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Complete Convergence for Weighted Sums of Negatively Superadditive Dependent Random Variables

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Abstract. Let $\{X_n, n\geq 1\}$ be a sequence of negatively superadditive dependent (NSD, in short) random variables and $\{a_{nk}, 1\leq k\leq n, n\geq 1\}$ be an array of real numbers. Under some suitable conditions, we present some results on complete convergence for weighted sums $\sum_{k=1}^{n} a_{nk}X_k$ of NSD random variables by using the Rosenthal type inequality. The results obtained in the paper generalize some corresponding ones for independent random variables and negatively associated random variables.

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1 Introduction

Firstly, let us recall the definitions of negatively associated random variables and negatively superadditive dependent random variables. The concept of negatively associated random variables was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [2] as follows.

Definition 1.1. A finite collection of random variables $X_1, X_2, \cdots, X_n$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1,2,\cdots,n\}$,

$$Cov\{f(X_i : i\in A_1), g(X_j : j\in A_2)\}\leq 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{X_n, n\geq 1\}$ is NA if every finite subcollection is NA.

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The next dependence notion is negatively superadditive dependence, which is weaker than negative association. The concept of negatively superadditive dependent random variables was introduced by Hu [3] as follows.

Definition 1.2. (cf. Kemperman [4]) A function $\phi:\mathbb{R}^n\to\mathbb{R}$ is called superadditive if $\phi(\mathbf{x}\vee\mathbf{y})+\phi(\mathbf{x}\wedge\mathbf{y})\geq\phi(\mathbf{x})+\phi(\mathbf{y})$ for all $\mathbf{x},\mathbf{y}\in R^n$, where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum.

Definition 1.3. (cf. Hu [3]) A random vector $\mathbf{X}=(X_1,X_2,\cdots,X_n)$ is said to be negatively superadditive dependent (NSD) if

$$E\phi(X_1,X_2,\cdots,X_n)\leq E\phi(X_1^*,X_2^*,\cdots,X_n^*), \tag{1.1}$$

where $X_1^*,X_2^*,\cdots,X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in (1.1) exist.

We will introduce the notion of sequences of NSD random variables and arrays of rowwise NSD random variables as follows.

Definition 1.4. A sequence $\{X_n,n\geq 1\}$ of random variables is said to be NSD if for all $n\geq 1$, $(X_1,X_2,\cdots,X_n)$ is NSD.

The concept of NSD random variables was introduced by Hu [3], which was based on the class of superadditive functions. Hu [3] gave an example illustrating that NSD does not imply NA, and Hu posed an open problem whether NA implies NSD. In addition, Hu [3] provided some basic properties and three structural theorems of NSD. Christofides and Vaggelatou [5] solved this open problem and indicated that NA implies NSD. Negatively superadditive dependent structure is an extension of negatively associated structure and sometimes more useful than it and can be used to get many important probability inequalities. Eghbal et al. [6] derived two maximal inequalities and strong law of large numbers of quadratic forms of NSD random variables under the assumption that $\{X_i,i\geq 1\}$ are a sequence of nonnegative NSD random variables with $EX_i^r<\infty$ for all $i\geq 1$ and some $r>1$. Eghbal et al. [7] provided some Kolmogorov inequality for quadratic forms $T_n=\sum_{1\leq i<j\leq n}X_iX_j$ and weighted quadratic forms $Q_n=\sum_{1\leq i<j\leq n}a_{ij}X_iX_j$, where $\{X_i,i\geq 1\}$ is a sequence of nonnegative NSD uniformly bounded random variables. Shen et al. [8] obtained Kolmogorov-type inequality and the almost sure convergence for NSD sequences. Shen et al. [9] established some convergence properties for weighted sums of NSD random variables. Since NSD random variables are much weaker than independent random variables and NA random variables, studying the limit behavior of NSD sequence is of interest. The main purpose of the main is to study the complete convergence for weighted sums of NSD random variables.

The following concept of stochastic domination will be used in the paper.

Definition 1.5. A sequence $\{X_n,n\geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n|>x)\leq CP(|X|>x)$$

for all $x \geq 0$ and $n \geq 1$.

Throughout the paper, let $X_1, X_2, \cdots$ be a sequence of NSD random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Denote $X^+ = \max\{0, X\}$ and $X^- = \max\{0, -X\}$. Let C be a positive constant which may be different in various places.

2 Preliminary lemmas

To prove the main results of the paper, we need the following important lemmas. The first two lemmas are from Hu [3].

Lemma 2.1 (cf. Hu [3]). *Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables, and let $\{f_n, n \geq 1\}$ be a sequence of nondecreasing functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of NSD random variables.*

Lemma 2.2 (cf. Hu [3]). *Let $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ be a NSD random vector, and let $\boldsymbol{X}^* = (X_1^*, X_2^*, \cdots, X_n^*)$ be an independent vector such that X_i^* and X_i have the same distribution for each i. Then for any nondecreasing convex function f,*

$$Ef\left(\max_{1\leq k\leq n}\sum_{i=1}^{k} X_i\right) \leq Ef\left(\max_{1\leq k\leq n}\sum_{i=1}^{k} X_i^*\right).$$

Lemma 2.2 is the so called comparison theorem on moments between the NSD and independent random variables. Similar to the proof of Theorem 2 of Shao [10] and by using Lemma 2.2, Wang et al. [11] get the following Rosenthal type maximal inequality for NSD random variables.

Lemma 2.3 (cf. Wang et al. [11]). *Let $\{X_n, n \geq 1\}$ be a sequence of NSD random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p \geq 2$. Then there exists a positive constant C_p depending only on p such that*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C_p\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} E{X_i}^2\right)^{p/2}\right\}, \quad n \geq 1.$$

The last one is a basic property for stochastic domination. For the proof, one can refer to Wu [12, 13], Shen [14] or Shen and Wu [15].

Lemma 2.4 (cf. Wang et al. [11]). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. Then for any $\alpha > 0$ and $b > 0$,*

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C_1[E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)],$$
$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$.

3 Main results and their proofs

In this section, we will present some results on complete convergence for weighted sums of NSD random variables.

Theorem 3.1. *Let $\{X_n, n\geq 1\}$ be a sequence of NSD random variables, which are stochastically dominated by a random variable X with $EX_n=0$ and $E|X|^p<\infty$ for some $p>1/\alpha$ and $1/2<\alpha\leq 1$. Let $\{a_{nk}, 1\leq k\leq n, n\geq 1\}$ be an array of real numbers such that $|a_{nk}|\leq C$ for $1\leq k\leq n$ and $n\geq 1$, where C is a positive constant. Then for any $\varepsilon>0$,*

$$\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}X_k\right|>\varepsilon n^{\alpha}\right)<\infty. \tag{3.1}$$

Proof. Without loss of generality, we assume that $a_{nk}\geq 0$ (otherwise, we will use a_{nk}^{+} and a_{nk}^{-} instead of a_{nk}, respectively). For fixed $n\geq 1$, denote

$$X_{nk}=-n^{\alpha}I(X_k<-n^{\alpha})+X_kI(|X_k|\leq n^{\alpha})+n^{\alpha}I(X_k>n^{\alpha}),\quad 1\leq k\leq n.$$

Since $EX_n=0$, $|a_{nk}|\leq C$ and $E|X|^p<\infty$, we have by Lemma 2.4 that

$$\begin{aligned}n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{k=1}^{j}a_{nk}EX_{nk}\right| &\leq n^{-\alpha}\sum_{k=1}^{n}|a_{nk}|E|X_k|I(|X_k|>n^{\alpha})+C\sum_{k=1}^{n}|a_{nk}|P(|X|>n^{\alpha})\\ &\leq Cn^{-\alpha}\sum_{k=1}^{n}|a_{nk}|E|X|I(|X|>n^{\alpha})+C\sum_{k=1}^{n}|a_{nk}|P(|X|>n^{\alpha})\\ &\leq Cn^{1-p\alpha}E|X|^p+Cn^{1-p\alpha}E|X|^p\to 0,\ \text{ as }\ n\to\infty.\end{aligned}$$

Hence for all n sufficiently large, we have

$$n^{-\alpha}\max_{1\leq j\leq n}\left|\sum_{k=1}^{j}a_{nk}EX_{nk}\right|<\frac{\varepsilon}{2}, \tag{3.2}$$

which implies that

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}X_k\right|>\varepsilon n^{\alpha}\right)\\ &\leq \sum_{n=1}^{\infty}n^{p\alpha-2}\sum_{k=1}^{n}P(|X_k|>n^{\alpha})+\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}X_{nk}\right|>\varepsilon n^{\alpha}\right)\\ &\leq C\sum_{n=1}^{\infty}n^{p\alpha-1}P(|X|>n^{\alpha})+C\sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j} a_{nk}(X_{nk}-EX_{nk})\right|>\frac{\varepsilon n^{\alpha}}{2}\right)\\ &\doteq CI+CJ.\end{aligned} \tag{3.3}$$

It is easy to check that

$$\begin{aligned} I &= \sum_{j=1}^{\infty} P(j^{\alpha} < |X| \le (j+1)^{\alpha}) \sum_{n=1}^{j} n^{p\alpha-1} \\ &\le C \sum_{j=1}^{\infty} P(j^{p\alpha} < |X|^p \le (j+1)^{p\alpha}) j^{p\alpha} \le CE|X|^p < \infty. \end{aligned}$$

Thus, it remains to show that $J < \infty$.

For fixed $n \ge 1$, it is easily seen that $\{a_{nk}(X_{nk} - EX_{nk}), 1 \le k \le n\}$ are still NSD random variables with mean zero from Lemma 2.1. For any $r \ge 2$, by Markov's inequality and Lemma 2.3, we can see that

$$\begin{aligned} J &\le \left(\frac{2}{\varepsilon}\right)^r \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 2} E\left(\max_{1 \le j \le n} \left| \sum_{k=1}^{j} a_{nk}(X_{nk} - EX_{nk}) \right|^r \right) \\ &\le C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 2} \left(\sum_{k=1}^{n} a_{nk}^2 EX_{nk}^2 \right)^{r/2} + C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 2} \sum_{k=1}^{n} |a_{nk}|^r E|X_{nk}|^r \\ &\doteq CJ_1 + CJ_2. \end{aligned} \tag{3.4}$$

If $p \ge 2$, then we can take r large enough such that $r > \max\{(p\alpha - 1)/(\alpha - 1/2), p\}$. Thus, $p\alpha - r\alpha - 2 + r/2 < -1$, which implies that

$$\begin{aligned} J_1 &\le C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 2 + r/2} \left[n^{2\alpha} P(|X| > n^{\alpha}) + EX^2 I(|X| \le n^{\alpha}) \right]^{r/2} \\ &\le C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 2 + r/2} \left[EX^2 I(|X| > n^{\alpha}) + EX^2 I(|X| \le n^{\alpha}) \right]^{r/2} \\ &\le C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 2 + r/2} < \infty. \end{aligned}$$

In the first inequality above, we used the fact $|a_{nk}| \le C$ for $1 \le k \le n$ and $n \ge 1$.

Since $r > p$, we have by the fact $|a_{nk}| \le C$ and Lemma 2.4 that

$$\begin{aligned} J_2 &\le C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 1} [n^{r\alpha} P(|X| > n^{\alpha}) + E|X|^r I(|X| \le n^{\alpha})] \\ &\le CE|X|^p + C \sum_{n=1}^{\infty} n^{p\alpha - r\alpha - 1} \sum_{k=1}^{n} E|X|^r I((k-1)^{\alpha} < |X| \le k^{\alpha}) \\ &\le CE|X|^p + C \sum_{k=1}^{\infty} E|X|^r I((k-1)^{\alpha} < |X| \le k^{\alpha}) k^{p\alpha - r\alpha} \\ &\le CE|X|^p + C \sum_{k=1}^{\infty} E|X|^p I((k-1)^{\alpha} < |X| \le k^{\alpha}) \\ &\le CE|X|^p < \infty. \end{aligned} \tag{3.5}$$

If $p<2$, then we take $r=2$. Since $r=2>p$, the inequality (3.5) still holds, that is to say $J_1=J_2<\infty$, which implies that $J<\infty$. This completes the proof of the theorem. □

Theorem 3.2. *Let $\{X_n,n\geq 1\}$ be a sequence of NSD random variables satisfying*

$$P(|X_n|\geq x)\leq M\int_x^{+\infty} e^{-\gamma t^2}dt \tag{3.6}$$

for all $n\geq 1$ and all $x\geq 0$, where M and γ are positive constants. Suppose that $EX_n=0$ and $EX_n^2<\infty$ for all $n\geq 1$. Let $\{a_{nk},1\leq k\leq n,n\geq 1\}$ be an array of real numbers such that $\sum_{k=1}^n a_{nk}^2=\mathcal{O}(n^\delta)$ for some $\delta>0$. Then for all $\beta>\frac{1+\delta}{2}$ and any $\varepsilon>0$,

$$\sum_{n=1}^\infty P\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^j a_{nk}X_k\right|>\varepsilon n^\beta\right)<\infty. \tag{3.7}$$

Proof. Without loss of generality, we assume that $a_{nk}\geq 0$ (otherwise, we will use a_{nk}^+ and a_{nk}^- instead of a_{nk}, respectively). By Markov's inequality and Lemma 2.3, we have that for any $\varepsilon>0$,

$$\sum_{n=1}^\infty P\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^j a_{nk}X_k\right|>\varepsilon n^\beta\right)\leq C\sum_{n=1}^\infty \frac{1}{n^{2\beta}}\sum_{k=1}^n a_{nk}^2 EX_k^2.$$

In the following, we will estimate EX_k^2. By (3.6), we can see that

$$\begin{aligned}EX_k^2&=\int_0^{+\infty}2xP(|X_k|\geq x)dx \;\leq\; \int_0^{+\infty}2x\left(M\int_x^{+\infty}e^{-\gamma t^2}dt\right)dx\\ &=M\int_0^{+\infty}e^{-\gamma t^2}\left(\int_0^t 2xdx\right)dt \;=\; M\int_0^{+\infty}t^2e^{-\gamma t^2}dt \;=\; \frac{M\sqrt{\pi}}{4\gamma^{3/2}}.\end{aligned}$$

By the statements above and the assumption $\sum_{k=1}^n a_{ak}^2=\mathcal{O}(n^\delta)$ for some $\delta>0$, we can get that

$$\sum_{n=1}^\infty P\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^j a_{nk}X_k\right|>\varepsilon n^\beta\right)\leq C\sum_{n=1}^\infty\frac{1}{n^{2\beta-\delta}}<\infty.$$

This completes the proof of the theorem. □

Remark 3.1. Hanson and Wright [16] obtained a bound on tail probabilities for quadratic forms in independent random variables under the condition (3.6). Wright [17] proved that the bound established by Hanson and Wright [16] for independent symmetric random variables also holds when the random variables are not symmetric but condition (3.6) is valid. Here, our Theorem 3.2 studied the complete convergence for arrays of rowwise NSD random variables based on the condition (3.6).

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