
BERNSTEIN-JÖRGENS THEOREM FOR A FOURTH ORDER PARTIAL DIFFERENTIAL EQUATION*

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Abstract We introduce a metric, conformal to the affine metric, on a convex graph, and consider the Euler equation of the volume functional. We establish a priori estimates for solutions and prove a Bernstein-Jörgens type result in the two dimensional case.

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1. Introduction

In this paper we study locally uniformly convex solutions of fourth order elliptic equations of the form

$$L[u] = U^{ij}w_{ij} = f, \quad (1.1)$$

in the n -dimensional Euclidean space, \mathbf{R}^n , where (U^{ij}) is the cofactor matrix of the Hessian matrix $(u_{ij}) = D^2u \geq 0$, $w = [\det D^2u]^\alpha$, $\alpha \neq 0$ is a constant, and f is a given function in \mathbf{R}^n . The operator L is the Euler operator (up to a constant) of the functional

$$J(u) = \int [\det D^2u]^{1+\alpha}. \quad (1.2)$$

Let $\mathcal{M} = \{(x, u(x)) \mid x \in \mathbf{R}^n\}$ be a locally uniformly convex hypersurface, given by the graph of u . We introduce a metric g on \mathcal{M} , defined by

$$g_{ij} = \rho u_{ij}, \quad (1.3)$$

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where $\rho = [\det D^2 u]^{(1+2\alpha)/n} > 0$. Then (1.2) is the volume functional of the metric g . Note that (1.1) can also be written in the form (suppose $f = 0$)

$$\Delta_g \rho = 0, \quad (1.4)$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric g .

There has been a growing interest in recent years in functionals involving curvatures of a hypersurface (or manifold). Well-known examples are the Willmore functional [1,2]

$$\int_{\mathcal{M}} H^2 d\sigma, \quad (1.5)$$

the functional proposed by Calabi [3-5]

$$\int_{\mathcal{M}} S^2 d\sigma \quad (1.6)$$

and the affine surface area functional [6,7]

$$\int_{\mathcal{M}} K^{1/(n+2)} d\sigma, \quad (1.7)$$

where H, S, K are respectively the mean curvature, the scalar curvature, and the Gauss curvature, and $d\sigma$ is the volume element on \mathcal{M} . The Euler equations of these functionals are strongly nonlinear fourth order partial differential equations.

Our knowledge on higher order nonlinear partial differential equations is limited up to date, although there are some isolated results. The study of the functional (1.2) may help to understand other functionals such as (1.5)-(1.7). Note that the metrics g in (1.3) are conformal to each other for different α . If $\alpha = -\frac{1}{2}$, the metric g in (1.3) is called the Schwarz-Pick metric [8]. When $\alpha = -\frac{n+1}{n+2}$, the metric

$$g_{ij} = g_{ij}^a = [\det D^2 u]^{-1/(n+2)} u_{ij} \quad (1.8)$$

is the affine metric (Berwald-Blaschke metric). In this case the equation (1.1) is the affine maximal surface equation for $f = 0$, and the affine mean curvature equation for general f . In [7] we proved interior estimates and solved the Bernstein problem in dimension two for the affine maximal surface equation.

In this paper we study the equation (1.1) with positive exponent $\alpha > 0$. We will first derive a priori estimates (Section 2) and then prove the Bernstein-Jörgens theorem for the equation (1.1), with $f \equiv 0$, in two dimensions (Section 3). In [9] Jörgens proved that an entire convex solution to the Monge-Ampère equation

$$\det D^2 u = 1 \quad (1.9)$$

must be a quadratic function if $n = 2$. Jörgens' result was extended to high dimensions by Calabi for $3 \leq n \leq 5$ and Pogorelov for all $n \geq 2$, see [10]. Jörgens' result can also

be used to obtain an alternative proof of the well known Bernstein theorem for minimal graphs [11]. Our result, Theorem 3.2, extends Jörgens' Theorem, since if u is a solution of (1.9), u trivially satisfies (1.1) with $f = 0$.

Finally, we give some remarks in Section 4. An interesting phenomenon pointed out there is that interior regularity for the equation (1.1) for non-positive functions f does not carry over to positive f . We also indicate, among others, an application of Bernstein's original theorem [12] to our equation (1.1).

2. Interior Estimates

Lemma 2.1 *Let $u \in C^4(\Omega) \cap C^0(\bar{\Omega})$ be a convex solution of (1.1), with positive α , in a domain $\Omega \subset \mathbf{R}^n$. Suppose $f \leq 0$ and*

$$u = 0 \quad \text{on } \partial\Omega, \quad \inf_{\Omega} u = -1. \quad (2.1)$$

Then

$$\det D^2u \geq C(-u)^\beta, \quad (2.2)$$

where $C, \beta > 0$ depend only on n, α .

Proof Let

$$z = \log \frac{w}{(-u)^\beta} - \frac{A}{2}|x|^2,$$

where $\beta > 1$ and $A < 1$ are constants to be determined. We have $z = \infty$ on $\partial\Omega$. Hence z attains the minimum at some interior point $x_0 \in \Omega$. At x_0 we have

$$z_i = \frac{w_i}{w} - \beta \frac{u_i}{u} - Ax_i = 0,$$

namely

$$\frac{u_i}{u} = \frac{1}{\beta} \left(\frac{w_i}{w} - Ax_i \right),$$

where $z_i = \frac{\partial z}{\partial x_i}$. The Hessian matrix (z_{ij}) at x_0 is nonnegative. Choosing the coordinates properly we may suppose (z_{ij}) is diagonal at x_0 . We have

$$\begin{aligned} 0 \leq u^{ij} z_{ij} &= u^{ii} \left[\frac{w_{ii}}{w} - \frac{w_i^2}{w^2} - \beta \frac{u_{ii}}{u} + \beta \frac{u_i u_j}{u^2} - A \right] \\ &= \frac{f}{wd} - u^{ii} \frac{w_i^2}{w^2} - \frac{\beta n}{u} + \frac{1}{\beta} u^{ii} \left(\frac{w_i}{w} - Ax_i \right)^2 - Au^{ii} \\ &\leq \frac{f}{wd} - \left(1 - \frac{2}{\beta} \right) u^{ii} \frac{w_i^2}{w^2} - \frac{\beta n}{u} - A(1 - 2A|x|^2)u^{ii} \\ &\leq \frac{f}{wd} - \frac{\beta n}{u} - \frac{1}{2} Au^{ii} \end{aligned}$$

if $\beta \geq 2$ and $A \leq \frac{1}{4}[\text{diam}(\Omega)]^{-1}$, where $d = \det D^2u$. By the assumption $f \leq 0$, we obtain

$$|u|u^{ii} \leq C.$$

Choosing $\beta = \max(2, n\alpha)$, we obtain $z(x) \geq z(x_0) \geq C$.

A well known result for the Monge-Ampère equation (1.9) is that if $n = 2$ and u is a subsolution of (1.9), then u is strictly convex [13]. Hence by Lemma 2.1 we have

Corollary 2.2 *Let u be as in Lemma 2.1. If $n = 2$, then u is strictly convex in Ω_δ for any $\delta > 0$, where $\Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}$.*

More precisely, Corollary 2.2 means that there is a monotone increasing function φ , $\varphi(r) > 0$ if $r > 0$, depending on Ω , δ , the gradient of u , and the lower bound C in (2.2), such that

$$u(x) \geq \varphi(|x - x_0|) + u(x_0) + Du(x_0)(x - x_0) \quad (2.3)$$

for any $x_0 \in \Omega_\delta$ and $x \in \Omega$. If Ω has a good shape, say, Ω is a normalized domain (see Section 3 for definition), then φ is independent of Ω .

Lemma 2.3 *Let $n = 2$ and u be as in Lemma 2.1. Then*

$$u^2 \det D^2u \leq C, \quad (2.4)$$

where C depends on α , $\sup_\Omega |Du|$, and $\inf_\Omega f$.

Proof Let

$$z = \log w(-u)^\beta + \frac{A}{2}|Du|^2.$$

Then $z = 0$ on $\partial\Omega$. Suppose z attains maximum at x_0 . Then we have, at x_0 ,

$$0 = z_i = \frac{w_i}{w} + \beta \frac{u_i}{u} + Au_k u_{ki}$$

and

$$0 \geq z_{ii} = \frac{w_{ii}}{w} - \frac{w_i^2}{w^2} + \beta \frac{u_{ii}}{u} - \beta \frac{u_i^2}{u^2} + Au_{ii}^2 + Au_k u_{kii}.$$

We may suppose (z_{ij}) is diagonal at x_0 . Observe that

$$u^{ii} u_{kii} = \frac{1}{\alpha} \frac{w_k}{w}.$$

We have

$$\begin{aligned} 0 \geq u^{ii} z_{ii} &= \frac{f}{wd} - u^{ii} \left(\beta \frac{u_i}{u} + Au_i u_{ii} \right)^2 + \frac{\beta n}{u} - \beta u^{ii} \frac{u_i^2}{u^2} + A\Delta u + Au_k \frac{w_k}{\alpha w} \\ &\geq \frac{f}{wd} - C \frac{u^{ii}}{u^2} + A(1 - 2A|Du|^2)\Delta u - \frac{C}{|u|}. \end{aligned}$$

Choosing $A = \frac{1}{4}[\sup_\Omega |Du|^2]^{-1}$, we obtain

$$\Delta u \leq \frac{C}{u^2} u^{ii} + \frac{C}{|u|} - \frac{\inf_\Omega f}{wd}.$$

Multiplying the above inequality by $u_{11}u_{22}$, we obtain

$$u^2 \det D^2 u \leq C.$$

Choosing $\beta = 2\alpha$, we obtain Lemma 2.3.

The boundedness of the gradient Du in Lemma 2.3 is not a restriction. Indeed, by (2.1) and the convexity of u , Du is bounded in Ω_δ for any $\delta > 0$. Hence by Corollary 2.2, u is strictly convex in $\Omega_{2\delta}$. Therefore for any point $x \in \Omega$, we can apply Lemma 2.3 to a level set of u at x , namely the set

$$S_u(x, h) = \{y \in \Omega \mid u(y) < u(x) + Du(x)(y - x) + h\}, \quad (2.5)$$

where $h > 0$ is such that $S_u(x, h)$ is precompact in Ω .

The following Hölder estimate for the linearized Monge-Ampère equation follows from [14].

Lemma 2.4 *Let u be a strictly convex function vanishing on the boundary $\partial\Omega$ and satisfying*

$$C_1 \leq \det D^2 u \leq C_2 \quad \text{in } \Omega \quad (2.6)$$

for some positive constants C_1, C_2 . Suppose $w \in C^2(\Omega)$ is a solution of the equation

$$U^{ij} w_{ij} = 0 \quad \text{in } \Omega,$$

where (U^{ij}) is the cofactor matrix of the Hessian $(D^2 u)$. Then there exists $\alpha' \in (0, 1)$ such that for any $\Omega' \subset\subset \Omega$,

$$\|w\|_{C^{\alpha'}(\Omega')} \leq C. \quad (2.7)$$

where α' depends only on n, C_1, C_2 , and C depends additionally on Ω', Ω .

Note that for a convex function u , the Hessian matrix $D^2 u$ is well-defined almost everywhere. The condition on the Monge-Ampère measure $\mu_u = \det D^2 u$ in [14] is weaker. Rather than the pinching condition (2.6), it is assumed in [14] that μ_u satisfies a uniform continuity condition. It is not hard to check that the Hölder continuity is still true for non-homogeneous equations, namely the equation (1.1) if the condition (2.6) holds.

With the a priori estimates (2.2) and (2.4), and the Hölder continuity (2.7), we therefore have the following Schauder estimate and $W^{4,p}$ estimate for the solutions of (1.1).

Theorem 2.5 ($W^{4,p}$ estimate) *Let $n = 2$ and $u \in C^4(\Omega) \cap C^0(\bar{\Omega})$ be a solution of (1.1) with $\alpha > 0$ and $f \leq 0$. If u satisfies (2.1), then for any $p > 1, \delta > 0$, we have the estimate*

$$\|u\|_{W^{4,p}(\Omega_\delta)} \leq C, \quad (2.8)$$

where $C > 0$ depends only on $\alpha, p, \delta, \Omega$, and $\max |f|$.

Proof By the strict convexity, Corollary 2.2, and the Hölder continuity of solutions of the linearized Monge-Ampère equation, Lemma 2.4, we have an a priori Hölder

estimate for $\det D^2 u$. By the Schauder estimate [15], we then have an a priori $C^{2,\alpha}$ estimate for u . Hence (1.1) becomes a uniformly elliptic equation and the $W^{4,p}$ estimates follows.

Further estimates and corresponding regularity for the solutions of (1.1) follow from standard elliptic regularity theory. In particular we have

Theorem 2.6 (Schauder estimate) *Let $n = 2$ and $u \in C^4(\Omega) \cap C^0(\overline{\Omega})$ be a solution of (1.1) with positive α . Suppose u satisfies (2.1), $f \leq 0$, $f \in C^{k,\alpha'}(\Omega)$, where $k \geq 0$, $\alpha' \in (0, 1)$. Then $u \in C^{k+4,\alpha'}(\Omega)$, and for any $\delta > 0$, we have the estimate*

$$\|u\|_{C^{4,\alpha'}(\Omega_\delta)} \leq C, \quad (2.9)$$

where $C > 0$ depends only on $\alpha, k, \alpha', \delta$, and Ω .

We remark that Theorems 2.5 and 2.6 hold for all dimensions if one has the estimates (2.2) and (2.4).

3. The Bernstein Problem

In this section we consider the equation

$$L[u] = 0 \quad \text{in } \mathbf{R}^2, \quad (3.1)$$

where L is the operator in (1.1). We want to prove that a solution to (3.1) must be a quadratic polynomial. For this purpose we need two known results. The first one is that for any given bounded convex domain $\Omega \subset \mathbf{R}^n$ ($n \geq 2$), there exists a unique ellipsoid E containing Ω , called the *minimum ellipsoid* of Ω , which attains the minimum volume among all ellipsoids containing Ω . Moreover,

$$\frac{1}{n}E \subset \Omega \subset E, \quad (3.2)$$

where $\frac{1}{n}E$ is the $\frac{1}{n}$ dilation of E with concentric centre.

Let T be a linear transformation leaving the centre of E invariant such that $T(E) = B$, the unit ball. Then we have

$$\frac{1}{n}B \subset T(\Omega) \subset B.$$

We call $T(\Omega)$ the normalized domain of Ω . A domain Ω is normalized if T is the identity mapping, that is its minimal ellipsoid is the unit ball.

We also need a lemma from [16].

Lemma 3.1 *Let u be a locally uniformly convex function in \mathbf{R}^n . Then for any $y \in \mathbf{R}^n$ and any $h > 0$, there is a point $x \in \mathbf{R}^n$ such that y is the centre of mass of the level set $S_u(x, h)$.*

Let u be an entire solution of (3.1). By subtracting a linear function we suppose u is nonnegative and $u(0) = 0$. For any $h > 0$, let $x_h \in \mathbf{R}^n$ ($n = 2$) such that the origin is the centre of mass of the level set $S_u(x_h, h)$. For $h > 1$ sufficiently large, let T_h be a linear transformation which normalizes the level set $S_u(x_h, h)$ such that the origin is the centre of mass of $\Omega_h = T_h(S_u(x_h, h))$. Let

$$u_h(y) = \delta_h \{u(x) - u(x_h) - Du(x_h)(x - x_h)\}, \quad (3.3)$$

where $y = T_h(x) \in \Omega_h$. Then $u_h \geq u_h(x_h) = 0$. We choose $\delta_h > 0$ such that $u_h = 1$ on $\partial\Omega_h$. Obviously, u_h satisfies the equation (3.1) since (3.1) is invariant under linear transformation.

By our interior estimate, Theorem 2.6, we have

$$C_1 \leq \det D^2 u_h \leq C_2 \quad (3.4)$$

and

$$C_1 |x|^2 \leq u_h(x) - Du_h(0)x \leq C_2 |x|^2 \quad (3.5)$$

for x near the origin. Let Λ_h and λ_h denote respectively the largest and the least eigenvalues of T_h . It is easy to check by rescaling that

$$\lambda_h \geq C_3 \delta_h^{1/2}, \quad (3.6)$$

$$\Lambda_h \leq C_4 \delta_h^{1/2}, \quad (3.7)$$

where C_3 depends only on C_1, C_2 in (3.5) and $\inf_{\partial B_1} u$, and C_4 depends only on C_1, C_2 in (3.5) and $\sup_{\partial B_1} u$. It follows, from (3.4)-(3.7), that

$$C_1 \leq \det D^2 u \leq C_2 \quad \text{in } \mathbf{R}^2 \quad (3.8)$$

for different C_1, C_2 , and

$$\frac{u(x)}{|x|} \rightarrow \infty \quad \text{as } x \rightarrow \infty. \quad (3.9)$$

By (3.6) and (3.7) we have furthermore

$$\sup_{\partial B_R(0)} u \leq C \inf_{\partial B_R(0)} u \quad (3.10)$$

for any $R \geq 1$, where $C > 0$ depends on C_i , $i = 1, 2, 3, 4$, but is independent of R .

Remark If u is a solution of (3.1) defined on a convex domain Ω such that $u \rightarrow \infty$ as $x \rightarrow \partial\Omega$, by (3.6) and (3.7) we must have $\Omega = \mathbf{R}^n$. Note that Lemma 3.1 is still applicable in this case.

Now we can reduce the Bernstein problem for (3.1) to the interior estimate, Theorem 2.6.

Theorem 3.2 *Let $n = 2$ and u be an entire solution of (3.1). Then u is a quadratic polynomial.*

Proof Let T_h be a linear transformation normalizing the level set $S_h = \{u < h\}$. Let $u_h(x) = \frac{1}{h}u(T^{-1}(x))$, $x \in \Omega_h = T(S_h)$. Then $u_h = 1$ on $\partial\Omega_h$, $u_h \geq u_h(0) = 0$. By (3.10), the origin

$$O \in \Omega_{h,\delta} = \{x \in \Omega_h \mid \text{dist}(x, \partial\Omega_h) > \delta\}. \quad (3.11)$$

By Theorem 2.6 we have $|D^3u_h| \leq C$ near the origin. It follows, for any fixed $x \in \mathbf{R}^n$,

$$|D^3u(x)| \leq C\Lambda_h^3 h |D^3u_h(T_h(x))|.$$

By (3.6) and (3.7), we have $\Lambda_h^2 h \leq C$. Hence

$$|D^3u(x)| \leq Ch^{-1/2}.$$

Letting $h \rightarrow \infty$, we conclude that $D^3u = 0$. Hence u is quadratic.

4. Remarks

4.1 Strict convexity

It is well known that the strict convexity of solutions is crucial for the regularity of the Monge-Ampère equation (1.9). This is the same for the affine mean curvature equation

$$U^{ij}[(\det D^2u)^{-(n+1)/(n+2)}] = 0.$$

In [7] we proved only for dimension two the strict convexity of solutions which vanish on the boundary. For high dimensions we found a non-strictly convex, affine maximal function

$$u(x) = (|x'|^9 + x_{10}^2)^{1/2}, \quad (4.1)$$

where $x' = (x_1, \dots, x_9)$. The graph of this function can be regarded as an *affine maximal cone*. It has the affine invariant property that for any $t > 0$, there is an affine transformation T_t such that

$$\frac{1}{t}u(T_t(x)) = u(x).$$

For the equation (1.1), in the case $\alpha > 0$, the strict convexity of solutions in two dimensions follows from (2.2) immediately.

The following example shows that sufficiently smooth boundary data may be necessary for C^∞ regularity. Consider in the upper half-space $\{y > 0\}$ of \mathbf{R}^2 the function

$$u(x, y) = x^\lambda / y \quad \lambda \geq 2.$$

Direct computation shows that u satisfies the Laplace-Beltrami equation (1.4) with respect to the Schwarz-Pick metric $g_{ij} = u_{ij}$. The function (4.1), in a domain in $\{x_{10} > 0\}$, also shows that sufficiently smooth boundary data are necessary for the C^∞ regularity for the affine maximal surface equation in high dimensions.

For the Monge-Ampère equation (1.9), there also exist non-strictly convex solutions in dimensions $n \geq 3$ [10]. However for two dimensions a solution to (1.9) must be strictly convex [13], as we indicated in Section 2.

4.2 A priori estimates

In Theorem 2.5 and 2.6 we have the condition $f \leq 0$. When $f > 0$, the estimate (2.2) is not true and Theorems 2.5 and 2.6 do not hold. Indeed when $n = 2$ and $\alpha = 1$, the operator L in (1.1) can also be written (for radial functions) as

$$L[u] = \frac{1}{r} \left[\left(\frac{u'^2}{r} u'' \right)'' - \left(\frac{u'}{r} u''^2 \right)' \right]. \quad (4.2)$$

Direct computation shows that when $u(r) = r^{8/3}$,

$$L[u] = \frac{5}{3} \left(\frac{8}{3} \right)^4. \quad (4.3)$$

However u is not C^3 smooth. This is a very interesting phenomenon for the regularity of higher order nonlinear partial differential equations.

For the affine mean curvature equation, we have shown that Theorem 2.5 or 2.6 holds in dimension two for any bounded function f or Hölder continuous f [17]. From the function (4.1), it is readily seen that Theorem 2.6 does not hold in high dimensions ($n \geq 10$); additional conditions on the boundary are needed for the interior regularity.

4.3 Reduction of smoothness

If $u \in C^2(\Omega)$ is locally uniformly convex, then the equation (1.1) is still meaningful in the distribution sense,

$$\int_{\Omega} U^{ij} \eta_{ij} w = \int_{\Omega} f \eta, \quad (4.5)$$

for all $\eta \in C_0^2(\Omega)$. However, our regularity proof in [7] continues to apply and we can infer $u \in W_{loc}^{4,p}(\Omega)$ if $f \in L^p(\Omega)$ for $p > 1$. Accordingly, in Theorems 2.5 and 2.6, we need only assume $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, with $u \in C^2(\mathbf{R}^2)$ for the validity of the Bernstein-Jörgens result, Theorem 3.2. Note that the example (4.3) also shows that the local uniform convexity cannot be relaxed to strict convexity for the regularity.

4.4 Bernstein's result

A well-known result for minimal surfaces by Bernstein is that a complete minimal graph in \mathbf{R}^3 must be a linear function. Indeed Bernstein proved the following deep result for two dimensional elliptic equations.

Theorem 4.1 *Suppose u is a solution to the elliptic equation*

$$\sum_{i,j=1}^2 a_{ij}(x) u_{ij} = 0 \quad \text{in } \mathbf{R}^2, \quad (4.6)$$

such that

$$|u(x)| = o(|x|) \quad \text{as } |x| \rightarrow \infty. \quad (4.7)$$

Then u is a constant.

Bernstein's proof [12] contains a gap, which was fixed in [18] and [19]. In Theorem 4.1 the operator is not required to be uniformly elliptic. Theorem 4.1 gives an alternative proof of Theorem 3.2. That is $w = [\det D^2 u]^\alpha$ is bounded in \mathbf{R}^n by (3.8) and so it is a constant. Therefore u is quadratic by Jörgens' result [9].

Theorem 4.1 also provides an alternative proof for the affine Bernstein problem in [7], which avoids the Caffarelli-Gutierrez theory. However this doesn't simplify the proof in [7] as all estimates proven there are still necessary.

As a final remark we indicate that in [7], the affine Bernstein problem is reduced to interior estimates by the rescaling $u_t(x) = \frac{1}{t}u(T^{-1}(x))$, where $t > 1$ is a constant and T is an affine transformation which normalizes the level set $\{u < t\}$. Since the affine mean curvature equation is invariant under different choices of coordinate systems, one can assume that $\text{dist}(0, \partial\Omega_t) \geq \delta_0$ for a fixed $\delta_0 > 0$, where $\Omega_t = T(\{u < t\})$. For the equation (3.1) such invariance is no longer true and we need the growth estimates (3.10) to ensure $\text{dist}(0, \partial\Omega_t) > 0$, see (3.11).

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