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Properties of Convergence for a Class of Generalized q-Gamma Operators

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Abstract. In this paper, a generalization of q-Gamma operators based on the concept of q-integer is introduced. We investigate the moments and central moments of the operators by computation, obtain a local approximation theorem and get the pointwise convergence rate theorem and also obtain a weighted approximation theorem. Finally, a Voronovskaya type asymptotic formula was given.

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1 Introduction

It is well known that the Gamma operators are given by

$$G_n(f;x)=\frac{1}{x^n\Gamma(n)}\int_0^\infty f(t/n)t^{n-1}e^{-t/x}dt,\ \ x\in[0,\infty). \tag{1.1}$$

In 2005, Zeng [9] obtained the approximation properties of G_n defined above, supposing f satisfies exponential growth condition. He studied the approximation properties to the locally bounded functions and the absolutely continuous functions and obtained some good properties.

Since the application of q-calculus in approximation theory is an active field, many researchers have performed studies in it, we mention some of them [3,5–8], these motivate us to introduce the q analogue of this kind of Gamma operators.

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Firstly, we recall some concepts of q-calculus. All of the results can be found in [4]. For any fixed real number $0<q\leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q=\begin{cases}\frac{1-q^k}{1-q}, & q\neq 1;\\ k, & q=1.\end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q!=\begin{cases}[k]_q[k-1]_q...[1]_q, & k=1,2,...;\\ 1, & k=0,\end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!},\quad (n\geq k\geq 0).$$

The q-improper integrals are defined as

$$\int_0^{\infty/A} f(x)d_qx=(1-q)\sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right)\frac{q^n}{A},\ A>0, \tag{1.2}$$

provided the sums converge absolutely.

The q-exponential function $E_q(x)$ is given as

$$E_q(x)=\sum_{k=0}^{\infty} q^{k(k-1)/2}\frac{x^k}{[k]_q!}=(1+(1-q)x)_q^{\infty},\ |q|<1,$$

where $(1-x)_q^{\infty}=\prod_{j=0}^{\infty}(1-q^jx)$.

The q-Gamma integral is defined as

$$\Gamma_q(t)=\int_0^{\infty/A} x^{t-1}E_q(-qx)d_qx,\ t>0, \tag{1.3}$$

which satisfies the following functional equations: $\Gamma_q(t+1)=[t]_q\Gamma_q(t),\ \Gamma_q(1)=1$.

For $f\in C[0,\infty)$, $q\in(0,1)$ and $n\in\mathbb{N}$, we introduce a generalization of q-Gamma operators $G_{n,q}(f,x)$ as

$$G_{n,q}(f;x)=\frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/A} f\left(\frac{t}{[n]_q}\right)t^{n-1}E_q\left(-\frac{qt}{x}\right)d_qt. \tag{1.4}$$

Obviously, $G_{n,q}(f;x)$ are positive linear operators. It is observed that for $q\to 1^-$, $G_{n,1^-}(f;x)$ become the Gamma operators defined in (1.1).

2 Some preliminary results

In this section, we give the following lemmas, which are need to prove our theorems:

Lemma 2.1. *For $q\in(0,1)$, $x\in[0,\infty)$ and $k=0,1,...$, we have*

$$G_{n,q}(t^k;x)=\frac{[n+k-1]_q!}{[n-1]_q![n]_q^k}x^k. \tag{2.1}$$

Proof. From (1.3) and (1.4), we have

$$\begin{aligned}G_{n,q}(t^k;x)&=\frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/A}\left(\frac{t}{[n]_q}\right)^k t^{n-1}E_q\left(-\frac{qt}{x}\right)d_qt\\&=\frac{x^k}{[n]_q^k\Gamma_q(n)}\int_0^{\infty/A}\left(\frac{t}{x}\right)^{n+k-1}E_q\left(-\frac{qt}{x}\right)d_q\left(\frac{t}{x}\right)\\&=\frac{\Gamma_q(n+k)x^k}{[n]_q^k[n-1]_q!}=\frac{[n+k-1]_q!}{[n-1]_q![n]_q^k}x^k.\end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.2. *For $q\in(0,1)$, $x\in[0,\infty)$, we have*

$$G_{n,q}(1;x)=1,\ \ G_{n,q}(t;x)=x,\ \ G_{n,q}\left(t^2;x\right)=\left(1+\frac{q^n}{[n]_q}\right)x^2, \tag{2.2}$$

$$G_{n,q}\left(t^3;x\right)=\left[1+\frac{q^n(2+q)}{[n]_q}+\frac{[2]_qq^{2n}}{[n]_q^2}\right]x^3, \tag{2.3}$$

$$G_{n,q}\left(t^4;x\right)=\left[1+\frac{(1+[2]_q+[3]_q)q^n}{[n]_q}+\frac{([2]_q+[3]_q+[2]_q[3]_q)q^{2n}}{[n]_q^2}+\frac{[2]_q[3]_qq^{3n}}{[n]_q^3}\right]x^4. \tag{2.4}$$

Proof. From Lemma 2.1, we get (2.2) easily. Next,

$$\begin{aligned}G_{n,q}\left(t^3;x\right)&=\frac{[n+2]_q!}{[n-1]_q![n]_q^3}x^3=\frac{[n+2]_q[n+1]_q}{[n]_q^2}x^3=\frac{\left([n]_q+[2]_qq^n\right)\left([n]_q+q^n\right)}{[n]_q^2}x^3\\&=\frac{[n]_q^2+\left([2]_q+1\right)q^n[n]_q+[2]_qq^{2n}}{[n]_q^2}x^3=\left[1+\frac{q^n(2+q)}{[n]_q}+\frac{[2]_qq^{2n}}{[n]_q^2}\right]x^3.\end{aligned}$$

Finally,

$$\begin{aligned}G_{n,q}\left(t^4;x\right)&=\frac{[n+3]_q!}{[n-1]_q![n]_q^4}x^4=\frac{[n+3]_q[n+2]_q[n+1]_q}{[n]_q^3}x^4\\&=\frac{\left([n]_q+[3]_qq^n\right)\left([n]_q+[2]_qq^n\right)\left([n]_q+q^n\right)}{[n]_q^3}x^4\\&=\left[1+\frac{(1+[2]_q+[3]_q)q^n}{[n]_q}+\frac{\left([2]_q+[3]_q+[2]_q[3]_q\right)q^{2n}}{[n]_q^2}+\frac{[2]_q[3]_qq^{3n}}{[n]_q^3}\right]x^4.\end{aligned}$$

Lemma 2.2 is proved. □

Remark 2.1. Let $n\in\mathbb{N}$ and $x\in[0,\infty)$, then for every $q\in(0,1)$, by Lemma 2.2, we have

$$G_{n,q}(1+t;x)=1+x. \tag{2.5}$$

Lemma 2.3. *For every $q\in(0,1)$ and $x\in[0,\infty)$, we have*

$$G_{n,q}\left((t-x)^2;x\right)=\frac{q^n}{[n]_q}x^2, \tag{2.6}$$

$$G_{n,q}\Big((t-x)^4;x\Big)=\frac{q^n(1-q)^2}{[n]_q}x^4+\frac{q^{2n}\left(q^3+3q^2-1\right)}{[n]_q^2}x^4+\frac{q^{3n}[2]_q[3]_q}{[n]_q^3}x^4. \tag{2.7}$$

Proof. Since $G_{n,q}\left((t-x)^2;x\right)=G_{n,q}\left(t^2;x\right)-2xG_{n,q}(t;x)+x^2$ and $G_{n,q}\left((t-x)^4;x\right)=G_{n,q}\left(t^4;x\right)-4xG_{n,q}\left(t^3;x\right)+6x^2G_{n,q}\left(t^2;x\right)-4x^3G_{n,q}(t;x)+x^4$, and from Lemma 2.2, We get Lemma 2.3 easily. □

Remark 2.2. Let the sequence $q=\{q_n\}$ satisfies $q_n\in(0,1)$ and $q_n\to 1$ as $n\to\infty$, then for any fixed $x\in[0,\infty)$, by Lemma 2.3, we have

$$\lim_{n\to\infty}L_{n,q_n}\left((t-x)^2;x\right)=0,\ \ \lim_{n\to\infty}[n]_q\sqrt{L_{n,q_n}\left((t-x)^4;x\right)}=O(1). \tag{2.8}$$

3 Local approximation

In this section, we establish direct and local approximation theorems in connection with the operators $L_{n,q}(f;x)$.

We denote the space of all real valued continuous bounded functions f defined on the interval $[0,\infty)$ by $C_B[0,\infty)$. The norm $||\cdot||$ on the space $C_B[0,\infty)$ is given by $||f||=\sup\{|f(x)|:x\in[0,\infty)\}$.

Further let us consider Peetre's $K-$functional:

$$K_2(f;\delta)=\inf_{g\in W^2}\{||f-g||+\delta||g''||\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$.

For $f\in C_B[0,\infty)$, the modulus of continuity of second order is defined by

$$\omega_2(f;\delta)=\sup_{0<h\le\delta}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|,$$

by [1, p.177], there exists an absolute constant $C>0$ such that

$$K_2(f;\delta)\le C\omega_2\left(f;\sqrt{\delta}\right),\ \ \delta>0. \tag{3.1}$$

Our first result is a direct local approximation theorem for the operators $G_{n,q}(f;x)$.

Theorem 3.1. *For $q\in(0,1)$, $x\in[0,\infty)$, $n\in\mathbb{N}$ and $f\in C_B[0,\infty)$, we have*

$$|L_{n,q}(f,x)-f(x)|\leq C\omega_2\left(f;\frac{q^{n/2}}{\sqrt{[n]_q}}x\right), \tag{3.2}$$

where C is a positive constant.

Proof. Let $g\in W^2$, by Taylor's expansion, we have

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t(t-u)g''(u)du,\ \ x,t\in[0,\infty).$$

Using (2.5), we get

$$G_{n,q}(g;x)=g(x)+G_{n,q}\left(\int_x^t(t-u)g''(u)du;x\right).$$

Hence, we have

$$\begin{aligned}\big|G_{n,q}(g;x)-g(x)\big|=&\left|G_{n,q}\left(\int_x^t(t-u)g''(u)du;x\right)\right|\leq G_{n,q}\left(\left|\int_x^t(t-u)|g''(u)|du\right|;x\right)\\ \leq&\frac{q^n}{[n]_q}x^2||g''||.\end{aligned} \tag{3.3}$$

On the other hand, using Lemma 2.2, we have

$$\big|G_{n,q}(f;x)\big|\leq\frac{1}{x^n\Gamma_q(n)}\int_0^{\infty/A}\left|f\left(\frac{t}{[n]_q}\right)\right|t^{n-1}E_q\left(-\frac{qt}{x}\right)d_qt\leq||f||. \tag{3.4}$$

Now (3.3) and (3.4) imply

$$\begin{aligned}|G_{n,q}(f;x)-f(x)|\leq&|G_{n,q}(f-g;x)-(f-g)(x)|+|G_{n,q}(g;x)-g(x)|\\ \leq&2||f-g||+\frac{q^n}{[n]_q}x^2||g''||.\end{aligned}$$

Hence taking infimum on the right hand side over all $g\in W^2$, we get

$$|G_{n,q}(f;x)-f(x)|\leq 2K_2\left(f;\frac{q^n}{[n]_q}x^2\right).$$

By (3.1), for every $q\in(0,1)$, we have

$$|L_{n,q}(f,x)-f(x)|\leq C\omega_2\left(f;\frac{q^{n/2}}{\sqrt{[n]_q}}x\right).$$

This completes the proof of Theorem 3.1. □

4 Rate of convergence

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$, satisfying the condition $|f(x)|\leq M_f(1+x^2)$, where M_f is a constant depending only on f. We denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f\in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $||f||_{x^2}=\sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$. We denote the usual modulus of continuity of f on the closed interval $[0,a],(a>0)$ by

$$\omega_a(f,\delta)=\sup_{|t-x|\leq\delta}\sup_{x,t\in[0,a]}|f(t)-f(x)|.$$

Obviously, for the function $f\in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 4.1. *Let $f\in C_{x^2}[0,\infty)$, $q\in(0,1)$ and $\omega_{a+1}(f,\delta)$ be the modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$. Then we have*

$$||G_{n,q}(f)-f||_{C[0,a]}\leq 4M_f\left(1+a^2\right)\frac{q^na^2}{[n]_q}+2\omega_{a+1}\left(f,\frac{q^{n/2}a}{\sqrt{[n]_q}}\right). \tag{4.1}$$

Proof. For $x\in[0,a]$ and $t>a+1$, we have $t-x\geq t-a>1$. Hence $(t-x)^2>1$. Thus $2+3x^2+2(t-x)^2\leq(2+3x^2)(t-x)^2+2(t-x)^2=(4+3x^2)(t-x)^2\leq(4+3a^2)(t-x)^2\leq4(1+a^2)(t-x)^2$. Hence, we obtain

$$|f(t)-f(x)|\leq 4M_f(1+a^2)(t-x)^2. \tag{4.2}$$

For $x\in[0,a]$ and $t\leq a+1$, we have

$$|f(t)-f(x)|\leq\omega_{a+1}(f;|t-x|)\leq\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta),\ \ \delta>0. \tag{4.3}$$

From (4.2) and (4.3), we get

$$|f(t)-f(x)|\leq 4M_f(1+a^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta). \tag{4.4}$$

For $x\in[0,a]$ and $t\geq0$, by Schwarz's inequality and Lemma 2.3, we have

$$\begin{aligned}&|G_{n,q}(f;x)-f(x)|\\&\leq G_{n,q}(|f(t)-f(x)|;x)\\&\leq 4M_f(1+a^2)G_{n,q}\left((t-x)^2;x\right)+\omega_{a+1}(f;\delta)\left[1+\frac{1}{\delta}\sqrt{G_{n,q}((t-x)^2;x)}\right]\\&\leq 4M_f\left(1+a^2\right)\frac{q^na^2}{[n]_q}+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\frac{q^{n/2}a}{\sqrt{[n]_q}}\right),\end{aligned}$$

by taking $\delta=\frac{q^{n/2}a}{\sqrt{[n]_q}}$, we get the assertion of Theorem 4.1. □

5 Weighted approximation and Voronovskaya type asymptotic formula

Now we will discuss the weighted approximation theorems.

Theorem 5.1. *Let the sequence $q=\{q_n\}$ satisfies $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$, for $f\in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty}||G_{n,q_n}(f)-f||_{x^2}=0. \tag{5.1}$$

Proof. By using the Korovkin theorem in [2], we see that it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}||G_{n,q_n}(t^v;x)-x^v||_{x^2},\ \ v=0,1,2. \tag{5.2}$$

Since $G_{n,q_n}(1;x)=1$ and $G_{n,q_n}(t;x)=x$, (5.2) holds true for $v=0$ and $v=1$.
Finally, for $v=2$, we have

$$||G_{n,q_n}(t^2;x)-x^2||_{x^2}=\sup_{x\in[0,\infty)}\frac{|G_{n,q_n}(t^2;x)-x^2|}{1+x^2}\le\frac{q^n}{[n]_q}\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\le\frac{q^n}{[n]_q},$$

since $\lim_{n\to\infty}q_n=1$, we get $\lim_{n\to\infty}\frac{q_n^n}{[n]_{q_n}}=0$, so the third condition of (5.2) holds for $v=2$ as $n\to\infty$, then the proof of Theorem 5.1 is completed. □

Finally, we give a Voronovskaya type asymptotic formula for $G_{n,q}(f;x)$ by means of the second and fourth central moments.

Theorem 5.2. *Let the sequence $q=\{q_n\}$ satisfies $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Then for $f\in C^2_{x^2}[0,\infty)$ and fix $x\in[0,\infty)$, the following equality holds*

$$\lim_{n\to\infty}[n]_q\left(G_{n,q}(f;x)-f(x)\right)=\frac{f''(x)}{2}x^2. \tag{5.3}$$

Proof. Let $x\in[0,\infty)$ be fixed. By the Taylor formula, we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t;x)(t-x)^2, \tag{5.4}$$

where $r(t;x)$ is the Peano form of the remainder, $r(t;x)\in C_{x^2}[0,\infty)$, using L'Hopital's rule, we have

$$\begin{aligned}\lim_{t\to x}r(t;x)&=\lim_{t\to x}\frac{f(t)-f(x)-f'(x)(t-x)-\frac{1}{2}f''(x)(t-x)^2}{(t-x)^2}\\&=\lim_{t\to x}\frac{f'(t)-f'(x)-f''(x)(t-x)}{2(t-x)}=\lim_{t\to x}\frac{f''(t)-f''(x)}{2}=0.\end{aligned}$$

Applying $G_{n,q}(f;x)$ to (5.4), we obtain

$$[n]_q\left(G_{n,q}(f;x)-f(x)\right)=\frac{f''(x)}{2}[n]_qG_{n,q}\left((t-x)^2;x\right)+[n]_qG_{n,q}\left(r(t;x)(t-x)^2;x\right).$$

By the Cauchy-Schwarz inequality, we have

$$G_{n,q}\left(r(t;x)(t-x)^2;x\right)\le\sqrt{G_{n,q}(r^2(t;x);x)}\sqrt{G_{n,q}((t-x)^4;x)}. \tag{5.5}$$

Since $r^2(x;x)=0$, then it follows from Theorem 5.1 that

$$\lim_{n\to\infty}K_{n,q}\left(r^2(t;x);x\right)=r^2(x;x)=0. \tag{5.6}$$

Now, from (5.5), (5.6) and (2.8), we get $\lim_{n\to\infty}[n]_qG_{n,q}\left(r(t;x)(t-x)^2;x\right)=0$. Since

$$\lim_{n\to\infty}[n]_qK_{n,q}\left((t-x)^2;x\right)=\lim_{n\to\infty}q^nx^2=x^2,$$

we get the desired result. $\square$

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