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Infinitely Many Clark Type Solutions to a $p(x)$-Laplace Equation*

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Abstract. In this paper, the following $p(x)$-Laplacian equation:
$$\Delta_{p(x)}u+V(x)|u|^{p(x)-2}u=Q(x)f(x,u),\quad x\in\mathbb{R}^N,$$
is studied. By applying an extension of Clark's theorem, the existence of infinitely many solutions as well as the structure of the set of critical points near the origin are obtained.

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1 Introduction

The Clark theorem [2] is an important tool in critical point theory, which is constantly and effectively applied to sublinear differential equations with symmetry. A variant of the Clark Theorem was given by Heinz in [8].

Theorem 1.1. *Let X be a Banach space, $\Phi\in C^1(X,\mathbb{R})$. Assume that Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0)=0$. If for any $k\in\mathbb{N}$, there exists a k-dimensional subsequence X^k of X and $\rho_k>0$ such that $\sup_{X^k\cap S_{\rho_k}}\Phi<0$, where $S_\rho=\{u\in X\,|\,\|u\|=\rho\}$, then Φ has a sequence of critical values $c_k<0$ satisfying $c_k\to 0$ as $k\to\infty$.*

Theorem 1.1 asserts the existence of a sequence of critical values $c_k<0$ satisfying $c_k\to 0$ as $k\to\infty$, without giving any information on the structure of the set of critical points. A

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very interesting question arising from Theorem 1.1 is whether there are a sequence of critical points u_k such that $\Phi(u_k)\to 0$ and $\|u_k\|\to 0$ as $k\to\infty$ under the assumptions of Theorem 1.1. In [11], the authors answered this question and gave the structure of the set of critical points near the original in the abstract setting of Clark's theorem. One of the results is the following.

Theorem 1.2. *Let X be a Banach space, $\Phi\in C^1(X,\mathbb{R})$. Assume that Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0)=0$. If for any $k\in\mathbb{N}$, there exists a k-dimensional subsequence X^k of X and $\rho_k>0$ such that $\sup_{X^k\cap S_{\rho_k}}\Phi<0$, where $S_\rho=\{u\in X|\,\|u\|=\rho\}$, then at least one of the following conclusions holds.*

(i) There exist a sequence of critical points u_k satisfying $\Phi(u_k)<0$ for all k and $\|u_k\|\to 0$ as $k\to\infty$.

(ii) There exists $r>0$ such that for any $0<a<r$, there exists a critical point u such that $\|u\|=a$ and $\Phi(u)=0$.

In [11], the authors got some variants of Clark Theorem which were applied to indefinite problems such as problems on periodic solutions of first order Hamiltonian systems. And the Theorem 1.2 is applied to a p-Laplace equation on $\mathbb{R}^N$. i.e.,

$$\begin{cases} -\Delta_p u+V(x)|u|^{p-2}u=Q(x)f(x,u), & x\in\mathbb{R}^N,\\ u\in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $p>1$. Assuming

(a1) there exists $\delta>0$, $1\le\gamma<p$, $C>0$ such that $f\in C(\mathbb{R}^N\times[-\delta,\delta],\mathbb{R})$, f is odd in u, $|f(x,u)|\le C|u|^{\gamma-1}$, and $\lim_{u\to 0}F(x,u)/|u|^p=+\infty$ uniformly in some ball $B_r(x_0)\subset\mathbb{R}^N$;

(a2) $V,Q\in C(\mathbb{R}^N,\mathbb{R}^1)$, $V(x)\ge\alpha_0$ and $0<Q(x)\le\beta_0$ for some $\alpha_0>0$, $\beta_0>0$, and $M\triangleq Q^{\frac{p}{p-\gamma}}V^{\frac{-\gamma}{p-\gamma}}\in L^1(\mathbb{R}^N)$.

With conditions (a1) and (a2), equation (1.1) has infinitely many solutions u_k such that $\|u_k\|_{L^\infty}\to 0$ as $k\to\infty$.

In this paper, the following $p(x)$-Laplacian equation $(p(x)>1)$ is considered:

$$\begin{cases} -\Delta_{p(x)} u+V(x)|u|^{p(x)-2}u=Q(x)f(x,u), & x\in\mathbb{R}^N,\\ u\in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where $\Delta_{p(x)}u=\mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p(x)}-2}\nabla \mathrm{u}),\mathrm{p(x)}\in \mathrm{C}(\mathbb{R}^{\mathrm{N}})$.

When $p(x)\equiv const.$, problem (1.2) is the equation (1.1), which was studied in [11]. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is a natural generalization of the classical Sobolev space $W^{1,p}(\Omega)$. The variable exponent $p(x)$-Laplacian equations arise from nonlinear elastic mechanics (see [15]) and electrorheological fluids(see [1,12]). And the

$p(x)$-Laplacian operator possesses more complicated nonlinearities, for example, it is inhomogeneous, so in the discussion, some special techniques needed will be given in Section 2.

Following the argument in [11] and using Theorem 1.2, we get the following result.

Theorem 1.3. *Assume*

(b1) there exists $\delta>0$, $1\le\gamma<p_-$, $C>0$ *such that* $f\in C(\mathbb{R}^N\times[-\delta,\delta],\mathbb{R})$, f *is odd in* u, $|f(x,u)|\le C|u|^{\gamma-1}$, *and* $\lim_{u\to 0}F(x,u)/|u|^{p(x)}=+\infty$ *uniformly in some ball* $B_r(x_0)\subset\mathbb{R}^N$,

(b2) $V,Q\in C(\mathbb{R}^N,\mathbb{R}^1)$, $V(x)\ge\alpha_0$ *and* $0<Q(x)\le\beta_0$ *for some* $\alpha_0>0$, $\beta_0>0$, *and* $M\triangleq Q^{\frac{p(x)}{p(x)-\gamma}}V^{\frac{-\gamma}{p(x)-\gamma}}\in L^1(\mathbb{R}^N)$.

Then equation(1.2) *has infinitely many solutions* u_k *such that* $\|u_k\|_{L^\infty}\to 0$ *as* $k\to\infty$.

2 Preliminaries

Let $\Omega\subset\mathbb{R}^N(N\ge 2)$, $p(x)\in C(\Omega)$, set

$$L^{p(x)}(\Omega)=\Big\{u:\ u \text{ is measureableand real}-\text{valued},\quad \int_\Omega|u|^{p(x)}dx<\infty\Big\},$$

The space $L^{p(x)}(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_{p(x)}\triangleq\inf\Big\{\lambda>0,\ \int_\Omega\Big|\frac{u}{\lambda}\Big|^{p(x)}dx\le 1\Big\}. \tag{2.1}$$

Proposition 2.1. (see [4])

(1) The space $L^{p(x)}(\Omega)$ is a separable, uniform convex Banach space, and its conjugate space is $L^{q(x)}$, where $\frac{1}{p(x)}+\frac{1}{q(x)}=1$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have

$$\Big|\int_\Omega uvdx\Big|\le 2\|u\|_{p(x)}\|v\|_{q(x)}.$$

(2) If $p_1,p_2\in C^+(\bar\Omega)$, $p_1(x)\le p_2(x)$ for any $x\in\bar\Omega$, then $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Proposition 2.2. (see [4,9]) Set $\rho(u)=\int_\Omega|u(x)|^{p(x)}dx$. If $u,u_k\in L^{p(x)}(\Omega)$, we have

(1) for $u\ne 0$, $\|u\|_{p(x)}=\lambda\Leftrightarrow\rho(\frac{u}{\lambda})=1$;

(2) $\|u\|_{p(x)}<1(=1;>1)\Leftrightarrow\rho(u)<1(=1,>1)$;

(3) if $\|u\|_{p(x)}>1$, then $\|u\|_{p(x)}^{p_-}\le\rho(u)\le\|u\|_{p(x)}^{p_+}$;

(4) if $\|u\|_{p(x)}<1$, then $\|u\|_{p(x)}^{p_+}\le\rho(u)\le\|u\|_{p(x)}^{p_-}$.

The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega)\triangleq\Big\{u\in L^{p(x)}(\Omega)\,\big|\,|\nabla u|\in L^{p(x)}(\Omega)\Big\}$$

and equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} \triangleq \|u\|_{p(x)} + ||\nabla u||_{p(x)}.$$

We denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ by $W_0^{1,p(x)}(\Omega)$ and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ \infty, & p(x) \geq N. \end{cases} \tag{2.2}$$

$$1 < p_- \triangleq \inf_{x\in\Omega} p(x) \leq p_+ \triangleq \sup_{x\in\Omega} p(x). \tag{2.3}$$

Proposition 2.3. (see [4])

(1) $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive Banach spaces.

(2) If $q \in C^+(\bar{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{p(x)}(\Omega)$ is compact and continuous.

(3) There is a constant $C > 0$ such that

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

Proposition 2.4. (see [5]) Define $I(u) = \int (|\nabla u|^{p(x)} + a(x)|u|^{p(x)})$, where $0 < a_- < a(x) < \infty$, use the norm

$$\|u\| = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N} \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + a(x)\left|\frac{u}{\lambda}\right|^{p(x)} \right) \leq 1 \right\}.$$

Let $u \in W^{1,p(x)}(\Omega)$, then

(*i*) $\|u\| < 1 (=1, >1) \Leftrightarrow I(u) < 1 (=1, >1)$;

(*ii*) if $\|u\| \geq 1$, then $\|u\|^{p_-} \leq I(u) \leq \|u\|^{p_+}$;

(*iii*) if $\|u\| \leq 1$, then $\|u\|^{p_+} \leq I(u) \leq \|u\|^{p_-}$;

(*iv*) $I(u_n) \to 0 \Leftrightarrow \|u_n\| \to 0$;

(*v*) $I(u_n) \to \infty \Leftrightarrow \|u_n\| \to \infty$.

Definition 2.1. $u \in W^{1,p(x)}(\mathbb{R}^N)$ is called a weak solution of problem (1.2) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla \phi dx + V(x)|u|^{p(x)-2} u\phi dx = \int_{\mathbb{R}^N} Q(x) f(x,u)\phi dx, \quad \forall \phi \in W^{1,p(x)}(\mathbb{R}^N).$$

Set

$$F(x,u) = \int_0^u f(x,s)ds, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R} \tag{2.4}$$

and

$$\Phi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u|^{p(x)} + V(x)|u|^{p(x)}) - \int_{\mathbb{R}^N} Q(x) F(x,u), \quad u \in W^{1,p(x)}(\mathbb{R}^N). \tag{2.5}$$

We know that the critical points of Φ are just the weak solutions of problem (1.2).

3 Proof of Theorem 1.1

Proof. We prove it in three steps:
Step 1. Construct proper functional and get the coerciveness. Choose $\hat{f}\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ so that $\hat{f}$ is odd in $u\in\mathbb{R}$, $\hat{f}(x,u)=f(x,u)$ for $x\in\mathbb{R}$ and $|u|<\delta/2$, and $\hat{f}(x,u)=0$ for $x\in\mathbb{R}^N$ and $|u|>\delta$. In order to obtain solutions of (1.2) we study

$$\begin{cases} -\Delta_{p(x)}u+V(x)|u|^{p(x)-2}u=Q(x)\hat{f}(x,u), & x\in\mathbb{R}^N,\\ u\in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{3.1}$$

which is the Euler equation of the functional

$$\Phi(u)=\int_{\mathbb{R}^N}\frac{1}{p(x)}(|\nabla u|^{p(x)}+V(x)|u|^{p(x)})-\int_{\mathbb{R}^N}Q(x)\hat{F}(x,u),\quad u\in X,$$

where X is the Banach space

$$X=\{u\in W^{1,p(x)}(\mathbb{R}^N)|\int_{\mathbb{R}^N}V(x)|u|^{p(x)}<\infty\}$$

endowed with the norm (see [5])

$$\|u\|=\inf\left\{\lambda>0:\int_{\mathbb{R}^N}\left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)}+V(x)\left|\frac{u}{\lambda}\right|^{p(x)}\right)\le 1\right\}. \tag{3.2}$$

and $\hat{F}(x,u)=\int_0^u\hat{f}(x,s)ds$. It is standard to check that $\Phi\in C^1(X,\mathbb{R})$, Φ is even, and $\Phi(0)=0$. For $u\in X$,

$$\begin{aligned}\int_{\mathbb{R}^N}Q(x)|\hat{F}(x,u)|&\le C_1\int_{\mathbb{R}^N}Q(x)|u|^{\gamma}=C_1\int_{\mathbb{R}^N}(QV^{-\frac{\gamma}{p(x)}})(V^{\frac{\gamma}{p(x)}}|u|^{\gamma})\\
&\le 2C_1\|QV^{-\frac{\gamma}{p(x)}}\|_{\frac{p(x)}{p(x)-\gamma}}\cdot\|V^{\frac{\gamma}{p(x)}}|u|^{\gamma}\|_{\frac{p(x)}{\gamma}} &(3.3)\\
&\le 2C_1\left(\int_{\mathbb{R}^N}|QV^{-\frac{\gamma}{p(x)}}|^{\frac{p(x)}{p(x)-\gamma}}\right)^{\frac{p_+-\gamma}{p_+}}\cdot\left(\int_{\mathbb{R}^N}|V^{\frac{\gamma}{p(x)}}|u|^{\gamma}|^{\frac{p(x)}{\gamma}}\right)^{\frac{\gamma}{p_+}} &(3.4)\\
&\le 2C_1\|M\|_{L^1(\mathbb{R}^N)}^{\frac{p_+-\gamma}{p_+}}\left(\int_{\mathbb{R}^N}|V|u|^{p(x)}\right)^{\frac{\gamma}{p_+}} &(3.5)\\
&\le C_2\|u\|^{\gamma}. &(3.6)\end{aligned}$$

Therefore,

$$\Phi(u)\ge\frac{1}{p_+}\|u\|^{p_-}-C_2\|u\|^{\gamma},\quad u\in X,$$

and then Φ is coercive and bounded below.

Step 2. The (PS) condition holds. Let $\{u_n\}$ be a (PS) sequence, thus $\Phi(u_n)$ is bounded and $\Phi'(u_n)\to 0$. Then $\{u_n\}$ is bounded. Assume $u_n \rightharpoonup u$ in $W^{1,p(x)}(\mathbb{R}^N)$ and strongly in $L^{p(x)}_{\rm loc}(\mathbb{R}^N)$. Then $\Phi'(u)\to 0$, and

$$\begin{aligned}\langle \Phi'(u_n)-\Phi'(u),u_n-u\rangle =& [\int_{\mathbb{R}^N}|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u)\cdot(\nabla u_n-\nabla u\\ &+\int_{\mathbb{R}^N}(V(x)|u_n|^{p(x)-2}u_n-|u|^{p(x)-2}u)\cdot(u_n-u)]\\ &-\int_{\mathbb{R}^N}Q(x)(\hat{f}(x,u_n)\hat{f}(x,u))\cdot(u_n-u)\\ &\triangleq I_1-I_2\to 0 \end{aligned}\tag{3.7}$$

For $\xi,\eta\in\mathbb{R}^N$, using the monotonous inequalities (see [6]),

$$(|\xi|^{p(x)-2}-|\eta|^{p(x)-2})\cdot(\xi-\eta)\geq\left\{\begin{array}{ll} c(|\xi|+|\eta|)^{p(x)-2}|\xi-\eta|^2, & 1<p(x)<2,\\ c|\xi-\eta|^{p(x)}, & p(x)\geq 2,\end{array}\right.\tag{3.8}$$

where $c>0$ is a constant.

For $p(x)\geq 2$, it is easy to see that

$$I_1\geq C_3\|u_n-u\|^{p_+}_{p(x)}.\tag{3.9}$$

In the case $1<p(x)<2$, for any $v,w\in L^{p(x)}(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|w|^{p(x)}&=\int_{\mathbb{R}^N}(|w|^{p(x)}|v|^{\frac{p(x)}{p(x)-2}})|v|^{\frac{p(x)}{2-p(x)}}\\ &\leq 2\|\,|w|^{p(x)}|v|^{\frac{p(x)}{p(x)-2}}\|_{\frac{2}{p(x)}}\cdot\|\,|v|^{\frac{p(x)}{2-p(x)}}\|_{\frac{2}{2-p(x)}}\\ &\leq 2\left[\int_{\mathbb{R}^N}(|w|^{p(x)}|v|^{\frac{p(x)}{p(x)-2}})^{\frac{2}{p(x)}}\right]^{\frac{p_+}{2}}\cdot\left[\int_{\mathbb{R}^N}|v|^{p(x)}\right]^{\frac{2-p_-}{2}}\\ &=2\left(\int_{\mathbb{R}^N}|v|^{p(x)-2}|w|^2\right)^{\frac{p_+}{2}}\cdot\left(\int_{\mathbb{R}^N}|v|^{p(x)}\right)^{\frac{2-p_-}{2}}\end{aligned}$$

Thus

$$\int_{\mathbb{R}^N}|v|^{p(x)-2}|w|^2\geq\frac{(\int_{\mathbb{R}^N}|w|^{p(x)})^{\frac{2}{p_+}}}{2^{\frac{2}{p_+}}(\int_{\mathbb{R}^N}|v|^{p(x)})^{\frac{2-p_-}{p_+}}}\geq\frac{\|w\|^2_{p(x)}}{2^{\frac{2}{p_+}}(\int_{\mathbb{R}^N}|v|^{p(x)})^{\frac{2-p_-}{p_+}}}\tag{3.10}$$

Substitute $w=|u_n-u|$, $v=|u_n|+|u|$ to (3.8), (3.10), we have for $1<p(x)<2$,

$$I_1\geq C_4\|u_n-u\|^2_{p(x)}.\tag{3.11}$$

Now we estimate I_2, for any $R>0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} Q(x)(\hat{f}(x,u_n)-\hat{f}(x,u))\cdot(u_n-u)\\
\le& C_5\int_{\mathbb{R}^N\setminus B_R(0)} Q(x)(|u_n|^\gamma+|u|^\gamma)+C_5\int_{B_R(0)}(|u_n|^{\gamma-1}+|u|^{\gamma-1})(u_n-u)\\
=& C_5\int_{\mathbb{R}^N\setminus B_R(0)}(V^{\frac{\gamma}{p(x)}}|u_n|^\gamma)(QV^{\frac{-\gamma}{p(x)}})+C_5\int_{\mathbb{R}^N\setminus B_R(0)}(V^{\frac{\gamma}{p(x)}}|u|^\gamma)(QV^{\frac{-\gamma}{p(x)}})\\
&+C_5\int_{B_R(0)}|u_n|^{\gamma-1}|u_n-u|+C_5\int_{B_R(0)}|u|^{\gamma-1}|u_n-u|\\
\le& 2C_5\left(\|V^{\frac{\gamma}{p(x)}}|u_n|^\gamma\|_{L^{\frac{p(x)}{\gamma}}(\mathbb{R}^N\setminus B_R(0))}+\|V^{\frac{\gamma}{p(x)}}|u|^\gamma\|_{L^{\frac{p(x)}{\gamma}}(\mathbb{R}^N\setminus B_R(0))}\right)\|QV^{\frac{-\gamma}{p(x)}}\|_{\frac{p(x)-\gamma}{\gamma}} \qquad (3.12)\\
&+C_5\left(\|u_n\|^{\gamma-1}_{L^\gamma(B_R(0))}+\|u\|^{\gamma-1}_{L^\gamma(B_R(0))}\right)\|u_n-u\|_{L^\gamma(B_R(0))}\\
\le& 2C_5\left[\left(\int_{\mathbb{R}^N\setminus B_R(0)}(V|u_n|^{p(x)})\right)^{\frac{\gamma}{p_+}}+\left(\int_{\mathbb{R}^N\setminus B_R(0)}(V|u|^{p(x)})\right)^{\frac{\gamma}{p_+}}\right]\|M\|^{\frac{p_+-\gamma}{p_+}}_{L^1(\mathbb{R}^N\setminus B_R(0))} \qquad (3.13)\\
&+C_5\left(\|u_n\|^{\gamma-1}_{L^\gamma(B_R(0))}+\|u\|^{\gamma-1}_{L^\gamma(B_R(0))}\right)\|u_n-u\|_{L^\gamma(B_R(0))}\\
\le& C_6\|M\|^{\frac{p_+-\gamma}{p_+}}_{L^1(\mathbb{R}^N\setminus B_R(0))}+C_6\|u_n-u\|_{L^\gamma(B_R(0))}. \qquad (3.14)
\end{aligned}
$$

where we used Hölder inequality (Proposition 2.1) in (3.12), Proposition 2.2 in (3.13), and the definition of X in (3.14). By the condition (b2), we have $\|M\|_{L^1(\mathbb{R}^N\setminus B_R(0))}\to 0$ as $R\to\infty$. $L^{p(x)}\hookrightarrow L^\gamma$ leads to $\|u_n-u\|_{L^\gamma(B_R(0))}\to 0$. Thus

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)(\hat{f}(x,u_n)-\hat{f}(x,u))\cdot(u_n-u). \qquad (3.15)$$

Combining with(3.7), (3.9), (3.11) and (3.15), $\{u_n\}$ converges strongly in X and the (PS) condition holds for Φ.

Step 3. Equation (1.2) has infinitely many Clark type solutions. For any $K>0$, there exists $\delta=\delta(K)>0$ such that if $u\in C_0^\infty(B_r(x_0))$ and $|u|_\infty<\delta$, then $Q(x)\hat{F}(x,u)\ge K|u(x)|^{p(x)}$, and thus

$$\Phi(u)\le\frac{1}{p_-}\|u\|^{p_-}-K\|u\|^{p_+}_{p(x)}.$$

This implies, for any $k\in\mathbb{N}$, if X^k is a k-dimensional subsequence of $C_0^\infty(B_r(x_0))$ and $\rho_k>0$ is sufficiently small, then $\sup_{X^k\cap S_{\rho_k}}\Phi<0$, where $S_\rho=\{u\in X:\|u\|=\rho\}$. Now we appeal to Theorem 1.2 to obtain infinitely many solutions $\{u_k\}$ for (3.1) such that $\|u_k\|\to 0$ as $k\to\infty$.

In the following, we show that $\|u_k\|_{L^\infty}\to 0$ as $k\to\infty$. In the case $1<p(x)<N$, denote $p*=\frac{Np(x)}{N-p(x)}$. Let u be a solution of (3.1) and constant $\alpha>0$. Let $T>0$ and set $u^T(x)=\max\{-T,\min\{u(x),T\}\}$. Multiplying both sides of (3.1) with $|u^T|^\alpha u^T$ implies

$$\int_{\mathbb{R}^N}|u^T|^\alpha|\nabla u^T|^{p(x)}=\int_{\mathbb{R}^N}\left(\frac{p(x)}{\alpha+p(x)}\right)^{p(x)}|\nabla|u^T|^{\frac{\alpha+p(x)}{p(x)}}|^{p(x)}\le C\int_{\mathbb{R}^N}|u^T|^{\alpha+1}. \qquad (3.16)$$

Since

$$\left(\int_{\mathbb{R}^N}|u^T|^{\frac{\alpha+p(x)}{p(x)}\frac{Np(x)}{N-p(x)}}\right)^{\frac{N-p_-}{Np_-}}\leq C\||u^T|^{\frac{\alpha+p(x)}{p(x)}}\|_{L^{\frac{Np(x)}{N-p(x)}}(\mathbb{R}^N)} \tag{3.17}$$

and

$$\|\nabla|u^T|^{\frac{\alpha+p(x)}{p(x)}}\|_{L^{p(x)}(\mathbb{R}^N)}\leq\left(\int_{\mathbb{R}^N}|\nabla|u^T|^{\frac{\alpha+p(x)}{p(x)}}|^{p(x)}\right)^{\frac{1}{p_+}}. \tag{3.18}$$

Combining (3.16)-(3.18) with Sobolev inequality, we have

$$\|u^T\|^{\frac{N(\alpha+p_-)}{N-p_-}\frac{N-p_-}{Np_-}}_{L^{\frac{N(\alpha+p_-)}{N-p_-}}(\mathbb{R}^N)}\leq\left(\int_{\mathbb{R}^N}|u^T|^{\frac{N(\alpha+p(x))}{N-p(x)}}\right)^{\frac{N-p_-}{Np_-}}\leq C\left(\int_{\mathbb{R}^N}|\nabla|u^T|^{\frac{\alpha+p(x)}{p(x)}}|^{p(x)}\right)^{\frac{1}{p_+}} \tag{3.19}$$

$$\leq C\left(\frac{\alpha+p(x)}{p(x)}\right)^{\frac{p(x)}{p_+}}\left(\int_{\mathbb{R}^N}|u^T|^{\alpha+1}\right)^{\frac{1}{p_+}}$$

$$\leq(C(\alpha+p(x)))^{\frac{p(x)}{p_+}}\|u^T\|^{\frac{\alpha+1}{p_+}}_{L^{\alpha+1}}. \tag{3.20}$$

Thus

$$\|u^T\|_{L^{\frac{N(\alpha+p_-)}{N-p_-}}(\mathbb{R}^N)}\leq(C(\alpha+p(x)))^{\frac{p(x)}{p_+}\frac{(N-p_+)Np_-}{N(\alpha+p_+)(N-p_-)}}\|u\|^{\frac{\alpha+1}{p_+}\frac{(N-p_+)p_-}{(\alpha+p_+)(N-p_-)}}_{L^{\alpha+1}(\mathbb{R}^N)}$$

$$\leq(C(\alpha+p_+))^{\frac{p_+}{\alpha+p_+}}\|u\|^{\frac{\alpha+1}{\alpha+p_+}\frac{(N-p_+)p_-}{(N-p_-)p_+}}_{L^{\alpha+1}(\mathbb{R}^N)}. \tag{3.21}$$

where $C\geq 1$, independent of u and α. Set $\alpha_0=p_-^*-1=\frac{Np_-}{N-p_-}-1$ and $\alpha_k=\frac{(\alpha_{k-1}+p_-)N}{N-p_-}-1$, that is $\alpha_k=\frac{(p_-^*/p_-)^{k+1}-1}{(p_-^*/p_-)-1}\alpha_0$, for $k=1,2,\cdots$. From (3.21), an iterating process leads to

$$\|u^T\|_{L^{\alpha_{k+1}+1}(\mathbb{R}^N)}\leq\exp\left(\Sigma_{i=0}^k\frac{p_-\ln(C(\alpha_i+p_-))}{\alpha_i+p_-}\right)\|u^T\|^{\nu^k}_{L^{p_-^*}(\mathbb{R}^N)}, \tag{3.22}$$

where $\nu^k=\Pi_{i=0}^k\frac{\alpha_i+1}{\alpha_i+p_+}\frac{(N-p_+)p_-}{(N-p_-)p_+}$. Sending T to infinity and then k to infinity, as a consequence, we have

$$\|u^T\|_{L^\infty(\mathbb{R}^N)}\leq\exp\left(\Sigma_{i=0}^\infty\frac{p_-\ln(C(\alpha_i+p_-))}{\alpha_i+p_-}\right)\|u^T\|^{\nu}_{L^{p_-^*}(\mathbb{R}^N)}, \tag{3.23}$$

where $\nu=\Pi_{i=0}^\infty\frac{\alpha_i+1}{\alpha_i+p_+}\frac{(N-p_+)p_-}{(N-p_-)p_+}$ is a number in $(0,1)$ and $\exp\left(\Sigma_{i=0}^k\frac{p_-\ln(C(\alpha_i+p_-))}{\alpha_i+p_-}\right)$ is a positive number. For the case $p(x)\geq N$ and $p^*=\infty$, the argument is similar and even simpler. Therefore, $\|u_k\|_{L^\infty}\to 0$ as $k\to\infty$, and u_k with k sufficiently large are solutions of (1.2).

$\square$

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