J. Partial Diff. Eqs. **18**(2005), 22–34
 Vol.18, No.1

CAUCHY PROBLEM FOR ONE-DIMENSIONAL P-LAPLACIAN EQUATION WITH POINT SOURCE*

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(Received Jun. 21, 2004)

Abstract We prove the existence, uniqueness and finite propagation of disturbance of continuous solutions to the Cauchy problem for one-dimensional p-Laplacian equation with point source.
Key Words p-Laplacian equation; point source; existence; uniqueness.
2000 MR Subject Classification 35K55, 35K65, 35R05.
Chinese Library Classification O175.26.

1. Introduction

This paper concerns with the Cauchy problem for the following p-Laplacian equation with point source

$$\frac{\partial u}{\partial t} = D(|Du|^{p-2}Du) + \delta(x), \qquad (x,t) \in Q, \tag{1.1}$$

$$u(x,0) = 0, \qquad x \in \mathbb{R}, \tag{1.2}$$

where $\delta(x)$ is the Dirac measure, $p \geq 2$, $D = \dfrac{\partial}{\partial x}$, $Q = \mathbb{R} \times (0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$.

The equation (1.1) is an important degenerate diffusion equation, which can be used to describe many phenomena in nature such as filtration and dynamics of biological groups and so on. During the past years, there were a tremendous amount of papers devoted to such kinds of equations without singular sources. However, as we know, the investigation about the equations with measure data is quite fewer. For the case $p = 2$, in [1] Li Huilai proved the existence of the solutions to parabolic equations with measure data, and in [2] Pang Zhiyuan, Wang Yaodong and Jiang Lishang studied the optimal control problems for semilinear diffusion equations with Dirac measure. F. Abergel, A. Decarreau and J. M. Rakotoson [3] dealt with a class of equations with measure data, and studied the existence and uniqueness of the solutions of the initial boundary

*This work is partially supported by the National Science Foundation of China, and partially supported by a Specific Foundation for Ph.D Specialities of Educational Department of China.

value problem in a bounded domain. Yuan Hongjun and Wu Gang [4] investigated the porous medium equation with Dirac measure and gave the existence of the weak solutions for Cauchy problem. In 1997, Lucio Boccardo, Andrea Dall'Aglio, Thierry Gallouët and Luigi Orsina [5] studied the initial boundary value problem for a class of nonlinear parabolic equations with measure data in general form in bounded domains.

In this paper, we deal with the Cauchy problem for one-dimensional p-Laplacian equation with point source, which is quite different from the initial boundary value problem in a bounded domain. Just as did in [5], we should first make an approximation of the Dirac measure. However, based on our approach technique, we require a C^∞–approximation rather than in L^q norm. Then we approximate the Cauchy problem by a sequence of bounded domains of the form $Q_{R,T} = (-R, R) \times (0, T)$. Finally, because of the degeneracy, we use parabolic regularization to approach the equation. Based on BV estimates, L^p–type estimates and weighted energy estimates, we establish the existence and uniqueness of continuous solutions of the problem (1.1), (1.2). Precisely, we have the following result

Theorem 1.1 *The Cauchy problem (1.1), (1.2) admits one and only one continuous solution with compact support.*

By the continuous solution, we mean the following

Definition 1.1 *A nonnegative function $u : Q \longmapsto \mathbb{R}$ is said to be a continuous solution of the Cauchy problem (1.1), (1.2), if for any $T \in (0, +\infty)$, $u \in L^\infty(Q_T) \cap L^\infty(0, T; W^{1,p}(\mathbb{R})) \cap BV(Q_T)$ and the following integral equalities*

$$-\iint_{Q_T} u\frac{\partial \varphi}{\partial t}dxdt = -\iint_{Q_T} |Du|^{p-2}DuD\varphi dxdt + \int_0^T \varphi(0,t)dt, \quad \forall \varphi \in C_0^\infty(Q_T), \tag{1.3}$$

and

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}} \psi(x)u(x,t)dxdt = 0, \qquad \forall \psi \in C_0^\infty(\mathbb{R}), \tag{1.4}$$

hold, where $Q_T = \mathbb{R} \times (0, T)$.

2. Proof of the Main Result

Just as mentioned above, to discuss the existence of continuous solutions of the problem (1.1), (1.2), we first consider the regularized problem

$$\frac{\partial u}{\partial t} = D\Big(\Big(|Du|^2 + \frac{1}{n}\Big)^{(p-2)/2} Du\Big) + \delta_\varepsilon(x), \qquad (x,t) \in Q_{R,T}, \tag{2.1}$$

$$u(x,0) = 0, \qquad x \in (-R, R), \tag{2.2}$$

$$u(\pm R,t)=0, \qquad t\in(0,T), \tag{2.3}$$

where $Q_{R,T}=(-R,R)\times(0,T)$, $R\equiv R(T)$ is a properly large enough positive constant depending only on T, and

$$\begin{aligned}
&\delta_\varepsilon(x)=\frac{1}{\varepsilon}j\Big(\frac{x}{\varepsilon}\Big), && 0<\varepsilon<1,\\
&j(x)=\begin{cases}\dfrac{1}{A}\,\mathrm{e}^{1/(|x|^2-1)}, & |x|<1,\\ 0, & |x|\ge 1,\end{cases}\\
&A=\int_{\mathbb{R}}\mathrm{e}^{1/(|x|^2-1)}dx.
\end{aligned}$$

Obviously,

$$0\le\delta_\varepsilon(x)\in C_0^\infty(\mathbb{R}),\quad \operatorname{supp}\delta_\varepsilon=\{x\in\mathbb{R};\ |x|\le\varepsilon\},\quad \int_{\mathbb{R}}\delta_\varepsilon(x)dx=1,$$
$$\lim_{\varepsilon\to0^+}\int_{\mathbb{R}}\delta_\varepsilon(x)\phi(x)dx=\phi(0),\quad \phi\in C(\mathbb{R}).$$

By virtue of the standard theory for parabolic equations, we see that the regularized problem (2.1) – (2.3) has a classical solution $u_{\varepsilon,R,n}\in C^\infty(\overline{Q}_{R,T})$, and the maximum principle shows that

$$0\le u_{\varepsilon,R,n}\le C, \tag{2.4}$$

where the constant C depends only on ε and R. From now on we denote by C various constant, which value may be different from line to line.

Lemma 2.1 *Let $u_{\varepsilon,R,n}$ be a solution of the regularized problem (2.1) – (2.3). Then*

$$\int_{-R}^{R}\left|\frac{\partial u_{\varepsilon,R,n}}{\partial t}\right|dx\le C, \tag{2.5}$$
$$\int_{-R}^{R}|Dv_{\varepsilon,R,n}|dx\le C, \tag{2.6}$$

where C is a positive constant independent of ε, R and n, and

$$v_{\varepsilon,R,n}=\Big(|Du_{\varepsilon,R,n}|^2+\frac{1}{n}\Big)^{(p-2)/2}Du_{\varepsilon,R,n}.$$

Proof Conveniently, denote $u=u_{\varepsilon,R,n}$. Differentiating (2.1) with respect to t, we have

$$\frac{\partial w}{\partial t}=aD^2w+DaDw, \tag{2.7}$$

where $w = \dfrac{\partial u}{\partial t}$ and

$$a = \left(|Du|^2 + \frac{1}{n}\right)^{(p-2)/2} \left[1 + (p-2)\left(|Du|^2 + \frac{1}{n}\right)^{-1} |Du|^2\right].$$

Clearly,

$$w(x,0) = \delta_\varepsilon(x), \qquad x \in (-R,R),$$
$$w(\pm R,t) = 0, \qquad t \in (0,\ T).$$

Multiply (2.7) by $\dfrac{w}{\sqrt{w^2+\eta}}$ and integrate over $Q_{R,t} = (-R,R) \times (0,t)$. Integrating by parts yields

$$\begin{aligned}
&\iint_{Q_{R,t}} \frac{w}{\sqrt{w^2+\eta}} \frac{\partial w}{\partial \tau} dx d\tau \\
&\quad = \iint_{Q_{R,t}} \frac{w}{\sqrt{w^2+\eta}} a D^2 w dx d\tau + \iint_{Q_{R,t}} \frac{w}{\sqrt{w^2+\eta}} DaDw dx d\tau \\
&\quad = -\iint_{Q_{R,t}} D\left(\frac{w}{\sqrt{w^2+\eta}}\right) aDw dx d\tau \\
&\qquad - \iint_{Q_{R,t}} \frac{w}{\sqrt{w^2+\eta}} DaDw dx d\tau + \iint_{Q_{R,t}} \frac{w}{\sqrt{w^2+\eta}} DaDw dx d\tau \\
&\quad = -\iint_{Q_{R,t}} \frac{\eta a |Dw|^2}{(w^2+\eta)^{3/2}} dx d\tau \le 0.
\end{aligned}$$

Notice

$$\begin{aligned}
&\iint_{Q_{R,t}} \frac{w}{\sqrt{w^2+\eta}} \frac{\partial w}{\partial \tau} dx d\tau \\
&\quad = \int_{-R}^{R} \int_0^t \frac{\partial}{\partial \tau} \left(\int_0^w \frac{s}{\sqrt{s^2+\eta}} ds\right) d\tau dx \\
&\quad = \int_{-R}^{R} \int_0^{w(x,t)} \frac{s}{\sqrt{s^2+\eta}} ds dx - \int_{-R}^{R} \int_0^{\delta_\varepsilon(x)} \frac{s}{\sqrt{s^2+\eta}} ds dx.
\end{aligned}$$

We have

$$\int_{-R}^{R} \int_0^{w(x,t)} \frac{s}{\sqrt{s^2+\eta}} ds dx \le \int_{-R}^{R} \int_0^{\delta_\varepsilon(x)} \frac{s}{\sqrt{s^2+\eta}} ds dx \le \int_{-R}^{R} \delta_\varepsilon(x) dx \le C.$$

Letting $\eta \to 0^+$, we see that

$$\int_{-R}^{R} |w(x,t)| dx \le C,$$

that is

$$\int_{-R}^{R} \left|\frac{\partial u}{\partial t}\right| dx \le C.$$

Combining the above with the equation (2.1), we have

$$\int_{-R}^{R}\Big|D\Big(\Big(|Du|^2+\frac{1}{n}\Big)^{(p-2)/2}Du\Big)+\delta_\varepsilon(x)\Big|dx\le C.$$

Thus

$$\int_{-R}^{R}\Big|D\Big(\Big(|Du|^2+\frac{1}{n}\Big)^{(p-2)/2}Du\Big)\Big|dx\le C.$$

Denoting $v_{\varepsilon,R,n}=(|Du_{\varepsilon,R,n}|^2+\frac{1}{n})^{(p-2)/2}Du_{\varepsilon,R,n}$ we obtain that

$$\int_{-R}^{R}|Dv_{\varepsilon,R,n}|dx\le C.$$

The proof is completed.

Lemma 2.2 *Let $u_{\varepsilon,R,n}$ be a solution of the regularized problem (2.1) – (2.3). Then*

$$\iint_{Q_{R,T}}|Du_{\varepsilon,R,n}|^p dxdt\le C, \tag{2.8}$$

where C is a positive constant depending only on ε, R, and T.

Proof Conveniently, denote $u=u_{\varepsilon,R,n}$. Multiplying (2.1) by u and integrating over $Q_{R,T}$ yield

$$\iint_{Q_{R,T}}\frac{\partial u}{\partial t}udxdt=-\iint_{Q_{R,T}}\Big(|Du|^2+\frac{1}{n}\Big)^{(p-2)/2}|Du|^2dxdt+\iint_{Q_{R,T}}\delta_\varepsilon(x)udxdt.$$

By virtue of (2.4) and (2.5), we further obtain

$$\begin{aligned}\iint_{Q_{R,T}}|Du|^p dxdt\le&\iint_{Q_{R,T}}\Big(|Du|^2+\frac{1}{n}\Big)^{(p-2)/2}|Du|^2dxdt\\=&-\iint_{Q_{R,T}}\frac{\partial u}{\partial t}udxdt+\iint_{Q_{R,T}}\delta_\varepsilon(x)udxdt\\\le& C.\end{aligned}$$

The proof is completed.

Utilizing the results in [6], we can easily obtain that the Hölder norms of $u_{\varepsilon,R,n}$ and $Du_{\varepsilon,R,n}$ are bounded. Combining this with (2.4), (2.5) and Lemma 2.2, we conclude that there exist a subsequence of $\{u_{\varepsilon,R,n}\}$, denoted by $\{u_{\varepsilon,R,n}\}$ itself, and a bounded nonnegative function

$$u_{\varepsilon,R}\in L^\infty(Q_{R,T})\cap L^\infty(0,T;W^{1,p}(-R,R))\cap BV(Q_{R,T}),$$

such that

$$u_{\varepsilon,R,n}\in BV(Q_{R,T}),$$

$$u_{\varepsilon,R,n} \longrightarrow u_{\varepsilon,R}, \qquad \text{uniformly in } Q_{R,T}, \tag{2.9}$$
$$Du_{\varepsilon,R,n} \longrightarrow Du_{\varepsilon,R}, \qquad \text{uniformly in } Q_{R,T}. \tag{2.10}$$

By an approximate process, we can easily get the existence of solutions of the following problem

$$\frac{\partial u}{\partial t} = D(|Du|^{p-2}Du) + \delta_\varepsilon(x), \qquad (x,t) \in Q_{R,T}, \tag{2.11}$$
$$u(x,0) = 0, \qquad x \in (-R,R), \tag{2.12}$$
$$u(\pm R,t) = 0, \qquad t \in (0,T). \tag{2.13}$$

On the other hand, following the idea of [7], we know that problem (2.11) – (2.13) has at most one solution. And according to (2.6), we obtain

$$\int_{-R}^{R} |Dv_{\varepsilon,R}|dx \le C, \tag{2.14}$$

where $v_{\varepsilon,R} = |Du_{\varepsilon,R}|^{p-2}Du_{\varepsilon,R}$, C is a positive constant independent of ε and R.

Now, we will prove the finite propagation of disturbances of $u_{\varepsilon,R}$, for which the following estimates are needed. For this purpose, we want the following weighted energy equality.

Lemma 2.3 *Let $u_{\varepsilon,R}$ be a solution of the problem (2.11) – (2.13). Then for any $0 \le \rho \in C^2(\overline{\Omega})$ with* $\mathrm{supp}\rho \cap \mathrm{supp}\delta_\varepsilon = \varnothing$*, we have*

$$\frac{1}{2}\int_{-R}^{R} \rho(x)u_{\varepsilon,R}^2(x,t)dx = -\iint_{Q_{R,t}} |Du_{\varepsilon,R}|^{p-2}Du_{\varepsilon,R}D(\rho(x)u_{\varepsilon,R}(x,\tau))dxd\tau, \tag{2.15}$$

where $Q_{R,t} = (-R,R) \times (0,t)$.

Proof Multiplying the equation (2.1) by $\rho(x)u_{\varepsilon,R,n}$ and then integrating over $Q_{R,t}$ yield

$$\begin{aligned}
&\frac{1}{2}\int_{-R}^{R} \rho(x)u_{\varepsilon,R,n}^2(x,t)dx \\
&\quad = -\iint_{Q_{R,t}} \left(|Du_{\varepsilon,R,n}|^2 + \frac{1}{n}\right)^{(p-2)/2} Du_{\varepsilon,R,n}D(\rho(x)u_{\varepsilon,R,n}(x,\tau))dxd\tau \\
&\quad = -\iint_{Q_{R,t}} \rho'(x)u_{\varepsilon,R,n}\left(|Du_{\varepsilon,R,n}| + \frac{1}{n}\right)^{(p-2)/2} Du_{\varepsilon,R,n}dxd\tau \\
&\qquad -\iint_{Q_{R,t}} \rho(x)\left(|Du_{\varepsilon,R,n}| + \frac{1}{n}\right)^{(p-2)/2} |Du_{\varepsilon,R,n}|^2dxd\tau.
\end{aligned}$$

Noticing (2.9), (2.10) and letting $n \to \infty$, we see that (2.15) holds.

The following two Lemmas are also required.

Lemma 2.4 (Hardy's inequality [8])

$$\int_{\mathbb{R}}(x)_+^k|u|^p dx \le C\int_{\mathbb{R}}(x)_+^{k+p}|Du|^p dx,$$

where $k\ge 0$, $p>1$, provided that the integrals on both sides exists.

Lemma 2.5 (Weighted Nirenberg' inequality [9])

$$\left(\int_{\mathbb{R}}(x)_+^k|u|^p dx\right)^{1/p} \le C\left(\int_{\mathbb{R}}(x)_+^k|Du|^p dx\right)^{a/p}\left(\int_{\mathbb{R}}(x)_+^k|u|^q dx\right)^{(1-a)/q},$$

where k is a nonnegative integer, $(x)_+=\max\{x,0\}$. Provided that the integral on the right hand side exists, and

$$\frac{1}{2}\le a<1,\quad \frac{1}{p}=\frac{1}{1+k}+a\Big(\frac{1}{p}-\frac{2}{1+k}\Big)+(1-a)\frac{1}{q}.$$

Proposition 2.1 *Let u be the solution of the approaching problem (2.11)–(2.13), then*

$$\operatorname{supp} u(\cdot,t)\subset[R_1,R_2],\qquad a.e.t\in(0,T),$$

where

$$R_1=-1-C_2T^\mu,\qquad R_2=1+C_3T^\mu,$$

with the positive constants C_2, C_3, μ depend only on p.

Proof Setting $\rho(x)=(x-y)_+^k$ in Lemma 2.3, where $y\in[1,R)$ is any fixed constant. Using Young's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\int_{-R}^{R}(x-y)_+^k u^2(x,t)dx\\
&=-\iint_{Q_{R,t}}|Du|^{p-2}DuD((x-y)_+^k u(x,\tau))dxd\tau\\
&=-\iint_{Q_{R,t}}(x-y)_+^k|Du|^p dxd\tau-k\iint_{Q_{R,t}}(x-y)_+^{k-1}u|Du|^{p-2}Dudxd\tau\\
&\le-\frac{1}{2}\iint_{Q_{R,t}}(x-y)_+^k|Du|^p dxd\tau+C\iint_{Q_{R,t}}(x-y)_+^{k-p}u^p dxd\tau.
\end{aligned}$$

Thus

$$\frac{1}{2}\int_{-R}^{R}(x-y)_+^k u^2(x,t)dx+\frac{1}{2}\iint_{Q_{R,t}}(x-y)_+^k|Du|^p dxd\tau\le C\iint_{Q_{R,t}}(x-y)_+^{k-p}u^p dxd\tau.$$

From this and

$$\iint_{Q_{R,t}}(x-y)_+^{k-p}|u|^p dxd\tau\le C\iint_{Q_{R,t}}(x-y)_+^k|Du|^p dxd\tau,$$

which is a consequence of Lemma 2.4, we obtain

$$\sup_{0<\tau\le t}\int_{-R}^{R}(x-y)_+^k u^2(x,\tau)dx \le C\iint_{Q_{R,t}}(x-y)_+^k|Du|^p dxd\tau, \tag{2.16}$$

$$\iint_{Q_{R,t}}(x-y)_+^k|Du|^p dxd\tau \le C\iint_{Q_{R,t}}(x-y)_+^{k-p}u^p dxd\tau. \tag{2.17}$$

Set

$$f_m(y)=\iint_{Q_{R,t}}(x-y)_+^m|Du|^p dxd\tau,\quad m=1,2,\cdots,$$

$$f_0(y)=\int_0^t\int_y^R|Du|^P dxd\tau.$$

From (2.16), (2.17), Lemma 2.5 and Hölder's inequality, we have

$$\begin{aligned}
f_{2p+1}(y)=&\iint_{Q_{R,t}}(x-y)_+^{2p+1}|Du|^p dxd\tau\\
\le& C\iint_{Q_{R,t}}(x-y)_+^{p+1}u^p dxd\tau\\
\le& C\int_0^t\left(\int_{-R}^{R}(x-y)_+^{p+1}|Du|^p dx\right)^a\left(\int_{-R}^{R}(x-y)_+^{p+1}u^2 dx\right)^{(1-a)p/2}d\tau\\
\le& C\int_0^t\left(\int_{-R}^{R}(x-y)_+^{p+1}|Du|^p dx\right)^a\left(\iint_{Q_{R,t}}(x-y)_+^{p+1}|Du|^p dxd\tau\right)^{(1-a)p/2}d\tau\\
=& C\left[f_{p+1}(y)\right]^{(1-a)p/2}\int_0^t\left(\int_{-R}^{R}(x-y)_+^{p+1}|Du|^p dx\right)^a d\tau\\
\le& Ct^{1-a}\left[f_{p+1}(y)\right]^{a+(1-a)p/2},
\end{aligned}$$

where

$$a=\frac{\dfrac{1}{2}+\dfrac{1}{p+2}-\dfrac{1}{p}}{\dfrac{1}{2}+\dfrac{2}{p+2}-\dfrac{1}{p}}.$$

Set $\gamma = a+(1-a)\dfrac{p}{2}$. Applying Hölder' inequality to the right side of the above inequality, we further obtain

$$\begin{aligned}
f_{2p+1}(y)\le& Ct^{1-a}\left(\iint_{Q_{R,t}}(x-y)_+^{p+1}|Du|^p dxd\tau\right)^{\gamma}\\
\le& Ct^{1-a}\left(\iint_{Q_{R,t}}(x-y)_+^{2p+1}|Du|^p dxd\tau\right)^{(p+1)\gamma/(2p+1)}
\end{aligned}$$

$$\begin{aligned}&\cdot\left(\int_0^t\int_y^R|Du|^P dxd\tau\right)^{p\gamma/(2p+1)}\\ \le& Ct^{1-a}\left[f_{2p+1}(y)\right]^{(p+1)\gamma/(2p+1)}\left[f_0(y)\right]^{p\gamma/(2p+1)}.\end{aligned}$$

Therefore

$$f_{2p+1}(y)\le Ct^{(1-a)/\sigma}[f_0(y)]^{p\gamma/(2p+1)\sigma},$$

where

$$\sigma=1-\frac{p+1}{2p+1}\gamma>0.$$

Using Hölder's inequality again gives

$$f_1(y)\le(f_{2p+1}(y))^{1/(2p+1)}[f_0(y)]^{2p/(2p+1)}\le Ct^{\lambda}[f_0(y)]^{\theta+1},$$

where

$$\lambda=\frac{1-a}{\sigma(2p+1)},\quad \theta=\frac{p\gamma}{\sigma(2p+1)^2}-\frac{1}{2p+1}>0.$$

Since $f_1'(y)=-f_0(y)$, we have

$$f_1'(y)\le -Ct^{-\lambda/(\theta+1)}[f_1(y)]^{1/(\theta+1)}.$$

If $f_1(1)=0$, then $Du(x,t)=0$ for $x\in[1,R]$, and hence from the boundary value condition, we see that $u(x,t)=0$ for $x\in[1,R]$, i.e. $\mathrm{supp}u(\cdot,\,t)\subset[-R,1]$. If $f_1(1)\neq 0$, then there exists an interval $(1,R^*)$, such that $f_1(y)>0$ in $(1,R^*)$, but $f_1(R^*)=0$. So, for $y\in(1,R^*)$,

$$\left(f_1(y)^{\theta/(\theta+1)}\right)'=\frac{\theta}{\theta+1}\frac{f_1'(y)}{f_1(y)^{1/(\theta+1)}}\le -Ct^{-\lambda/(\theta+1)}.$$

Integrating the above inequality over $(1,R^*)$, we obtain

$$f_1(R^*)^{\theta/(\theta+1)}-f_1(1)^{\theta/(\theta+1)}\le -Ct^{-\lambda/(\theta+1)}(R^*-1).$$

Therefore

$$R^*\le 1+Ct^{\lambda/(\theta+1)}f_1(1)^{\theta/(\theta+1)}=1+Ct^{\mu}\le 1+CT^{\mu},$$

which implies

$$\mathrm{supp}u(\cdot,\,t)\subset[-R,1+CT^{\mu}].$$

Similarly

$$\mathrm{supp}u(\cdot,\,t)\subset[-1-CT^{\mu},R].$$

The proof is completed.

Combining (2.14) with Corollary 2.1, we then see that there exists a positive constant C independent of ε and R, such that

$$0\le\sup_{\Omega_{R,T}}u_{\varepsilon,R}\le C,$$

$$|Du_{\varepsilon,R}| \le C.$$

Owing to Corollary 2.1 we know $u_{\varepsilon,R}$ has compact support. Consequently, we can extend the domain of definition respect to x to the whole $\mathbb{R}$. So, we get

$$0 \le \sup_{\Omega_T} u_\varepsilon \le C, \tag{2.18}$$

$$|Du_\varepsilon| \le C, \tag{2.19}$$

where C is a positive constant independent of ε and R, and u_ε is a solution of the problem

$$\frac{\partial u}{\partial t} = D(|Du|^{p-2}Du) + \delta_\varepsilon(x), \qquad (x,t) \in Q_T, \tag{2.20}$$

$$u(x,0) = 0, \qquad x \in \mathbb{R}. \tag{2.21}$$

In order to prove the main result, we also need the following Hölder estimate.

Lemma 2.6 *Let u_ε be a solution of the problem (2.20), (2.21). Then*

$$|u_\varepsilon(x_1, t_1) - u_\varepsilon(x_2, t_2)| \le C(|x_1 - x_2| + |t_1 - t_2|^{1/2}), \tag{2.22}$$

where C is a positive constant independent of ε.

Proof Since (2.19) implies that

$$|u_\varepsilon(x_1,t) - u_\varepsilon(x_2,t)| \le C|x_1 - x_2|, \quad \forall (x_1,t),(x_2,t) \in Q_T, \tag{2.23}$$

it remains to prove

$$|u(x,t_1) - u(x,t_2)| \le C|t_1 - t_2|^{1/2}, \quad \forall (x,t_1),(x,t_2) \in Q_T. \tag{2.24}$$

For any $(x,t_1) and (x,t_2) \in Q_T$, satisfy $\Delta t = t_2 - t_1 > 0$, and $x + l \le 1$, where denote $l = \Delta t^{1/2}$, we integrate (2.20) over $(x, x+l) \times (t_1, t_2)$. Integrating by parts gives

$$\int_x^{x+l} (u_\varepsilon(z, t_2) - u_\varepsilon(z, t_1))\, dz = \int_{t_1}^{t_2} |Du_\varepsilon|^{p-2} Du_\varepsilon \Big|_x^{x+l} dt + \int_{t_1}^{t_2}\int_x^{x+l} \delta_\varepsilon(z) dz dt.$$

Using the mean value theorem for integrals, we see that

$$\int_x^{x+l} (u_\varepsilon(z, t_2) - u_\varepsilon(z, t_1))\, dz = l(u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1)),$$

for some $x^* \in [x, x+l]$. This, together with (2.18) and (2.19) gives

$$|u_\varepsilon(x^*, t_2) - u_\varepsilon(x^*, t_1)| \le Cl.$$

Thus

$$\begin{aligned}
&|u_\varepsilon(x,\,t_2)-u_\varepsilon(x,\,t_1)|\\
&\qquad\le|u_\varepsilon(x,\,t_2)-u_\varepsilon(x^*,\,t_1)|+|u_\varepsilon(x^*,\,t_2)-u_\varepsilon(x^*,\,t_1)|+|u_\varepsilon(x^*,\,t_1)-u_\varepsilon(x,\,t_1)|\\
&\qquad\le Cl=C\Delta^{1/2}.
\end{aligned}$$

The proof is completed.

According to Ascoli-Arzelá's Theorem, there exist a subsequence of $\{u_\varepsilon\}$, supposed to be $\{u_\varepsilon\}$ itself, a bounded nonnegative function

$$u\in L^\infty(Q_T)\cap L^\infty(0,T;W^{1,p}(\mathbb{R}))\cap BV(Q_T)$$

and a function $\chi\in L^{p/(p-1)}(Q_T)$ such that

$$\begin{aligned}
&u_\varepsilon\to u, && \text{uniformly in}\quad Q_T,\\
&Du_\varepsilon\to Du, && \text{uniformly in}\quad Q_T,\\
&|Du_\varepsilon|^{p-2}Du_\varepsilon\to\chi, && \text{weakly in}\quad L^{p/(p-1)}(Q_T).
\end{aligned}$$

Now, we prove the main result

The Proof of Theorem 1.1 It is easy to verify that

$$-\iint_{Q_T}u\frac{\partial\varphi}{\partial t}dxdt=-\iint_{Q_T}\chi D\varphi dxdt+\int_0^T\varphi(0,t)dt,\quad\forall\varphi\in C_0^\infty(Q_T).\tag{2.25}$$

For any $v\in L^p(0,T;W^{1,p}(\mathbb{R}))$ and $\psi\in C_0^\infty(Q_T)$, $0\le\psi\le1$, supp$\psi\subset Q_T$, we can easily obtain

$$\iint_{Q_T}\psi(|Du_\varepsilon|^{p-2}Du_\varepsilon-|Dv|^{p-2}Dv)D(u_\varepsilon-v)dxdt\ge0.\tag{2.26}$$

Multiplying (2.20) by ψu_n and integrating over Q_T yield

$$\begin{aligned}
&\iint_{Q_T}\psi|Du_\varepsilon|^pdxdt\\
&\quad=\frac12\iint_{Q_T}u_\varepsilon^2\frac{\partial\psi}{\partial t}dxdt-\iint_{Q_T}u_\varepsilon|Du_\varepsilon|^{p-2}Du_\varepsilon D\psi dxdt+\iint_{Q_T}\delta_\varepsilon(x)u_\varepsilon\psi dxdt.
\end{aligned}\tag{2.27}$$

From (2.27) and (2.26), we further obtain

$$\begin{aligned}
&\frac12\iint_{Q_T}u_\varepsilon^2\frac{\partial\psi}{\partial t}dxdt-\iint_{Q_T}u_\varepsilon|Du_\varepsilon|^{p-2}Du_\varepsilon D\psi dxdt+\iint_{Q_T}\delta_\varepsilon(x)u_\varepsilon\psi dxdt\\
&\qquad-\iint_{Q_T}\psi|Du_\varepsilon|^{p-2}Du_\varepsilon Dvdxdt-\iint_{Q_T}\psi|Dv|^{p-2}DvD(u_\varepsilon-v)dxdt\ge0.
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}&\frac{1}{2}\iint_{Q_T} u^2\frac{\partial\psi}{\partial t}dxdt - \iint_{Q_T} u\chi D\psi dxdt + \iint_{Q_T}\delta(x)u\psi dxdt\\&\quad - \iint_{Q_T}\psi\chi Dvdxdt - \iint_{Q_T}\psi|Dv|^{p-2}DvD(u-v)dxdt \ge 0.\end{aligned} \tag{2.28}$$

Take $\varphi = \psi u$ in (2.25), then

$$\frac{1}{2}\iint_{Q_T} u^2\frac{\partial\psi}{\partial t}dxdt = \iint_{Q_T} u\chi D\psi dxdt + \iint_{Q_T}\psi\chi Dudxdt - \iint_{Q_T}\delta(x)u\psi dxdt. \tag{2.29}$$

Substituting this into (2.28), then we deduce that

$$\iint_{Q_T}\psi(\chi - |Dv|^{p-2}Dv)D(u-v)dxdt \ge 0. \tag{2.30}$$

Take $v = u - \lambda\varphi$, $\lambda \ge 0$ with $\lambda \ge 0$, $\varphi \in C_0^\infty(Q_T)$ in (2.30),

$$\iint_{Q_T}\psi(\chi - |D(u-\lambda\varphi)|^{p-2}D(u-\lambda\varphi))D\varphi dxdt \ge 0.$$

Letting $\lambda \to 0$, we obtain

$$\iint_{Q_T}\psi(\chi - |Du|^{p-2}Du)D\varphi dxdt \ge 0, \qquad \forall\varphi \in C_0^\infty(Q_T).$$

If we take $\lambda \le 0$, then we can obtain the opposite inequality. Therefore, if we choose ψ such that $\operatorname{supp}\varphi \subset \operatorname{supp}\psi$ and $\psi = 1$ on $\operatorname{supp}\varphi$, then

$$\iint_{Q_T}|Du|^{p-2}DuD\varphi dxdt = \iint_{Q_T}\chi D\varphi dxdt, \quad \forall\varphi \in C_0^\infty(Q_T).$$

Consequently,

$$-\iint_{Q_T} u\frac{\partial\varphi}{\partial t}dxdt = -\iint_{Q_T}|Du|^{p-2}DuD\varphi dxdt + \int_0^T \varphi(0,t)dt,$$

that is, equality (1.3) holds. We can easily see that equality (1.4) is also fulfilled.

Next, we prove the uniqueness of the solution. Set u_1, u_2 to be two solutions of the Cauchy problem (1.1), (1.2). Let $z = u_1 - u_2$. Then for arbitrary $\varphi \in C_0^\infty(Q_T)$, the following integral holds

$$\iint_{Q_t}\frac{\partial z}{\partial t}\varphi dxdt = -\iint_{Q_t}\Big(|Du_1|^{p-2}Du_1 - |Du_2|^{p-2}Du_2\Big)D\varphi dxdt.$$

Choose $\varphi = H_\eta(z)$, where

$$H_\eta(s) = \frac{s}{\sqrt{s^2+\eta}}, \qquad H_\eta'(s) = \frac{\eta}{(s^2+\eta)^{3/2}},$$

then

$$\iint_{Q_t} \frac{\partial}{\partial t}\mathcal{H}_\eta(z)dxdt = -\iint_{Q_t}\Big(|Du_1|^{p-2}Du_1 - |Du_2|^{p-2}Du_2\Big)DzH_\eta'(z)dxdt,$$

where

$$\mathcal{H}_\eta(s) = \int_k^s H_\eta(\tau)d\tau, \qquad \lim_{\eta\to 0^+}\mathcal{H}_\eta(s) = |s|.$$

Notice that the right side is no larger than zero, therefore

$$\int_{\mathbb{R}} \mathcal{H}_\eta(z)dxdt - \int_{\mathbb{R}} \mathcal{H}_\eta(z_0)dxdt \le 0.$$

Finally, letting $\eta \to 0^+$, we obtain

$$\int_{\mathbb{R}} |z|dxdt \le 0.$$

Hence $u_1 = u_2$. The proof is complete.

Acknowledgement The authors would like to express their deep thanks to Professor Yuan Hongjun and Professor Yin Jingxue for helpful discussion of this paper.

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