, which is verified with order $2-\alpha$ $(0<\alpha<1)$. Then, collocation method is introduced for the two-dimensional space approximation. Unconditional stability of the scheme is proved. To make the method more efficient, the alternating direction implicit method is introduced to reduce the computational cost. At last, numerical experiments are carried out to verify the effectiveness of the scheme.

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1 Introduction

The time fractional diffusion equation (TFDE) is obtained by replacing the first-order time derivative with fractional derivative of order $\alpha\,(0<\alpha<1)$. This model equation governs the evolution for the probability density function that describes anomalously diffusing particles. Examples for sub-diffusive transport include turbulent flow, chaotic dynamics charge transport in amorphous semiconductors [1, 2], NMR diffusometry in disordered

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materials [3], and dynamics of a bead in polymer network [4]. Literatures on TFDE are mainly based on Riemann-Liouville derivative or Caputo derivative. In addition to that, Erdélyi-Kober fractional derivative and other generalized fractional derivatives are also important for some special problems. In [5], Sandev et al. investigated a kind of diffusion equation with a Hilfer-generalized Riemann-Liouville time fractional derivative. By using methods of separation of variables and Laplace transform, the solution of the fractional diffusion equation is obtained in terms of Mittag-Leffler-type functions and Foxs H-function. In [6], the solution of space-time fractional diffusion equations with a generalized Riemann-Liouville time fractional derivative and Riesz-Feller space fractional derivative is studied. The Laplace and Fourier transform methods are applied to solve the proposed fractional diffusion equation and the solutions are expressed by using the Mittag-Leffler functions and the Fox H-function. In [7], Pagnini proposed a new fractional diffusion model for generalized grey Brownian motion (ggBm) driven by a fractional integral equation in the sense of Erdélyi-Kober. The ggBm is a parametric class of stochastic processes that provides models for both fast and slow anomalous diffusion. For this reason, this new model is called Erdélyi-Kober fractional diffusion. A linear space-time fractional reaction-diffusion equation with composite time fractional derivative and Riesz-Feller space fractional derivative is investigated in [8] by Garg. Laplace and Fourier transforms is applied to obtain its solution. The fractional derivatives used in these papers are different with the classical Riemann-Liouville or Caputo fractional derivatives. In addition to these, there are still other definitions proposed for some special applications.

The notion "generalized operator of fractional integration" appeared first in the papers of the jubilarian professor S. L. Kalla in the years 1969–1979 [9, 10]. The basic ideas of the generalized fractional calculus is surveyed in [11, 12] and further generalizations of fractional integrals and derivatives is presented by Agrawal in [13]. Due to the special formulation, almost all known fractional integrals and derivatives, such as Hadamard operator and the Erdélyi-Kober operator, in various areas of analysis happened to fall in the framework of this generalized fractional calculus. In this paper, we consider the time fractional diffusion equation with generalized fractional derivative proposed by Agarwal. Analytical solutions of fractional differential equations are generally expressed using Mittag-Leffler-type functions and Fox H-functions, which are difficult to evaluation. Thus numerical method is a powerful tool for solving fractional differential equations. There are already abundant of papers working on numerical methods for time fractional diffusion with classical Caputo or Riemann-Liouville derivatives, such as [14, 15, 17–21, 26], to name a few. But there are still very few papers on time diffusion equations with generalized fractional derivatives. In [6], numerical scheme and Grünwald-Letnikov approximation is used to solve the space-time fractional diffusion equation with a generalized Riemann-Liouville time fractional derivative and Riesz-Feller space fractional derivative. In 2013, Xu [22] propose a numerical scheme for one-dimensional time fractional advection-diffusion equations with generalized fractional derivative. Later, a similar numerical method and an analytical solution are proposed for generalized time fractional diffusion equation in [23]. The numerical methods are based on finite difference method for both time and space domain and convergence results are shown through numerical examples for some special case of the equation.

Spectral method, which has been confirmed to be exponentially accurate for smooth problems, has been applied to fractional differential equations by many authors [24–30]. To our knowledge, there is still no works on spectral method for fractional differential equations with generalized fractional derivatives. In this paper, we first present a finite difference formula for generalized Caputo derivative. Then spectral collocation method is introduced for the discretization of the space domain in two-dimensional. To reduce the computational cost of the method, alternating direction implicit (ADI) schemes are introduced for the two-dimensional fractional diffusion equation.

The rest of the current paper is organized as follows: In Section 2, we introduce the new generalized fractional calculus and the problem we concern. In Section 3, finite difference approximation formula is derived for generalized Caputo derivative and collocation method in two-dimension is proposed for space discretization. Stability of the scheme is analyzed. In Section 4, alternating direction implicit method is introduced for reducing computational cost. Numerical examples and remarks are given in Section 5. Finally, we draw our conclusions in Section 6.

2 Preliminaries and notations

First, we introduce some notations and definitions of the generalized fractional operators which is proposed in [13] by Om P. Agrawal. There are several different ways to define fractional derivatives. Here we only present left Riemann-Liouville derivative and left Caputo derivative and later we will focus on the left Caputo derivative in the rest of the paper.

Definition 2.1 (Generalized fractional integral)**.** The left/forward weighted/scaled fractional integral of order $\alpha>0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$ is defined as

$$(I^{\alpha}_{a+;[z;w]}f)(x)=\frac{[w(x)]^{-1}}{\Gamma(\alpha)}\int_a^x\frac{w(t)z'(t)f(t)}{[z(x)-z(t)]^{1-\alpha}}dt. \tag{2.1}$$

Definition 2.2. The left/forward weighted/scaled derivative of order 1 of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$ is defined as

$$(D_{[z,w,L]}f)(x)=[w(x)]^{-1}\left[\left(\frac{1}{z'(x)}D_x\right)(w(x)f(x))\right]. \tag{2.2}$$

Definition 2.3. The left/forward weighted/scaled derivative of integer order $m\geqslant 1$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$ is defined as

$$(D^m_{[z,w,L]}f)(x)=[w(x)]^{-1}\left[\left(\frac{1}{z'(x)}D_x\right)^m(w(x)f(x))\right](x). \tag{2.3}$$

Definition 2.4 (Generalized Riemann-Liouville derivative)**.** The left/forward weighted generalized Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$ is defined as

$$(D^{\alpha}_{a+;[z,w,1]}f)(x)=D^{m}_{z,w,L}\left(I^{m-\alpha}_{a+;[z;w]}f\right)(x),\quad m\in\mathbb{N},\ \ m-1<\alpha<m. \tag{2.4}$$

Definition 2.5 (Generalized Caputo derivative)**.** The left/forward weighted generalized Caputo fractional derivative of order $\alpha>0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$ is defined as

$$(D^{\alpha}_{a+;[z,w,2]}f)(x)=\left(I^{m-\alpha}_{a+;[z;w]}D^{m}_{z,w,L}f\right)(x),\quad m\in\mathbb{N},\ \ m-1<\alpha<m. \tag{2.5}$$

Remark 2.1. In the definitions of generalized fractional operators, more general kernels and weight functions are used. It generalizes nearly all the existing fractional operators, such as the Riemann-Liouville derivative, the Grünwald-Letnikov derivative, the Caputo derivative, the Erdélyi-Kober-type fractional operator and the Hadamard type fractional operators. For the semi-group property and composite rule of these new generalized fractional operators, we refer to [13].

Remark 2.2. The function $z(t)$ in the definition is usually a monotone function in applications and the weight function $w(t)$ is usually a positive function. This will also be our assumption in the analysis part.

With the help of the definitions, we present our generalized time fractional diffusion equation (GTFDE) as follows,

$$\begin{cases} (D^{\alpha}_{0+;[z,w,2]}u)(x,y,t)=\Delta u(x,y,t)+f(x,y,t), & (x,y)\in\Omega, t\in(0,T],\\ u(x,y,0)=u_0(x,y), & (x,y)\in\Omega,\\ u(x,y,t)|_{\partial\Omega}=g(x,y,t), & (x,y)\in\partial\Omega, t\in(0,T]. \end{cases} \tag{2.6}$$

3 Finite difference/collocation scheme for GTFDE

First, we introduce a finite difference approximation to discretize the generalized time-fractional derivative. We assume $z(t)$ is a strictly monotone functions in $(0,T]$, $w(t)>0$, $0<\alpha<1$. Let $t_j=j\tau, j=0,1,2,\cdots,N$ and $\tau=\frac{T}{N}$. Then the generalized Caputo derivative is expressed as

$$\begin{aligned}(D^{\alpha}_{0+;[z;w,2]}u)(t_k)&=\frac{[w(t)]^{-1}}{\Gamma(1-\alpha)}\int_0^{t_k}\frac{(w(s)u(s))'}{[z(t_k)-z(s)]^{\alpha}}ds\\ &=\frac{[w(t)]^{-1}}{\Gamma(1-\alpha)}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\frac{(w(s)u(s))'}{[z(t_k)-z(s)]^{\alpha}}ds.\end{aligned}$$

Let $\zeta=z(s)$, then $s=z^{-1}(\zeta)$. By changing integration variable in the formula, we obtain

$$\begin{aligned}(D^{\alpha}_{0+;[z;w,2]}u)(t_k)&=\frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)}\sum_{j=0}^{k-1}\int_{z(t_j)}^{z(t_{j+1})}\frac{\big(w(z^{-1}(\zeta))u(z^{-1}(\zeta))\big)'}{[z(t_k)-\zeta]^{\alpha}}dz^{-1}(\zeta)\\&=\frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)}\sum_{j=0}^{k-1}\int_{z(t_j)}^{z(t_{j+1})}\frac{\frac{\partial\left(w(z^{-1}(\zeta))u(z^{-1}(\zeta))\right)}{\partial s}\frac{\partial s}{\partial\zeta}}{[z(t_k)-\zeta]^{\alpha}}d\zeta\\&=\frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)}\sum_{j=0}^{k-1}\frac{w(t_{j+1})u(t_{j+1})-w(t_j)u(t_j)}{z(t_{j+1})-z(t_j)}\int_{z(t_j)}^{z(t_{j+1})}\frac{d\zeta}{[z(t_k)-\zeta]^{\alpha}}+r^k_{\tau},\end{aligned}$$

where r^k is the truncation error of the approximation formula, defined as below

$$r^k_{\tau}=\frac{[w(t_k)]^{-1}}{\Gamma(1-\alpha)}\sum_{j=0}^{k-1}\int_{z(t_j)}^{z(t_{j+1})}\Big(\frac{\frac{\partial(w(z^{-1}(\zeta))u(z^{-1}(\zeta)))}{\partial\zeta}}{[z(t_k)-\zeta]^{\alpha}}-\frac{\frac{w(t_{j+1})u(t_{j+1})-w(t_j)u(t_j)}{z(t_{j+1})-z(t_j)}}{[z(t_k)-\zeta]^{\alpha}}\Big)d\zeta. \tag{3.1}$$

Denote $t_{j+1/2}=\frac{t_j+t_{j+1}}{2}$, $a_{j+1/2}=\big(z(t_k)-z(t_j)\big)^{1-\alpha}-\big(z(t_k)-z(t_{j+1})\big)^{1-\alpha}$. The discretized generalized Caputo derivative is obtained,

$$(D^{\alpha}_{0+;[z;w,2]}u)(t_k)=\frac{[w(t_k)]^{-1}}{\Gamma(2-\alpha)}\sum_{j=0}^{k-1}a_{j+1/2}\frac{w(t_{j+1})u(t_{j+1})-w(t_j)u(t_j)}{z(t_{j+1})-z(t_j)}+r^k_{\tau}. \tag{3.2}$$

Remark 3.1. The finite difference formula for generalized Caputo derivative is very similar as the L_1 formula for classical Caputo derivative. If $z(t)=t$ and $w(t)=1$, the formula reduce to L_1 formula. The truncation error of L_1 formula for classical Caputo derivative has been proved in [24] to be of order $2-\alpha$. But it is more hard to prove the truncation error of the finite difference formula (3.2). In the discretized formula (3.2), the term $\frac{\partial(wu)}{\partial\zeta}$ is approximated using finite difference formula on a nonuniform grids $\zeta_j=z(t_j), j=0,1,\cdots,N$. The methods for proving truncation errors on uniform grids fail when used to estimate the truncation term (3.1). In this paper, we will not prove the truncation error but show the approximation accuracy through numerical examples instead.

Furthermore, the approximation formula can be rewritten in a more convenient form,

$$\begin{aligned}&\sum_{j=0}^{k-1}a_{j+1/2}\frac{w(t_{j+1})u(t_{j+1})-w(t_j)u(t_j)}{z(t_{j+1})-z(t_j)}\\&=\frac{a_{k-\frac12}}{z^k-z^{k-1}}w^ku^k-\sum_{j=1}^{k-1}\left(\frac{a_{j+\frac12}}{z^{j+1}-z^j}-\frac{a_{j-\frac12}}{z^j-z^{j-1}}\right)w^ju^j-\frac{a_{\frac12}}{z^1-z^0}w^0u^0.\end{aligned} \tag{3.3}$$

We introduce the following notations,

$$\eta^k = \frac{1}{w^k\Gamma(2-\alpha)}; \quad w^k = w(t_k); \quad z^k = z(t_k);$$
$$u^k = u(x,y,t^k); \quad f^k = f(x,y,t_k); \quad b_j = \frac{a_{j-\frac{1}{2}}}{z^j - z^{j-1}}.$$

Using the finite difference formula, time semi-discretized formulation of the equation is obtained,

$$\frac{1}{\eta^k}\Big(b_k w^k u^k - \sum_{j=1}^{k-1}\Big(b_{j+1}-b_j\Big) w^j u^j - b_1 w^0 u^0\Big) + r_\tau^k = \Delta u^k + f^k. \tag{3.4}$$

Next, we present the collocation scheme for the space domain. Suppose the equation is defined in a rectangle domain $\Omega = [0,L_x]\times[0,L_y]$. $x_0,x_1,\cdots,x_{N_x}$ are collocation points in x-direction and $y_0,y_1,\cdots,y_{N_y}$ are collocation points in y-direction. We define the following one-dimensional cardinal function based on Lagrange functions,

$$\phi_j(x) = \prod_{k=0,k\neq j}^{N_x} \frac{(x-x_k)}{(x_j-x_k)}, \quad j = 0,1,\cdots,N_x,$$
$$\psi_j(y) = \prod_{k=0,k\neq j}^{N_y} \frac{(y-y_k)}{(y_j-y_k)}, \quad j = 0,1,\cdots,N_y,$$

then $\Phi_{ij}(x,y) = \phi_i(x)\psi_j(y)$, $(i = 0,1,\cdots,N_x; j = 0,1,\cdots,N_y)$, forms a basis of the two-dimensional polynomial spaces. Denote u_{N_x} as the interpolation of u at N_x+1 collocation points $x_0,x_1,\cdots,x_{N_x}$; denote u_{N_y} as the interpolation of u at N_y+1 collocation points $y_0,y_1,\cdots,y_{N_y}$ and u_N as the interpolation of u in the two-dimensional space. Define two notations

$$u_{ij}^k = u(x_i,y_j,t_k), \quad f_{ij}^k = f(x_i,y_j,t_k),$$

then we have the following approximation formulas,

$$\frac{\partial^2}{\partial x^2}u(x,y,t_k) \approx \frac{\partial^2}{\partial x^2}u_N(x,y,t_k) = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} u_{ij}^k \phi_i''(x)\psi_j(y);$$
$$\frac{\partial^2}{\partial y^2}u(x,y,t_k) \approx \frac{\partial^2}{\partial y^2}u_N(x,y,t_k) = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} u_{ij}^k \phi_i(x)\psi_j''(y);$$
$$\Delta u(x,y,t^k) \approx \Delta u_N(x,y,t_k) = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} u_{ij}^k \Big(\phi_i''(x)\psi_j(y) + \phi_i(x)\psi_j''(y)\Big).$$

We assume $\Delta u(x,y,t^k) = \Delta u_N(x,y,t_k) + r_s^k$. Replacing Δu with Δu_N in the semi-discretized scheme and making it holds at any (x_i,y_j,t_k), $i = 0,1,\cdots,N_x$; $j = 0,1,\cdots,N_y$; $k =$

$0,1,\cdots,N$, the full-discretized scheme is obtained,

$$\begin{aligned}&\frac{1}{\eta^k}\Big(b_k w^k u_{ij}^k-\sum_{m=1}^{k-1}(b_{m+1}-b_m)w^m u_{ij}^m-b_1 w^0 u_{ij}^0\Big)\\=&\sum_{m=0}^{N_x}u_{mj}^k\phi_m''(x_i)+\sum_{n=0}^{N_y}u_{in}^k\psi_n''(y_j)+f_{ij}^k+r_\tau^k+r_s^k,\end{aligned}\tag{3.5}$$

where $i=0,1,\cdots,N_x$; $j=0,1,\cdots,N_y$; $k=0,1,\cdots,N$.

Assuming v_{ij}^k is the numerical solution, omitting the truncation error, the scheme is obtained as follows,

$$\begin{aligned}&\frac{1}{\eta^k}\Big(b_k w^k v_{ij}^k-\sum_{m=1}^{k-1}(b_{m+1}-b_m)w^m v_{ij}^m-b_1 w^0 v_{ij}^0\Big)\\=&\sum_{m=0}^{N_x}v_{mj}^k\phi_m''(x_i)+\sum_{n=0}^{N_y}v_{in}^k\psi_n''(y_j)+f_{ij}^k,\end{aligned}\tag{3.6}$$

where $i=0,1,\cdots,N_x$; $j=0,1,\cdots,N_y$; $k=0,1,\cdots,N$ and

$$v_{ij}^0=u_0(x_i,y_j);\quad i=0,1,\cdots,N_x;\quad j=0,1,\cdots,N_y;\tag{3.7}$$

$$v_{0j}^k=g(x_0,y_j,t_k);\quad v_{N_xj}^k=g(x_{N_x},y_j,t_k);\quad j=0,1,\cdots,N_y;\quad k=0,1,\cdots,N;\tag{3.8}$$

$$v_{i0}^k=g(x_i,y_0,t_k);\quad v_{iN_y}^k=g(x_i,y_{N_y},t_k);\quad i=0,1,\cdots,N_x;\quad k=0,1,\cdots,N.\tag{3.9}$$

Although there are many different choices of the collocation points, zeros of orthogonal polynomials have been shown to be a good choice in many works on spectral method [31, 32]. In this paper, we choose Legendre-Gauss-Lobatto points as our collocation points.

We denote the Gaussian integration weights corresponding to the collocation points x_i by $\kappa_{x,i}$, $i=0,1,\cdots,N_x$, and denote the Gaussian integration weights corresponding to the collocation points y_j by $\kappa_{y,j}$, $j=0,1,\cdots,N_y$. We set $\kappa_{ij}=\kappa_{x,i}\kappa_{y,j}$. In order to carry on the analysis, we define the discretized norm as follows,

$$\|v\|_h=\sqrt{\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\kappa_{ij}v_{ij}^2}.$$

Theorem 3.1. *Let v_{ij}^k be solution of Eqs.* (3.6)-(3.9). *Assuming $u(x,t)$ and $f(x,t)$ are smooth enough functions and $g(x,y,t)=0$ for any $(x,y)\in\partial\Omega$, then the scheme* (3.6)-(3.9) *is unconditionally stable, and the following inequality holds for any $k\geq 1$,*

$$w^k\|v^k\|_h\leq w^0\|v^0\|_h+\sum_{m=1}^{k-1}\eta^m\left(\frac{1}{b_m}-\frac{1}{b_{m+1}}\right)\|f^m\|_h+\frac{\eta^k\|f^k\|_h}{b_k}.\tag{3.10}$$

Proof. We rewrite the scheme (3.6) in the following form,

$$\begin{aligned} &b_k w^k v_{ij}^k - \sum_{m=1}^{k-1}(b_{m+1}-b_m)w^m v_{ij}^m - b_1 w^0 v_{ij}^0 \\ =&\eta^k \sum_{m=0}^{N_x} v_{mj}^k \phi_m''(x_i) + \eta^k \sum_{n=0}^{N_y} v_{in}^k \psi_n''(y_j) + \eta^k f_{ij}^k, \end{aligned} \tag{3.11}$$

where $i=0,1,\cdots,N_x$; $j=0,1,\cdots,N_y$; $k=0,1,\cdots,N$. Multiplying the Eq. (3.11) by $\kappa_{ij}v_{ij}^k$ from both sides and making a summation for $i=0,1,\cdots,N_x$; $j=0,1,\cdots,N_y$, it yields

$$\begin{aligned} &\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\left(b_k w^k v_{ij}^k - \sum_{m=1}^{k-1}(b_{m+1}-b_m)w^m v_{ij}^m - b_1 w^0 v_{ij}^0\right)\kappa_{ij}v_{ij}^k \\ =&\eta^k \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\left(\sum_{m=0}^{N_x} v_{mj}^k \phi_m''(x_i) + \sum_{n=0}^{N_y} v_{in}^k \psi_n''(y_j)\right)\kappa_{ij}v_{ij}^k + \eta^k \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\kappa_{ij} f_{ij}^k v_{ij}^k. \end{aligned} \tag{3.12}$$

Since $v_{ij}^k = \sum_{m=0}^{N_x} v_{mj}^k \phi_m(x_i) = \sum_{m=0}^{N_y} v_{im}^k \psi_m(y_j)$, using the boundary condition $g(x,y,t)=0$ for any $(x,y)\in\partial\Omega$, we have

$$\begin{aligned} \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{m=0}^{N_x} v_{mj}^k \phi_m''(x_i) v_{ij}^k \kappa_{ij} =& \sum_{j=0}^{N_y}\sum_{i=0}^{N_x}\kappa_{ij}\left(\sum_{m=0}^{N_x} v_{mj}^k \phi_m''(x_i)\right)\left(\sum_{m=0}^{N_x} v_{mj}^k \phi_m(x_i)\right) \\ =& -\sum_{j=0}^{N_y}\sum_{i=0}^{N_x}\kappa_{ij}\left(\sum_{m=0}^{N_x} v_{mj}^k \phi_m'(x_i)\right)^2 \le 0. \end{aligned} \tag{3.13}$$

Correspondingly, the following equation is obtained for the second term in the right hand side of Eq. (3.12),

$$\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{n=0}^{N_y} v_{in}^k \psi_n''(y_j)\kappa_{ij}v_{ij}^k = -\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\kappa_{ij}\left(\sum_{n=0}^{N_y} v_{in}^k \psi_n'(y_j)\right)^2 \le 0. \tag{3.14}$$

Combining Eqs. (3.12)-(3.14), the following inequality is obtained,

$$\begin{aligned} &b_k w^k \|v^k\|_h^2 = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\left(b_k w^k v_{ij}^k\right)\kappa_{ij}v_{ij}^k \\ \le& \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\left(\sum_{m=1}^{k-1}(b_{m+1}-b_m)w^m v_{ij}^m + b_1 w^0 v_{ij}^0\right)\kappa_{ij}v_{ij}^k + \eta^k \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\kappa_{ij} f_{ij}^k v_{ij}^k. \end{aligned} \tag{3.15}$$

Recalling the Cauchy-Schwarz inequality, we obtain

$$b_k w^k \|v^k\|_h \le \sum_{m=1}^{k-1}(b_{m+1}-b_m)w^m\|v^m\|_h + b_1 w^0\|v^0\|_h + \eta^k\|f^k\|_h. \tag{3.16}$$

Next, we prove the theorem using induction method. First, when $k=1$, we have

$$w^1\|v^1\|_h \le w^0\|v^0\|_h + \frac{\eta^1}{b_1}\|f^1\|_h.$$

The inequality (3.10) holds. We assume inequality (3.10) holds for any $n=1,2,\cdots,k-1$. From Eq. (3.16), we have,

$$\begin{aligned}
w^k\|v^k\|_h \le & \sum_{m=1}^{k-1}\frac{b_{m+1}-b_m}{b_k}w^m\|v^m\|_h + \frac{b_1}{b_k}w^0\|v^0\|_h + \frac{\eta^k}{b_k}\|f^k\|_h \\
\le & \frac{b_k-b_{k-1}}{b_k}\left[w^0\|v^0\|_h + \sum_{m=1}^{k-2}\left(\frac{1}{b_m}-\frac{1}{b_{m+1}}\right)\eta^m\|f^m\|_h + \frac{\eta^{k-1}\|f^{k-1}\|_h}{b_{k-1}}\right] \\
& + \frac{b_{k-1}-b_{k-2}}{b_k}\left[w^0\|v^0\|_h + \sum_{m=1}^{k-3}\left(\frac{1}{b_m}-\frac{1}{b_{m+1}}\right)\eta^m\|f^m\|_h + \frac{\eta^{k-2}\|f^{k-2}\|_h}{b_{k-2}}\right] \\
& + \cdots \quad \cdots \quad \cdots \\
& + \frac{b_3-b_2}{b_k}\left[w^0\|v^0\|_h + \sum_{m=1}^{1}\left(\frac{1}{b_m}-\frac{1}{b_{m+1}}\right)\eta^m\|f^m\|_h + \frac{\eta^2\|f^2\|_h}{b_2}\right] \\
& + \frac{b_2-b_1}{b_k}\left[w^0\|v^0\|_h + \frac{\eta^1\|f^1\|_h}{b_1}\right] + \frac{b_1}{b_k}w^0\|v^0\|_h + \frac{\eta^k}{b_k}\|f^k\|_h \\
\le & w^0\|v^0\|_h + \sum_{m=1}^{k-1}\left(\frac{1}{b_m}-\frac{1}{b_{m+1}}\right)\eta^m\|f^m\|_h + \frac{\eta^k\|f^k\|_h}{b_k}.
\end{aligned}$$

The theorem is proved. □

Remark 3.2. From Theorem 3.1, the full discretized scheme is stable to the initial condition and the source term. Due to the difficulty in proving truncation error of the finite difference formula (3.2), we leave the convergence of the method open. The effectiveness of the method will be illustrated through numerical examples.

4 ADI scheme for 2D GTFDE

In order to reduce the computational cost, we introduce alternating direction implicit method for the 2D GTFDE. We define two differential operators $\partial^2_{N_x,x}$ and $\partial^2_{N_y,y}$ such that

$$\partial^2_{N_x,x}v_{ij} = \sum_{m=0}^{N_x} v_{mj}\phi''_m(x_i); \quad \partial^2_{N_y,y}v_{ij} = \sum_{n=0}^{N_y} v_{in}\psi''_n(y_i).$$

Scheme (3.6) is rewritten as,

$$\frac{b^k w^k}{\eta^k}v^k_{ij} - \partial^2_{N_x,x}v_{ij} - \partial^2_{N_y,y}v_{ij} = \frac{1}{\eta^k}\sum_{m=1}^{k-1}(b_{m+1}-b_m)w^m v^m_{ij} + \frac{b_1 w^0 v^0_{ij}}{\eta^k} + f^k_{ij}. \tag{4.1}$$

Adding a small term $\frac{\tau\eta^k}{b^k w^k}\partial^2_{N_x,x}\partial^2_{N_y,y}\big(\frac{v^k_{ij}-v^{k-1}_{ij}}{\tau}\big)$ to the left hand side of scheme (4.1) and rearranging the equation, we have

$$\Big(\mathcal{I}-\frac{\eta^k}{b^k w^k}\partial^2_{N_x,x}\Big)\Big(\mathcal{I}-\frac{\eta^k}{b^k w^k}\partial^2_{N_y,y}\Big)v^k_{ij}=\frac{1}{b^k w^k}\sum_{m=1}^{k-1}\Big(b_{m+1}-b_m\Big)w^m v^m_{ij}+\frac{b_1 w^0}{b^k w^k}v^0_{ij}$$
$$+\frac{\eta^k f^k_{ij}}{b^k w^k}+\Big(\frac{\eta^k}{b^k w^k}\Big)^2\partial^2_{N_x,x}\partial^2_{N_y,y}v^{k-1}_{ij}, \tag{4.2}$$

where $\mathcal{I}$ is an identical operator.

Introducing an intermediate variable $v^{k*}_{ij}=\big(\mathcal{I}-\frac{\eta^k}{b^k w^k}\partial^2_{N_y,y}\big)v^k_{ij}$, the solution of equation (4.2) can be separated as two independent steps.

Step 1. For any fixed $j\in\{1,2,\cdots,N_y-1\}$, one solve the following equation to obtain v^{k*}_{ij},

$$\begin{cases}\big(\mathcal{I}-\frac{\eta^k}{b^k w^k}\partial^2_{N_x,x}\big)v^{k*}_{ij}=\frac{1}{b^k w^k}\sum_{m=1}^{k-1}\big(b_{m+1}-b_m\big)w^m v^m_{ij}+\frac{b_1 w^0}{b^k w^k}v^0_{ij}\\ \qquad\qquad +\frac{\eta^k}{b^k w^k}f^k_{ij}+\big(\frac{\eta^k}{b^k w^k}\big)^2\partial^2_{N_x,x}\partial^2_{N_y,y}v^{k-1}_{ij};\\ v^{k*}_{0j}=\big(\mathcal{I}-\frac{\eta^k}{b^k w^k}\partial^2_{N_y,y}\big)v^k_{0j};\\ v^{k*}_{N_x j}=\big(\mathcal{I}-\frac{\eta^k}{b^k w^k}\partial^2_{N_y,y}\big)v^k_{N_x j}.\end{cases} \tag{4.3}$$

Step 2. For any fixed $i\in\{1,2,\cdots,N_x-1\}$, one solve the following equation to obtain v^k_{ij},

$$\begin{cases}\big(\mathcal{I}-\frac{\eta^k}{b^k w^k}\partial^2_{N_y,y}\big)v^k_{ij}=v^{k*}_{ij};\\ v^k_{i0}=g(x_i,y_0,t_k);\\ v^k_{iN_y}=g(x_i,y_{N_y},t_k).\end{cases} \tag{4.4}$$

Remark 4.1. From the definition of b_k, $b_k\sim\tau^{-\alpha}$. The term $\partial^2_{N_x,x}\partial^2_{N_y,y}\big(\frac{v^k_{ij}-v^{k-1}_{ij}}{\tau}\big)$ is bounded, thus $\frac{\tau\eta^k}{b^k w^k}\partial^2_{N_x,x}\partial^2_{N_y,y}\big(\frac{v^k_{ij}-v^{k-1}_{ij}}{\tau}\big)\sim\tau^{1+\alpha}$. If the original scheme is of order $\tau^{2-\alpha}$, just as the classical fractional derivative with $z(t)=t$ and $w(t)=1$, the convergence order of this ADI scheme will be $\tau^{\min(1+\alpha,2-\alpha)}$.

5 Numerical examples

Example 5.1. First, we consider the direct approximation to generalized Caputo derivative using proposed finite difference scheme. Let $u(t)=t^6$, we choose several different pairs of $z(t),w(t)$, with which the exact generalized fractional derivative can be expressed analytically. Then, we compute the approximated generalized Caputo derivative and compare them with the exact solution.

- Test 1. $z(t)=t, w(t)=1$, then $D^{\alpha}_{t,[z,w,2]}=\frac{\Gamma(7)}{\Gamma(7-\alpha)}t^{6-\alpha}$;
- Test 2. $z(t)=\sqrt{t}, w(t)=1$, then $D^{\alpha}_{t,[z,w,2]}=\frac{\Gamma(13)}{\Gamma(13-\alpha)}t^{6-\alpha/2}$;
- Test 3. $z(t)=t^2, w(t)=1$, then $D^{\alpha}_{t,[z,w,2]}=\frac{6}{\Gamma(4-\alpha)}t^{6-2\alpha}$;
- Test 4. $z(t)=\sqrt{t}, w(t)=1+t$, then $D^{\alpha}_{t,[z,w,2]}=\frac{t^{6-\alpha/2}}{1+t}\left(\frac{\Gamma(13)}{\Gamma(13-\alpha)}+\frac{\Gamma(15)}{\Gamma(15-\alpha)}t\right)$.

Error and convergence order for this four tests are shown in Table 1.

From Table 1, we observe that the convergence order of the proposed finite difference formula is $2-\alpha$ for both classical and generalized Caputo derivatives.

Table 1: Error and convergence of the finite difference approximation to generalized Caputo derivative, N is number of elements.

Test 1: $z(t)=t$, $w(t)=1$						
	$\alpha=0.2$		$\alpha=0.5$		$\alpha=0.8$	
N	max error	order	max error	order	max error	order
100	6.5e-04	-	6.5e-03	-	4.4e-02	-
200	2.0e-04	1.70	2.4e-03	1.46	2.0e-02	1.18
300	9.9e-05	1.72	1.3e-03	1.47	1.2e-02	1.19
400	6.0e-05	1.73	8.5e-04	1.48	8.6e-03	1.19

Test 2: $z(t)=\sqrt{t}$, $w(t)=1$						
	$\alpha=0.2$		$\alpha=0.5$		$\alpha=0.8$	
N	max error	order	max error	order	max error	order
100	8.2e-04	-	1.0e-02	-	8.5e-02	-
200	2.5e-04	1.70	3.7e-03	1.46	3.7e-02	1.18
300	1.3e-04	1.72	2.0e-03	1.47	2.3e-02	1.19
400	7.6e-05	1.73	1.3e-03	1.48	1.6e-02	1.19

Test 3: $z(t)=t^2$, $w(t)=1$						
	$\alpha=0.2$		$\alpha=0.5$		$\alpha=0.8$	
N	max error	order	max error	order	max error	order
100	4.5e-04	-	3.7e-03	-	2.0e-02	-
200	1.4e-04	1.70	1.3e-03	1.46	9.0e-03	1.19
300	6.9e-05	1.72	7.3e-04	1.47	5.5e-03	1.19
400	4.2e-05	1.72	4.8e-04	1.48	3.9e-03	1.19

Test 4: $z(t)=\sqrt{t}$, $w(t)=1+t$						
	$\alpha=0.2$		$\alpha=0.5$		$\alpha=0.8$	
N	max error	order	max error	order	max error	order
100	9.7e-04	-	1.2e-02	-	1.0e-01	-
200	3.0e-04	1.70	4.3e-03	1.46	4.4e-02	1.18
300	1.5e-04	1.72	2.4e-03	1.47	2.7e-02	1.19
400	9.0e-05	1.72	1.6e-03	1.48	1.9e-02	1.19

Example 5.2. We consider the two-dimensional time fractional diffusion equation,

$$D^{\alpha}_{t,[z,w,2]}u(x,y,t)=\Delta u(x,y,t)+f(x,y,t), \quad (x,y)\in[0,1]\times[0,1]. \tag{5.1}$$

In order to verify the convergence of the proposed method, we choose the exact solution as $u(x,y,t)=t^3\sin(2\pi x)\cos(2\pi y)$. We set $z(t)=\sqrt{t}, w(t)=1$, then the source term $f(x,y,t)$ is recovered as,

$$f(x,y,t)=\left(\frac{720}{\Gamma(7-\alpha)}t^{3-\alpha/2}+8\pi^2t^3\right)\sin(2\pi x)\cos(2\pi y).$$

First, we fix the space discretization and verify the convergence of time discretization. We set $N_x=N_y=20$ in the following examples. Results are shown in Table 2.

Table 2: Error and convergence for the time discretization, N is number of elements.

	$\alpha=0.2$		$\alpha=0.5$		$\alpha=0.8$	
N	max error	order	max error	order	max error	order
100	2.8e-06	-	3.2e-05	-	2.6e-04	-
200	8.3e-07	1.73	1.1e-05	1.48	1.1e-04	1.19
300	4.1e-07	1.74	6.2e-06	1.49	6.9e-05	1.20
400	2.5e-07	1.75	4.0e-06	1.49	4.9e-05	1.20
500	1.7e-07	1.75	2.9e-06	1.49	3.7e-05	1.20

Next, we fix the number of element of time discretization as $N=20000$. We solve the equation with different $N_x(N_y)$, the results are shown in Fig. 1. From the results, the error decay exponentially when the space error is the dominate error and tend to constant when the time error becomes the dominate error.

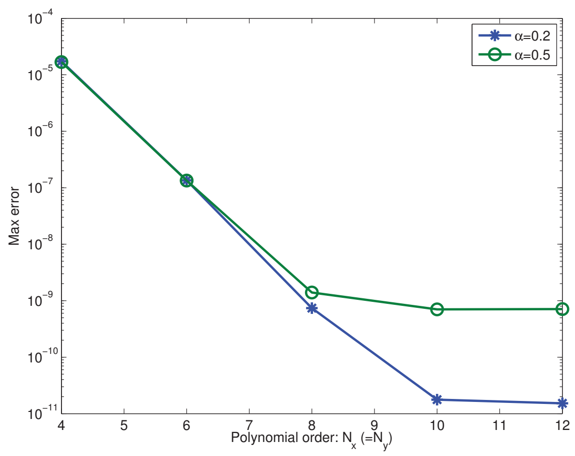

Figure 1: Error and convergence for different order of collocation polynomials.

Example 5.3. In this example, we solve the following generalized time fractional diffusion equation using ADI/collocation method and observe the convergence and time cost.

$$\begin{cases} D^{\alpha}_{t,[\sqrt{t},1,2]}u(x,y,t)=\Delta u(x,y,t)+(\frac{24t^{2-\alpha/2}}{\Gamma(5-\alpha)}+2t^2)\sin(x)\sin(y), \\ (x,y)\in\Omega=[0,\pi]\times[0,\pi],\ \ t\in(0,1/2]; \\ u(x,y,0)=0; \quad u(x,y,t)|_{\partial\Omega}=0. \end{cases} \tag{5.2}$$

First, we set $z(t)=\sqrt{t}$ and $w(t)=1$. It's easy to check that the exact solution is $u(x,y,t)=t^2\sin(x)\sin(y)$. To observe the convergence of the method with different α, the order of collocation polynomials is chosen as $N_x=N_y=20$ and the order of fractional derivative is set as $\alpha=0.2,0.5,0.8$ separately. The equation is solved using different number of time steps and the results are shown in Table 3. From the results, the convergence order is $\min\{1+\alpha,2-\alpha\}$.

Table 3: Error and convergence of the ADI/collocation method for different α.

	$\alpha=0.2$		$\alpha=0.5$		$\alpha=0.8$	
N	max error	order	max error	order	max error	order
20	2.9e-03	-	2.2e-04	-	4.7e-03	-
40	1.3e-03	1.16	7.7e-05	1.48	2.1e-03	1.17
80	5.9e-04	1.17	2.8e-05	1.47	9.1e-04	1.18
160	2.6e-04	1.18	1.0e-05	1.47	4.0e-04	1.19

Next, we compare the computational cost of finite difference/collocation method and ADI/collocation method under the same hardware platform. We solve the same equation using finite difference/collocation method and ADI/collocation method and list the error and time cost of both methods in Table 4. For both method we set $N=200, N_x=N_y=30$.

Table 4: Error and convergence of the ADI/collocation method for different α.

	FDM/Collocation		ADI/Collocation	
α	max error	time(second)	max error	time(second)
$\alpha=0.2$	2.4e-06	12.05	2.2e-04	2.47
$\alpha=0.5$	3.2e-05	11.82	7.2e-06	2.24
$\alpha=0.8$	3.1e-04	11.88	3.1e-04	2.21

From Table 4, for $\alpha>0.5$, almost same errors are obtained by using the two methods. But the computational cost of ADI/collocation method is obviously less than finite difference/collocation method. When $\alpha<0.5$, the error of ADI/collocation method is obviously large than FDM/collocation.

Example 5.4. In this example, we consider the time fractional diffusion equation without source term. We choose several pairs of $z(t)$,$w(t)$and observe what's the difference

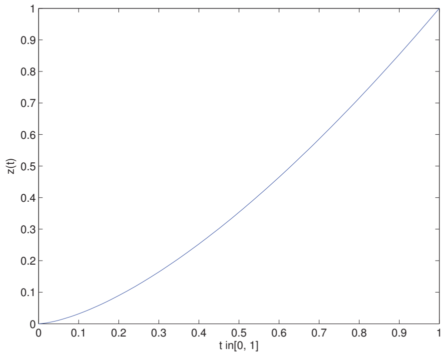

Figure 2: $z(t)=t\sqrt{t}$.

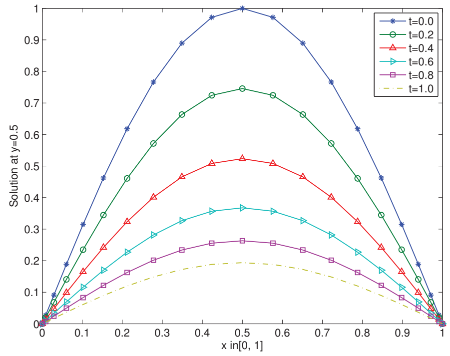

Figure 3: Solution of (5.3) with $z(t)=t\sqrt{t}$, $w(t)=1$.

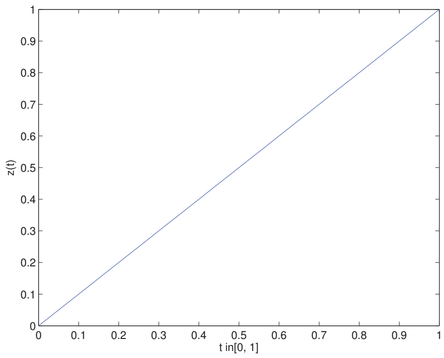

Figure 4: $z(t)=t$.

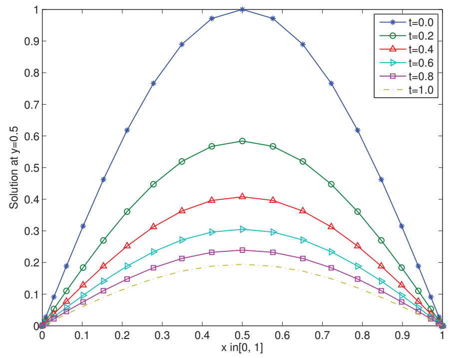

Figure 5: Solution of (5.3) with $z(t)=t$, $w(t)=1$.

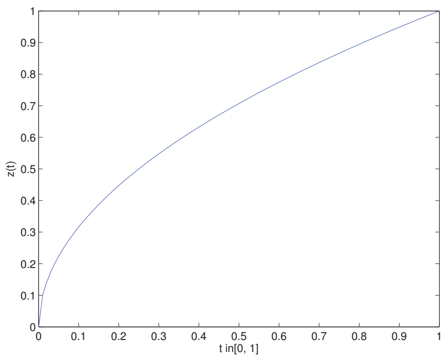

Figure 6: $z(t)=\sqrt{t}$.

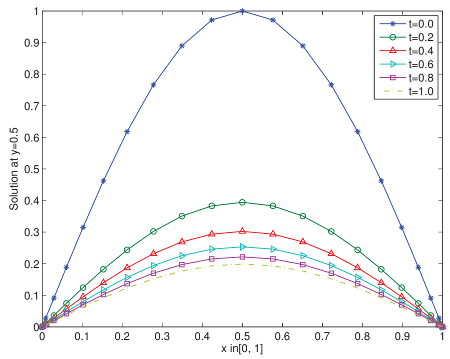

Figure 7: Solution of (5.3) with $z(t)=\sqrt{t}$, $w(t)=1$.

between generalized fractional diffusion and classical fractional diffusion. Results are shown in Figs. 2-7.

$$\begin{cases} D^{\alpha}_{0+,[z,w,2]}u(x,y,t)=\frac{1}{10}\Delta u(x,y,t), \quad (x,y)\in\Omega=(0,1)\times(0,1), \ t\in(0,1], \\ u(x,y,0)=\sin(\pi x)\sin(\pi y), \\ u(x,y,t)|_{\partial\Omega}=0. \end{cases} \tag{5.3}$$

Compared with the classical case $z(t)=t,w(t)=1$, the case $z(t)=t\sqrt{t}$, $w(t)=1$ diffuses slower at the initial time and the case $z(t)=\sqrt{t}$, $w(t)=1$ diffuses much faster than the classical case.

6 Conclusion

In this paper, we introduced the generalized fractional operators and proposed a finite difference/collocation method for two-dimensional time fractional diffusion equation with generalized fractional operator. First, finite difference approximation formula was derived for the generalized time fractional derivative. Then, we introduced collocation method for the two-dimensional space approximation and proved the unconditional stability of the scheme. Alternating direction implicit method was introduced to reduce the computational cost. At last, numerical experiments were carried out. We tested the convergence of the finite difference approximation to generalized Caputo derivative for several different pairs of $z(t),w(t)$. The convergence order is shown to be of order $2-\alpha$, $(0<\alpha<1)$, which is coincide with the classical case. The methods converge exponentially in space and algebraically in time, with convergence order $2-\alpha$ for FDM/collocation method and $\min\{2-\alpha,1+\alpha\}$ for ADI/collocation method.

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