J. Partial Diff. Eqs. **15**(2002), 1–6

Vol.15, No.1

THE ISOENERGY INEQUALITY FOR HARMONIC MAPS FROM ROTATIONAL SYMMETRIC MANIFOLDS

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(Received Mar. 5, 2001)

Abstract Let u be a harmonic map from a rotational symmetric manifold M and B a unit ball in M, let $E(u|_B)$ be the energy of the map $u|_B$ and $E(u|_{\partial B})$ the energy of the map $u|_{\partial B}$, then we obtain the relationship which is called the isoenergy inequality between $E(u|_B)$ and $E(u|_{\partial B})$.

Key Words Harmonic Map; isonergy inequality; rotational symmetric manifold.
2000 MR Subject Classification 58E20, 58G11.
Chinese Library Classification O192, O176.3.

1. Introduction

Suppose that M and N are two Riemannian manifolds of dimmensions m and n respectively, and that $u : M \to N$ is a harmonic map which is a solution of the Euler-Langrange equation of the Dirichlet integral

$$E(u) = \int_M |\nabla u|^2 dv.$$

Let $M = R^m$, and B a unit ball in R^m. We define $E(u|_B)$ and $E(u|_{\partial B})$ to be the energy of the map u and the energy of the restriction of u to ∂B respectively. Choe([1]) obtained the relationship between $E(u|_B)$ and $E(u|_{\partial B})$ which is called the isoenergy inequality.

If N is nonpositively curved, then

$$(m-1)E(u|_B) \leq E(u|_{\partial B})$$

1. Author is supported by the National Natural Science Foundation of China (No: 10001033, 10071084) and the President Foundation of Graduate School of the Chinese Academy of Sciences;
2. Author is supported by the Outstanding Young Scientist Foundation of China.

and the equality holds when $N = R^n, u$ is a linear map. If N is any Riemann manifold of dimension ≥ 3 and u is a stationary harmonic map, then

$$(m-2)E(u|_B) \leq E(u|_{\partial B})$$

and the equality holds if $N = S^{m-1} \subset R^m, u(x) = x/|x|$.

In this paper, we consider the relationship between $E(u|_B)$ and $E(u|_{\partial B})$ when M is a rotational symmetric manifold (see [1]). We first derive several monotonicity formulas for harmonic maps from rotational symmetric manifolds by the method used in [1] and [2]. Using these formulas we get several isoenergy inequalities which generalize Choe's result in [3]. Let $M(m \geq 3)$ be a rotational symmetric manifold, i.e., $M = (R^m, ds^2)$, where $ds^2 = dr^2 + f^2(r)d\theta^2, f(r) > 0$ for $r > 0, f'(0) = 1$ and $d\theta^2$ is the standard metric on S^{m-1}. Let $u : M \to N$ be a stationary harmonic map. We prove the following results

(1) If M has the nonpositive radical sectional curvature, then

$$(m-2)E(u|_B) \leq f(1)E(u|_{\partial B}).$$

In particular, If $f(r) = \sinh r$, i.e., M is a space form with the constant curvature -1, then

$$(m-2)E(u|_B) \leq \left(\frac{e^2-1}{2e}\right) E(u|_{\partial B}).$$

If M has the nonnegative radical sectional curvature and $f'(1) > 0$, then

$$f'(1)(m-2)E(u|_B) \leq f(1)E(u|_{\partial B}).$$

In particular, If $f(r) = \sin r$, i.e., M is a space form with the constant curvature 1, then

$$(m-2)E(u|_B) \leq (\tan 1)E(u|_{\partial B}).$$

(2) If M has the nonpositive radical sectional curvature, then

$$f^{m-3}(1)E(u|_B) \leq E(u|_{\partial B}) \int_0^1 f^{m-3}(r)dr.$$

In the case that $f(r) = r$, i.e., $M = R^m$, we reprove Choe's results in [3].

2. Monotonicity Formulas

Let $u : M \to N$ be a weakly harmonic map, u is called stationary, if for any smooth vector field X with compact support in M, $\{\Phi_s\}$ is the 1-parameter family of transforms of M generated by X, then its energy is critical with respect to the domain variations $u \circ \Phi_s$, i.e., $\frac{d}{ds}E(u \circ \Phi_s)|_{s=0} = 0$. It is proved in [2] (also see [4]) that

$$\frac{d}{ds}E(u \circ \Phi_s)|_{s=0} = -\int_M [|\nabla u|^2 \mathrm{div}(X) - 2\sum_{i=1}^m \langle du(\nabla_i X), du\left(\frac{\partial}{\partial x^i}\right)\rangle]dV, \tag{2.1}$$

where $\nabla_i X = X_{,i}$ is the covariant derivative of X along $\dfrac{\partial}{\partial x^i}$, and $\mathrm{div}(X) = \sum_{i=1}^{m} X^i_{,i}$ is the divergence of X. In local coordinates $X^k_{,i} = \dfrac{\partial X^k}{\partial x^i} + \Gamma^k_{ij} X^j$, $\nabla_i \dfrac{\partial}{\partial x^j} = \sum_{k=1}^{m} \Gamma^k_{ij} \dfrac{\partial}{\partial x^k}$, and $\Gamma^k_{ij} = \dfrac{1}{2} g^{kl} \left(\dfrac{\partial g_{li}}{\partial x^j} + \dfrac{\partial g_{lj}}{\partial x^i} - \dfrac{\partial g_{ij}}{\partial x^l} \right)$. We say that M_0 is a rotational symmetric manifold if $M_0 = (R^m, ds_0^2)$, and $ds_0^2 = dr^2 + f^2(r) d\theta^2$ where $f(r) > 0$ for $r > 0, f(0) = 0, f'(0) = 1$, and $d\theta^2$ is the standard metric on S^{m-1}. We choose normal coordinates of S^{m-1} at θ, then $d\theta^2 = \sum_{i=2}^{m} (d\theta^i)^2$. For simplicity we write r as θ^1, then $ds_0^2 = d\theta_1^2 + f^2(\theta^1) \sum_{i=2}^{m} (d\theta^i)^2$. Calculating directly we have

$$(\Gamma_0)^k_{ij} = \begin{cases} 0, & i, j, k \neq 1, \\ \dfrac{f'(r)}{f(r)}, & i = 1, j = k \neq 1, \\ -f'(r) f(r), & k = 1, i = j \neq 1, \\ 0, & i = 1, j \neq k, \\ 0, & k = 1, i \neq j, \\ 0, & i = j = k = 1. \end{cases} \tag{2.2}$$

Suppose that M is a Riemannian manifold, if there exists a smooth function $\varphi > 0$ so that $ds_M^2 = \varphi^2 ds_0^2$, then we say that (M, ds_M^2) is conformal to M_0. As we know that

$$\Gamma^k_{ij} = (\Gamma_0)^k_{ij} + \frac{1}{2} \left(\delta_{ki} \frac{\partial \log \varphi^2}{\partial \theta^j} + \delta_{kj} \frac{\partial \log \varphi^2}{\partial \theta^i} - g_{ij} g^{kk} \frac{\partial \log \varphi^2}{\partial \theta^k} \right). \tag{2.3}$$

We set $X = \eta(r) g(r) \dfrac{\partial}{\partial r}$, where

$$\eta(r) = \begin{cases} 1 & \text{if } r \leq t', \\ \dfrac{t - r}{t - t'} & \text{if } t' < r < t, \\ 0 & \text{if } r \geq t. \end{cases} \tag{2.4}$$

then we have

$$X^j_{,i} = \frac{\partial X^j}{\partial \theta^i} + \Gamma^i_{i1} X^1, \tag{2.5}$$

$$X^j_{,i} = \begin{cases} \eta'(r) g(r) + \eta(r) g'(r) + \eta(r) g(r) \dfrac{\partial \log \varphi}{\partial r}, & i = j = 1, \\ \eta(r) g(r) \dfrac{f'(r)}{f(r)} + \eta(r) g(r) \dfrac{\partial \log \varphi}{\partial r}, & i = j \neq 1, \\ \eta(r) g(r) \dfrac{\partial \log \varphi}{\partial \theta^i}, & i \neq j = 1, \\ -\dfrac{1}{f^2(r)} \eta(r) g(r) \dfrac{\partial \log \varphi}{\partial \theta^j}, & i = 1 \neq j, \\ 0, & i \neq j\ , i \neq k,\ j \neq 1. \end{cases} \tag{2.6}$$

$$\nabla_i X = \left(\eta(r)g(r)\frac{f'(r)}{f(r)} + \eta(r)g(r)\frac{\partial \log\varphi}{\partial r}\right)\frac{\partial}{\partial\theta^i} + \eta(r)g(r)\frac{\partial\log\varphi}{\partial\theta^i}\frac{\partial}{\partial r}, i \neq 1,$$

$$\nabla_1 X = \left(\eta'(r)g(r) + \eta(r)g'(r) + \eta(r)g(r)\frac{\partial\log\varphi}{\partial r}\right)\frac{\partial}{\partial r} - \sum_{k=2}^m \frac{\eta(r)g(r)}{f^2(r)}\frac{\partial\log\varphi}{\partial\theta^k}\frac{\partial}{\partial\theta^k}.$$

So, we have

$$\mathrm{div}_M X = (\eta'(r)g(r) + \eta(r)g'(r)) + (m-1)\eta(r)g(r)\frac{f'(r)}{f(r)} + m\eta(r)g(r)\frac{\partial\log\varphi}{\partial r}, \quad (2.7)$$

$$\begin{aligned}\sum_{i=1}^m &\langle du(\nabla_i X),\, du\left(\frac{\partial}{\partial\theta^i}\right)\rangle \\ &= \eta(r)g(r)\left(\frac{f'(r)}{f(r)} + \frac{\partial\log\varphi}{\partial r}\right)|\nabla u|^2 \\ &\quad + \varphi^{-2}\left(\eta'(r)g(r) + \eta(r)g'(r) - \eta(r)g(r)\frac{f'(r)}{f(r)}\right)\left(\frac{\partial u}{\partial r}\right)^2 \\ &\quad + \eta(r)g(r)\left(1 - \frac{1}{f^2(r)}\right)\sum_{k=2}^m\frac{\partial\log\varphi}{\partial\theta^k}\langle du\left(\frac{\partial}{\partial\theta^k}\right), du\left(\frac{\partial}{\partial r}\right)\rangle. \quad (2.8)\end{aligned}$$

Substituting (2.7) and (2.8) into (2.1), we obtain

$$\begin{aligned}&\int_M |\nabla u|^2\left[\eta(r)\left(g'(r) + (m-3)g(r)\frac{f'(r)}{f(r)} + (m-2)g(r)\frac{\partial\log\varphi}{\partial r}\right) + \eta'(r)g(r)\right]dv \\ &\quad - 2\int_M\left[\eta'(r)g(r) - \eta(r)\left(g(r)\frac{f'(r)}{f(r)} - g'(r)\right)\right]\varphi^{-2}\left(\frac{\partial u}{\partial r}\right)^2 dv \\ &\quad + 2\int_M \eta(r)g(r)\left(\frac{1}{f^2(r)} - 1\right)\sum_{i=2}^m\frac{\partial\log\varphi}{\partial\theta^i}\langle du\left(\frac{\partial}{\partial\theta^i}\right), du\left(\frac{\partial}{\partial r}\right)\rangle dv = 0. \quad (2.9)\end{aligned}$$

Let $B_t = \{x \in M | r(x) < t\}$ and $B = B_t|_{t=1}$.

(i) Choosing $g(r) = f(r)$, using (2.4) and letting $t' \to t$, we have (2.9) becomes

$$\begin{aligned}&(m-2)\int_{B_t}|\nabla u|^2\varphi^{-1}\frac{\partial(f(r)\varphi)}{\partial r}dv - \int_{\partial B_t}|\nabla u|^2 f(t)d\sigma + 2\int_{\partial B_t}\varphi^{-2}f(t)\left(\frac{\partial u}{\partial r}\right)^2 d\sigma \\ &\quad + 2\int_{B_t}\left(\frac{1}{f(r)} - f(r)\right)\sum_{i=2}^m\frac{\partial\log\varphi}{\partial\theta^i}\langle du\left(\frac{\partial}{\partial\theta^i}\right), du\left(\frac{\partial}{\partial r}\right)\rangle dv = 0. \quad (2.10)\end{aligned}$$

(ii) Choosing $g(r) = f^{3-m}(r)\int_0^r f^{m-3}(t)dt$ in (2.9), using (2.4) and letting $t' \to t$ we have

$$\begin{aligned}&\int_{B_t}|\nabla u|^2\left(1 + (m-2)g(r)\frac{\partial\log\varphi}{\partial r}\right)dv - \int_{\partial B_t}|\nabla u|^2 g(t)d\sigma \\ &\quad + 2\int_{\partial B_t}\left(\frac{\partial u}{\partial r}\right)^2\varphi^{-2}g(t)d\sigma + 2\int_{B_t}\left(\frac{\partial u}{\partial r}\right)^2\varphi^{-2}f^{2-m}(r)s(r)dv \\ &\quad + 2\int_{B_t}g(r)\left(\frac{1}{f^2(r)} - 1\right)\sum_{i=2}^m\frac{\partial\log\varphi}{\partial\theta^i}\langle du\left(\frac{\partial}{\partial\theta^i}\right), du\left(\frac{\partial}{\partial r}\right)\rangle dv = 0, \quad (2.11)\end{aligned}$$

where $s(r) = (m-2)\int_0^r f^{m-3}(t)(f'(r) - f'(t))dt$.

Lemma 1 *Let M be an m-dimensional conformal rotational symmetric manifold ($m \geq 3$), if $u : M \to N$ is a stationary harmonic map, then we have the formulas (2.10) and (2.11).*

3. Isoenergy Inequalities

In the monotonicity formulas (2.10) and (2.11), choosing $\varphi = 1$ and $t = 1$, we have

Proposition 2 *If $u : M_0 \to N$ is a stationary harmonic map, we have the following formulas*

$$(m-2)\int_B |\nabla u|^2 f'(r)dv = f(1)\int_{\partial B}\left(|\nabla u|^2 - 2\left|\frac{\partial u}{\partial r}\right|^2\right)d\sigma \tag{3.1}$$

and

$$\begin{aligned} F(1)\int_{\partial B}|\nabla u|^2 d\sigma =& F'(1)\int_B\left(|\nabla u|^2 + 2f^{2-m}(r)s(r)\left|\frac{\partial u}{\partial r}\right|^2\right)dv \\ &+ 2F(1)\int_{\partial B}\left|\frac{\partial u}{\partial r}\right|^2 d\sigma, \end{aligned} \tag{3.2}$$

where $F(t) = \int_0^t f^{m-3}(r)dr$.

Using (3.1) we obtain

Theorem 3 *Let $m \geq 3$ and let $u : M_0 \to N$ be a stationary harmonic map.*

(1) *If M_0 has the nonpositive radical sectional curvature, then*

$$(m-2)E(u|_B) \leq f(1)E(u|_{\partial B}). \tag{3.3}$$

(2) *If M_0 has the nonnegative radical sectional curvature and $f'(1) > 0$, then*

$$f'(1)(m-2)E(u|_B) \leq f(1)E(u|_{\partial B}). \tag{3.4}$$

Proof (1) Because of the radical sectional curvature $K(r) = -\dfrac{f''(r)}{f(r)} \leq 0$, we have $f'(r) \geq f'(0) = 1$. Let $\bar{\nabla}u$ denote the gradient of u on ∂B, then

$$E(u|_{\partial B}) = \int_{\partial B}\left|\bar{\nabla}u\right|^2 d\sigma = \int_{\partial B}\left(|\nabla u|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)d\sigma.$$

Hence by (3.1) we have

$$\begin{aligned} (m-2)\int_B |\nabla u|^2 dv \leq& (m-2)\int_B |\nabla u|^2 f'(r)dv = f(1)\int_{\partial B}\left(\left|\bar{\nabla}u\right|^2 - \left|\frac{\partial u}{\partial r}\right|^2\right)d\sigma \\ \leq& f(1)\int_{\partial B}\left|\bar{\nabla}u\right|^2 d\sigma. \end{aligned}$$

(2) Since $K(r)=-\dfrac{f''(r)}{f(r)}\geq 0, f'(1)>0$, we have $f'(r)\geq f'(1)>0\,(r\leq 1)$. By (3.1) we obtain

$$(m-2)f'(1)\int_B|\nabla u|^2dv\leq(m-2)\int_B|\nabla u|^2f'(r)dv\leq f(1)\int_{\partial B}\left|\bar\nabla u\right|^2d\sigma.$$

Using (3.2) we have

Theorem 4 *Let $m\geq 3$ and let $u:M_0\to N$ be a stationary harmonic map. If M_0 has nonpositive radical sectional curvature, then*

$$f^{m-3}(1)E(u|_B)\leq E(u|_{\partial B})\int_0^1 f^{m-3}(r)dr. \tag{3.5}$$

Proof Since $K(r)=-\dfrac{f''(r)}{f(r)}\leq 0$, we have $f'(r)\geq f'(t)(r\geq t), s(r)\geq 0$. By (3.2) we have

$$F'(1)\int_B|\nabla u|^2dv\leq F(1)\int_{\partial B}\left|\bar\nabla u\right|^2d\sigma.$$

By Theorem 3 we have

Corollary 5 (1) *If $f(r)=r$, i.e., $M_0=R^m$, then*

$$(m-2)E(u|_B)\leq E(u|_{\partial B}).$$

(2) *If $f(r)=\mathrm{sinh}r$, i.e., M_0 is a space form with the constant curvature -1, then*

$$(m-2)E(u|_B)\leq\left(\frac{e^2-1}{2e}\right)E(u|_{\partial B}).$$

(3) *If $f(r)=\sin r$, i.e., M_0 is a space form with the constant curvature 1, then*

$$(m-2)E(u|_B)\leq(\tan 1)E(u|_{\partial B}).$$

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