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THE CAUCHY PROBLEM OF NONLINEAR SCHRÖDINGER-BOUSSINESQ EQUATIONS IN $H^s(R^d)$

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Abstract In this paper, the local well posedness and global well posedness of solutions for the initial value problem (IVP) of nonlinear Schrödinger-Boussinesq equations is considered in $H^s(R^d)$ by resorting Besov spaces, where real number $s \geq 0$.
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1. Introduction

We consider the existence and uniqueness of the local solutions and global solutions for the following initial value problem (IVP) of nonlinear Schrödinger-Boussinesq equations

$$i\epsilon_t + \Delta\epsilon - n\epsilon - A|\epsilon|^p\epsilon = 0, \tag{1.1}$$

$$n_{tt} - \Delta(n - \Delta n + Bn^{K+1} + |\epsilon|^2) = 0, \quad x \in R^d, t \in R, \tag{1.2}$$

$$\epsilon(x,0) = \epsilon_0(x), \quad n(x,0) = n_0(x), \quad n_t(x,0) = \Delta\phi_0(x), \quad x \in R^d, \tag{1.3}$$

where A and B are constants, K is a positive integer, real number $p > 0$; ϵ and ϵ_0 are complex functions; n, n_0 and ϕ_0 are real functions; Δ is Laplacian operator in R^d.

The nonlinear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in nonlinear media, propagation of Langmuir waves in plasmas, etc. (see [1] and the references therein). Boussinesq equation as a model of long waves is derived in the studies of the propagation of long waves on the surface of shallow water[2], the nonlinear string [3] and the shape-memory alloys[4], etc. The nonlinear Schrödinger-Boussinesq equations (1.1)(1.2) is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction in a plasma [5-7] and diatomic lattice system [8], etc.

The Solitary wave solutions and integrability of nonlinear Schrödinger-Boussinesq equations has been considered by several authors, see [5, 6, 9] and the references therein. In [10] Guo established the existence and uniqueness of global solution for IVP (1.1)–(1.3) in H^k (integer $k \geq 4$) with $d = 1$ and $A = 0$. In [11] the existence and uniqueness of global solution for Cauchy problem of dissipative Schrödinger-Boussinesq equations in H^k (integer $k \geq 4$) with $d = 3$ is proved by Guo and Shen. For damped and dissipative Schrödinger-Boussinesq equations with initial boundary value, the existence of global attractors and the finiteness of the Hausdorff and the fractal dimensions of the attractor is established by Guo and Chen ([12], d=1) and Li and Chen ([13], $d \leq 3$), respectively.

In this paper, the local well-posedness in H^s, the conservation of energy and the global well-posedness in H^s (real number $s \geq 1$ and $d = 1, 2, 3$) of IVP (1.1)–(1.3) is proved.

Definition 1(admissible pair) *The pair* (q, r) *is admissible if* $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{r})$; $2 \leq r \leq \infty$ *for* $d = 1$, $2 \leq r \leq \infty$ *for* $d = 2$, $2 \leq r < \frac{2d}{d-2}$ *for* $d \geq 3$.

Definition 2*(condition $P(m)$) For a positive integer m, it is called that p satisfies the condition $P(m)$ if either p is an even integer, or p is not an even integer and $p + 1 > m$.*

The main theorems of this paper are stated as follows.

Theorem 1 *Suppose that* $\epsilon_0, n_0, \phi_0 \in H^s(R^d)$, $0 \leq s < \frac{d}{2}$, *K is an integer, p satisfies the condition* $P([s] + 1)$, $0 < p$, $K \leq \frac{4}{d-2s}$; *then for any admissible pair* (q, r), *there exists* $T = T(\epsilon_0, n_0, \phi_0) > 0$ *and a unique solution* (ϵ, n) *of IVP (1.1)–(1.3) such that*

$$\epsilon, n, (-\Delta)^{-1} n_t \in L^q\left(0, T; B^s_{r,2}(R^d)\right) \cap C\left([0, T]; H^s(R^d)\right)$$

Moreover, this solution has the following additional properties.

(I) *Let* p, $K < \frac{4}{d-2s}$. *If* $\epsilon_{0j}, n_{0j}, \phi_{0j}$ *are sequences in* $H^s(R^d)$ *with* $(\epsilon_{0j}, n_{0j}, \phi_{0j}) \to (\epsilon_0, n_0, \phi_0)$, *then there exists* $\tilde{T} = \tilde{T}(\epsilon_0, n_0, \phi_0) \in (0, T]$, *such that the solutions* $(\epsilon_j, n_j) \to (\epsilon, n)$ *and* $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ *in* $L^q\left(0, \tilde{T}; L^r(R^d)\right)$, *where* (ϵ_j, n_j) *are solutions of IVP (1.1)–(1.3) with* $(\epsilon_0, n_0, \phi_0)$ *replaced by* $(\epsilon_{0j}, n_{0j}, \phi_{0j})$. *If* $s \geq 1$, *then* $(\epsilon_j, n_j) \to (\epsilon, n)$ *and* $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ *in* $C\left([0, \tilde{T}]; H^{s-1}(R^d)\right) \cap L^q\left(0, \tilde{T}; B^{s-1}_{r,2}\right)$. *Moreover, if p satisfies the condition* $P([s] + 2)$, *then* $(\epsilon_j, n_j) \to (\epsilon, n)$ *and* $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ *in* $C\left([0, \tilde{T}]; H^s(R^d)\right) \cap L^q\left(0, \tilde{T}; B^s_{r,2}\right)$.

(II) *There exists* $T^\star = T^\star(\epsilon_0, n_0, \phi_0) > 0$ *such that the solution* $\epsilon, n, (-\Delta)^{-1} n_t \in C\left([0, T^\star); H^s(R^d)\right) \cap L^q_{loc}\left(0, T^\star; B^s_{r,2}(R^d)\right)$. *If* $T^\star < \infty$, *then*

$$\lim_{t \to T^\star} \left\{ \|(-\Delta)^{\frac{s}{2}} \epsilon(\cdot, t)\|_{L^2} + \|(-\Delta)^{\frac{s}{2}} n(\cdot, t)\|_{L^2} + \|(-\Delta)^{\frac{s-2}{2}} n_t(\cdot, t)\|_{L^2} \right\} = +\infty.$$

Theorem 2 *Suppose that $\epsilon_0, n_0, \phi_0 \in H^s(R^d)$, $s \geq \frac{d}{2}$, K is an integer, p satisfies the condition $P([s]+1)$, $0 < p, K < \infty$; then for any admissible pair (q,r), there exists $T = T(\epsilon_0, n_0, \phi_0) > 0$ and a unique solution (ϵ, n) of IVP (1.1)–(1.3) such that*

$$\epsilon, n, (-\Delta)^{-1} n_t \in L^q\left(0, T; B^s_{r,2}(R^d)\right) \cap C\left([0,T]; H^s(R^d)\right)$$

Moreover, this solution has the following additional properties.

(I) *If $\epsilon_{0j}, n_{0j}, \phi_{0j}$ are sequences in $H^s(R^d)$ with $(\epsilon_{0j}, n_{0j}, \phi_{0j}) \to (\epsilon_0, n_0, \phi_0)$, then there exists $\tilde{T} = \tilde{T}(\epsilon_0, n_0, \phi_0) \in (0, T]$, such that the solutions $(\epsilon_j, n_j) \to (\epsilon, n)$ and $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ in $L^q(0, \tilde{T}; L^r(R^d))$, where $(\epsilon_j,\ n_j)$ are solutions of IVP (1.1)–(1.3) with $(\epsilon_0, n_0, \phi_0)$ replaced by $(\epsilon_{0j}, n_{0j}, \phi_{0j})$. If $s \geq 1$, then $(\epsilon_j, n_j) \to (\epsilon, n)$ and $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ in $C([0,\tilde{T}]; H^{s-1}(R^d)) \cap L^q(0,\tilde{T}; B^{s-1}_{r,2})$. Moreover, if p satisfies the condition $P([s]+2)$, then $(\epsilon_j, n_j) \to (\epsilon, n)$ and $(-\Delta)^{-1}\partial_t n_j \to (-\Delta)^{-1} n_t$ in $C([0,\tilde{T}]; H^s(R^d)) \cap L^q(0,\tilde{T}; B^s_{r,2})$.*

(II) *There exists $T^\star = T^\star(\epsilon_0, n_0, \phi_0) > 0$ such that the solution $\epsilon, n, (-\Delta)^{-1} n_t \in C([0,T^\star); H^s(R^d)) \cap L^q_{loc}(0, T^\star; B^s_{r,2}(R^d))$. If $T^\star < \infty$, then*

$$\lim_{t \to T^\star} \left\{ \|(-\Delta)^{\frac{s}{2}} \epsilon(\cdot,t)\|_{L^2} + \|(-\Delta)^{\frac{s}{2}} n(\cdot,t)\|_{L^2} + \|(-\Delta)^{\frac{s-2}{2}} n_t(\cdot,t)\|_{L^2} \right\} = +\infty.$$

Theorem 3 *Suppose that s and K are integers.*

(I) *Let $0 \leq s < \frac{d}{2}$, $0 < p,\ K \leq \frac{4}{d-2s}$, $P([s]+1)$ be replaced by $P(s)$ and $P([s]+2)$ be replaced by $P(s+1)$, then all the conclusions of Theorem 1 are valid if Besov space $B^s_{r,2}$ is replaced by Sobolev space H^s_r.*

(II) *Let $s \geq \frac{d}{2}$, $0 < p,\ K < \infty$, $P([s]+1)$ be replaced by $P(s)$ and $P([s]+2)$ be replaced by $P(s+1)$, then all the conclusions of Theorem 2 are valid if Besov space $B^s_{r,2}$ is replaced by Sobolev space H^s_r.*

Theorem 4 *Let integer $m \geq 1$, p satisfy the condition $P(J)$ where $J = \max\{2, m\}$,*

$$1 \leq d \leq 3, \quad B > 0, \quad K \text{ is an even integer}, \tag{1.4}$$

$$0 < p,\ K < \left\{ \begin{array}{ll} \infty, & d = 1,2, \\ 4, & d = 3, \end{array} \right. \quad \max\{p - \frac{4}{d}, 0\} A \geq 0 \tag{1.5}$$

and $\epsilon_0,\ n_0,\ \phi_0 \in H^m$. Then for any $T \in (0,\infty)$, there exists a unique solution (ϵ, n) of IVP (1.1)–(1.3) such that $\epsilon,\ n,\ (-\Delta)^{-1} n_t \in C([0,T]; H^m(R^d)) \cap L^q(0,T; H^m_r(R^d))$, where (q,r) is any admissible pair. Moreover, for all $t \in [0,T]$, we get that $\|\epsilon(\cdot,t)\|_{L^2} = \|\epsilon_0\|_{L^2}$,

$$\begin{aligned} E(t) = \int_{R^d} \Big\{ |\nabla \epsilon|^2 + n|\epsilon|^2 + \frac{2A}{p+2}|\epsilon|^{p+2} + \frac{1}{2}\Big(|(-\Delta)^{-\frac{1}{2}} n_t|^2 + n^2 + |\nabla n|^2 + \frac{2B}{K+2} n^{K+2}\Big)\Big\} dx \\ = E(0), \end{aligned}$$

$$\|\epsilon(\cdot,t)\|_{H^1} + \|(-\Delta)^{-1} n_t(\cdot,t)\|_{H^1} + \|n(\cdot,t)\|_{H^1} \leq C(T,\ \|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},\ \|\phi_0\|_{H^1}).$$

Theorem 5 *Suppose that real number $s \geq 1$, ϵ_0, n_0, $\phi_0 \in H^s$, d, B, K, p, A satisfy the conditions (1.4)(1.5), p satisfies the condition $P([s]+1)$ and (q,r) is any admissible pair. Then for any $0 < T < \infty$ there exists a unique solution (ϵ, n) of IVP (1.1)–(1.3) such that*

$$\epsilon,\ \ n,\ \ (-\Delta)^{-1}n_t \in L^q(0,T;B^s_{r,2}(R^d)) \cap C([0,T];H^s(R^d)).$$

Remark Consider Cauchy problem of the following generalized Boussinesq equation

$$u_{tt} - \Delta(u - \Delta u + Bu^{K+1}) = 0,\ \ x \in R^d, t \in R, \tag{1.6}$$

$$u(x,0) = u_0(x),\ \ u_t(x,0) = \Delta u_1(x),\ \ x \in R^d, \tag{1.7}$$

where B is a constant and K is an integer, the local well-posedness and global well-posedness have been investigated by several authors [14-18, etc.]. As the corollary of the Theorem 1–5, we obtain that the problem (1.6)(1.7) is local well-posed in $H^s(R^d)$ with real $s \in [0,\infty)$ and $0 < K \begin{cases} \leq \frac{4}{d-2s}, & 0 \leq s < \frac{d}{2} \\ < \infty, & s \geq \frac{d}{2} \end{cases}$, and global well-posed in $H^s(R^d)$ with real $s \in [1,\infty)$, $1 \leq d \leq 3$, $B > 0$, K is even integer and $2 \leq K < \begin{cases} \infty, & d = 1,2 \\ 4, & d = 3 \end{cases}$. This results improve partially the results of [16, 18] by removing the condition which the initial data is sufficiently small.

Throughout this paper, we will have occasion to use a variety of function spaces; Lebesgue space $L^r = L^r(R^d)$; Sobolev spaces $H^s = H^s(R^d), H^s_r = H^s_r(R^d)$; homogeneous Sobolev spaces $\dot{H}^s = \dot{H}^s(R^d) = (-\Delta)^{-s/2}L^2(R^d)$, $\dot{H}^s_r = \dot{H}^s_r(R^d) = (-\Delta)^{-s/2}$ $L^r(R^d)$; Besov spaces $B^s_{r,b} = B^s_{r,b}(R^d)$; homogeneous Besov spaces $\dot{B}^s_{r,b} = \dot{B}^s_{r,b}(R^d)$; and the spaces $L^q(0,T;X)$ and $C([0,T];X)$, the norm of space $L^q(0,T;X)$ denotes by $\|\cdot\|_{L^q_T X}$, where X is one of the spaces just mentioned. In order to simplify the exposition, different positive constants might be denoted by the same letter C; if necessary, by $C(\cdot,\cdot)$ denote the constant depending only on the quantities appearing in parenthesis. For any number $r \geq 1$, its dual number is $r' = \frac{r}{r-1}$.

The plan of this paper is as follows. In section 2, we give $L^q(0,T;\, L^r(R^d))$ estimates for the inhomogeneous linear Schrödinger equation and Boussinesq equation. In section 3, we show the local well-posed results of IVP (1.1)–(1.3). In Section 4, we show the global well-posed results of IVP (1.1)–(1.3).

2. Estimates of Linear Equation

Resorting Besov spaces to study the well-posedness of Cauchy problem for coupled NLS-Boussinesq equation rely on a delicate balance between estimates of the linear Schrödinger equation and the linear Boussinesq equation and estimates for the nonlinear term. A large amount of work has been devoted to time-space estimates of evolution, see [19, 20] and the references therein. In this section, we give the necessary estimates for the linear Schrödinger equation and the linear Boussinesq equation.

First, we consider the linear Schrödinger equation

$$i\epsilon_t + \Delta\epsilon = g, \quad \epsilon(x,0) = \epsilon_0(x). \tag{2.1}$$

The solution of (2.1) is $\epsilon = S(t)\epsilon_0 - i\int_0^t S(t-\tau)g(\tau)d\tau$, where $S(t)$ is a unitary group, and $S(t)\epsilon_0 = C_0\int_{R^d} e^{i<x,\xi>-i|\xi|^2 t}\hat{\epsilon}_0(\xi)d\xi$. Let us set $S_I(g) = \int_0^t S(t-\tau)g(\tau)d\tau$.

Now, we consider the linear Boussinesq equation

$$u_{tt} + \Delta^2 u = \Delta g, \quad u(x,0) = u_0(x), \ \ u_t(x,0) = \Delta u_1(x). \tag{2.2}$$

The solution of (2.2) is

$$u = B_c(t)u_0 + B_s(t)\Delta u_1 + \int_0^t B_s(t-\tau)\Delta g(\tau)d\tau,$$

where $B_c(t)$ and $B_s(t)$ are semigroups,

$$B_c(t)n = C_0\int_{R^d} e^{i<x,\xi>}\cos(t|\xi|^2)\hat{n}(\xi)d\xi,$$

$$B_s(t)n = C_0\int_{R^d} e^{i<x,\xi>}\frac{\sin(t|\xi|^2)}{|\xi|^2}\hat{n}(\xi)d\xi.$$

Since

$$\cos(t|\xi|^2) = \frac{1}{2}(e^{it|\xi|^2} + e^{-it|\xi|^2}), \quad \sin(t|\xi|^2) = \frac{1}{2i}(e^{it|\xi|^2} - e^{-it|\xi|^2}),$$

we have

$$B_c(t)\psi = \frac{1}{2}(S(t)+S(-t))\psi \quad \text{and} \quad B_s(t)\Delta\psi = -\frac{1}{2i}(S(t)-S(-t))\psi.$$

Let us set

$$B_{SI}(g) = \int_0^t B_s(t-\tau)\Delta g(\tau)d\tau.$$

Due to [19, 20] and the references therein, we have the following lemma.

Lemma 2.1 *Let (q,r) and (γ,ρ) be any admissible pairs.*

(I) *If $\psi \in L^2(R^d)$, then $S(\cdot)\psi$, $B_c(\cdot)\psi$, $B_s(\cdot)\Delta\psi \in L^q(0,\infty;L^r)$; there exists a constant C such that*

$$\|S(\cdot)\psi\|_{L^q_\infty L^r} + \|B_c(\cdot)\psi\|_{L^q_\infty L^r} + \|B_s(\cdot)\Delta\psi\|_{L^q_\infty L^r} \le C\|\psi\|_{L^2}, \quad \forall\psi \in L^2. \tag{2.3}$$

(II) If $\psi \in \dot{H}^s, s \in R$, then $S(\cdot)\psi$, $B_c(\cdot)\psi$, $B_s(\cdot)\Delta\psi \in L^q(0,\infty;\dot{B}^s_{r,2})$; there exists a constant C such that

$$\|S(\cdot)\psi\|_{L^q_\infty \dot{B}^s_{r,2}} + \|B_c(\cdot)\psi\|_{L^q_\infty \dot{B}^s_{r,2}} + \|B_s(\cdot)\Delta\psi\|_{L^q_\infty \dot{B}^s_{r,2}} \le C\|\psi\|_{\dot{H}^s}, \quad \forall\psi \in \dot{H}^s. \tag{2.4}$$

(III) *If* $g \in L^{\gamma'}(0,T;L^{\rho'})$, *then* $S_I(g)$, $B_{SI}(g) \in L^q(0,T;L^r) \cap C([0,T];L^2)$; *there exists a constant* C, *independent of* T, *such that*

$$\|S_I(g)\|_{L^q_T L^r} + \|B_{SI}(g)\|_{L^q_T L^r} \le C\|g\|_{L^{\gamma'}_T L^{\rho'}}, \quad \forall g \in L^{\gamma'}(0,T;L^{\rho'}). \tag{2.5}$$

(IV) *If* $g \in L^{\gamma'}(0,T;\dot{B}^s_{\rho',2})$, $s \in R$, *then* $S_I(g)$, $B_{SI}(g) \in L^q(0,T;\dot{B}^s_{r,2}) \cap C([0,T];\dot{H}^s)$; *there exists a constant* C, *independent of* T, *such that*

$$\|S_I(g)\|_{L^q_T \dot{B}^s_{r,2}} + \|B_{SI}(g)\|_{L^q_T \dot{B}^s_{r,2}} \le C\|g\|_{L^{\gamma'}_T \dot{B}^s_{\rho',2}}, \quad \forall g \in L^{\gamma'}(0,T;\dot{B}^s_{\rho',2}). \tag{2.6}$$

3. Local Solution

In this section, we study the local well-posedness for IVP (1.1)–(1.3). First we establish some estimates of nonlinear terms in Besov spaces. We define $\rho(J)$ as follows.

$$\frac{1}{\rho(J)} = \begin{cases} \frac{d+sJ}{d(J+2)}, & 0 \le s < \frac{d}{2} \\ \frac{1}{2}\left(\frac{1}{2} + \max\left\{\frac{1}{2} - \frac{1}{d}, \frac{1}{2} - \frac{2}{(J+2)d}, \frac{1}{J+2}\right\}\right), & s \ge \frac{d}{2}, \end{cases}. \tag{3.1}$$

Lemma 3.1 *Let* J *be an integer,* $\rho = \rho(J)$, $v \in B^s_{\rho,2}$, $s \in R^+$, $\rho(J)$ *be defined in (3.1).*

(I) *Suppose that* $0 \le s < \dfrac{d}{2}$, $0 < J \le \dfrac{4}{d-2s}$, *then* $\|v^{J+1}\|_{\dot{B}^s_{\rho',2}} \le C\|v\|^{J+1}_{\dot{B}^s_{\rho,2}}$. (3.2)

(II) *Suppose that* $s \ge \dfrac{d}{2}$, $0 < J < \infty$, *then* $\|v^{J+1}\|_{B^s_{\rho',2}} \le C\|v\|^{K+1}_{B^s_{\rho,2}}$. (3.3)

Proof Estimate (3.2) has been obtained in Theorem 3.1 of [19]. Only estimates (3.3) must be proved. Let $m = [s]+1$. By the proof of Theorem 3.1 of [19], we have

$$\|v^{J+1}\|_{\dot{B}^s_{\rho',2}} \le C \sum_{k=1}^{\min\{m,K+1\}} \|v\|^{J+1-k}_{L^{\rho_\star}} \sum_{j_1+\cdots+j_k=m,\ j_q\ge 1} \prod_{q=1}^{k} \|v\|_{\dot{B}^{s/bq}_{\rho q,2bq}}.$$

where

$$\frac{1}{b_q} = \frac{j_q}{m}, \quad \frac{1}{\rho_q} = \frac{1}{\rho_\star} + \frac{1}{b_q}\left(\frac{1}{\rho} - \frac{1}{\rho_\star}\right), \quad \frac{1}{\rho_\star} = \frac{1}{J}\left(1 - \frac{2}{\rho}\right), \quad q = 1,\cdots,k.$$

If $s \ge \dfrac{d}{2}$, then

$$\frac{1}{\rho} \ge \frac{1}{\rho_\star} = \frac{1}{J}\left(1 - \frac{2}{\rho}\right) > \frac{1}{\rho} - \frac{s}{d}, \quad \frac{1}{\rho_q} - \frac{s}{b_q d} > \frac{1}{\rho} - \frac{s}{d}.$$

By the imbedding theorems of Besov spaces (see [19, 21]), we have

$$B^s_{\rho,2} \subset B^{s/b_q}_{\rho_q,2b_q}, \quad B^s_{\rho,2} \subset H^s_\rho \subset L^{\rho_\star}$$

Thus

$$\|v^{J+1}\|_{B^s_{\rho',2}} = \|v^{J+1}\|_{L^{\rho'}} + \|v^{J+1}\|_{\dot{B}^s_{\rho',2}} \le C\|v\|^{J+1}_{B^s_{\rho,2}},$$

where we have used the fact

$$\|v^{J+1}\|_{L^{\rho'}} \le \|v\|^J_{L^{\rho^\star}}\|v\|_{L^\rho}.$$

This lemma is proved.

Remark 3.1 Throughout this paper, if the nonlinear function $f(v) = v^{J+1}$ is replaced by nonlinear function $F(v) = |v|^p v \in C^m$, all estimates such as (3.2)–(3.7) (4.13) (4.18) still hold, where $m = [s]+1$, J is a positive integer, p is a real number.

Lemma 3.2 *Let J be an integer, $\rho = \rho(J)$, $\frac{2}{\gamma} = d\left(\frac{1}{2} - \frac{1}{\rho}\right)$, $n_l \in L^\gamma(0,T;B^s_{\rho,2})$, $s \in R^+$, $l = 1,\ 2$, $\rho(J)$ be defined in (3.1), (q,r) be any admissible pair.*

(I) *Suppose that $0 \le s < \frac{d}{2}$, $0 < J \le \frac{4}{d-2s}$, then there exists a constant C such that*

$$\|S_I(n_1^{J+1}) - S_I(n_2^{J+1})\|_{L^q_T L^r} \le CT^{1-\frac{J+2}{\gamma}}\Big\{\|n_1\|^J_{L^\gamma_T \dot{B}^s_{\rho,2}} + \|n_2\|^J_{L^\gamma_T \dot{B}^s_{\rho,2}}\Big\}\|n_1 - n_2\|_{L^\gamma_T L^\rho}, \quad (3.4)$$

$$\|S_I(n_1^{J+1})\|_{L^q_T \dot{B}^s_{r,2}} \le CT^{1-\frac{J+2}{\gamma}}\|n_1\|^{J+1}_{L^\gamma_T \dot{B}^s_{\rho,2}}, \quad (3.5)$$

(II) *Suppose that $s \ge \frac{d}{2}$, $0 < J < \infty$, then there exists a constant C such that*

$$\|S_I(n_1^{J+1}) - S_I(n_2^{J+1})\|_{L^q_T L^r} \le CT^{1-\frac{J+2}{\gamma}}\Big\{\|n_1\|^J_{L^\gamma_T B^s_{\rho,2}} + \|n_2\|^J_{L^\gamma_T B^s_{\rho,2}}\Big\}\|n_1 - n_2\|_{L^\gamma_T L^\rho}, \quad (3.6)$$

$$\|S_I(n_1^{J+1})\|_{L^q_T B^s_{r,2}} \le CT^{1-\frac{J+2}{\gamma}}\|n_1\|^{J+1}_{L^\gamma_T B^s_{\rho,2}}. \quad (3.7)$$

Proof First we establish estimate (3.4) (3.6). From Lemma 2.1, we see that

$$\begin{aligned}\|S_I(n_1^{J+1}) - S_I(n_2^{J+1})\|_{L^q_T L^r} &\le C\|n_1^{J+1} - n_2^{J+1}\|_{L^{\gamma'}_T L^{\rho'}}\\ &\le C\Big\{\|n_1^J(n_1 - n_2)\|_{L^{\gamma'}_T L^{\rho'}} + \|n_2^J(n_1 - n_2)\|_{L^{\gamma'}_T L^{\rho'}}\Big\}.\end{aligned}$$

By using the imbedding theorems of Besov spaces and Hölder's inequality, it follows that

$$\|n_1^J(n_1 - n_2)\|_{L^{\gamma'}_T L^{\rho'}} \le C\Big(\int_0^T \|n_1\|^{J\gamma'}_{L^{\rho^\star}}\|n_1 - n_2\|^{\gamma'}_{L^\rho}dt\Big)^{1/\gamma'}$$

$$\le \begin{cases} C\Big(\int_0^T \|n_1\|^{J\gamma'}_{\dot{B}^s_{\rho,2}}\|n_1 - n_2\|^{\gamma'}_{L^\rho}dt\Big)^{1/\gamma'} \le CT^{1-\frac{J+2}{\gamma}}\|n_1\|^J_{L^\gamma_T \dot{B}^s_{\rho,2}}\|n_1 - n_2\|_{L^\gamma_T L^\rho}, & 0 \le s < \dfrac{d}{2},\\ C\Big(\int_0^T \|n_1\|^{J\gamma'}_{B^s_{\rho,2}}\|n_1 - n_2\|^{\gamma'}_{L^\rho}dt\Big)^{1/\gamma'} \le CT^{1-\frac{J+2}{\gamma}}\|n_1\|^J_{L^\gamma_T B^s_{\rho,2}}\|n_1 - n_2\|_{L^\gamma_T L^\rho}, & s \ge \dfrac{d}{2}.\end{cases}$$

Using the same argument we have

$$\|n_2^J(n_1 - n_2)\|_{L^{\gamma'}_T L^{\rho'}} \le \begin{cases} CT^{1-\frac{J+2}{\gamma}}\|n_2\|^J_{L^\gamma_T \dot{B}^s_{\rho,2}}\|n_1 - n_2\|_{L^\gamma_T L^\rho}, & 0 \le s < \dfrac{d}{2},\\ CT^{1-\frac{J+2}{\gamma}}\|n_2\|^J_{L^\gamma_T B^s_{\rho,2}}\|n_1 - n_2\|_{L^\gamma_T L^\rho}, & s \ge \dfrac{d}{2}.\end{cases}$$

This proves (3.4) (3.6). From Lemma 2.1 and Lemma 3.1, we see that

$$\|S_I(n_1^{J+1})\|_{L_T^q \dot{B}^s_{r,2}} \leq C\|n_1^{J+1}\|_{L_T^{\gamma'} \dot{B}^s_{\rho',2}}$$

$$\leq C\Big(\int_0^T \|n_1\|^{(J+1)\gamma'}_{\dot{B}^s_{\rho,2}} dt\Big)^{1/\gamma'} \leq CT^{1-\frac{J+2}{\gamma}} \|n_1\|^{J+1}_{L_T^{\gamma} \dot{B}^s_{\rho,2}}, \quad 0 \leq s < \frac{d}{2},$$

$$\|S_I(n_1^{J+1})\|_{L_T^q B^s_{r,2}} \leq C\|n_1^{J+1}\|_{L_T^{\gamma'} B^s_{\rho',2}}$$

$$\leq C\Big(\int_0^T \|n_1\|^{(J+1)\gamma'}_{B^s_{\rho,2}} dt\Big)^{1/\gamma'} \leq CT^{1-\frac{J+2}{\gamma}} \|n_1\|^{J+1}_{L_T^{\gamma} B^s_{\rho,2}}, \quad s \geq \frac{d}{2},$$

where we have also used Hölder's inequality. Thus, (3.5) (3.7) is proved. This lemma is proved.

Proof of Theorem 1 We split n into its positive and negative frequency parts according to $n_\pm = n \pm i(-\Delta)^{-1} n_t$. Then (1.1)–(1.3) can be rewritten as

$$i\epsilon_t + \Delta\epsilon - (\frac{n_+ + n_-}{2})\epsilon - A|\epsilon|^p\epsilon = 0, \tag{3.8}$$

$$(i\partial_t \pm \Delta)n_\pm \mp \{\frac{n_+ + n_-}{2} + B(\frac{n_+ + n_-}{2})^{K+1} + |\epsilon|^2\} = 0, \tag{3.9}$$

$$\epsilon(x,0) = \epsilon_0(x), \ \ n_\pm(x,0) = n_0 \mp i\phi_0 = \tilde{n}_\pm, \quad x \in R^d. \tag{3.10}$$

Problem (3.8)–(3.10) is rewritten in a standard way as the integral equation

$$\epsilon = S(t)\epsilon_0 - i\int_0^t S(t-t')\{(\frac{n_+ + n_-}{2})\epsilon + A|\epsilon|^p\epsilon\}dt', \tag{3.11}$$

$$n_\pm = S(\pm t)\tilde{n}_\pm \mp i\int_0^t S(\pm(t-t'))\{\frac{n_+ + n_-}{2} + B(\frac{n_+ + n_-}{2})^{K+1} + |\epsilon|^2\}dt'. \tag{3.12}$$

Solving the equations (3.11)(3.12) by contraction mapping argument solves IVP (1.1)–(1.3).

Let $T>0$, $F_1>0$ are constants to be selected later, and

$$E_1 = E_1(T,F_1) = \Big\{(\epsilon, n_+, n_-) \,\Big|\, \epsilon \in \cap_{j=1}^2 L^{\gamma_j}(0,T; B^s_{\rho_j,2}), \ \ n_\pm \in \cap_{j=0,1,3} L^{\gamma_j}(0,T; B^s_{\rho_j,2}),$$

$$\sum_{j=1}^2 \|\epsilon\|_{L_T^{\gamma_j} \dot{B}^s_{\rho_j,2}} + \sum_{j=0,1,3} (\|n_+\|_{L_T^{\gamma_j} \dot{B}^s_{\rho_j,2}} + \|n_-\|_{L_T^{\gamma_j} \dot{B}^s_{\rho_j,2}}) \leq F_1\Big\},$$

where $\rho_0 = \rho(0) = 2$, $\rho_1 = \rho(1)$, $\rho_2 = \rho(p)$, $\rho_3 = \rho(K)$, $\frac{2}{\gamma_j} = d(\frac{1}{2} - \frac{1}{\rho_j})$ $(j = 0,1,2,3)$ and $\rho(J)$ is defined by (3.1). It is obvious that (γ_j, ρ_j) $(j = 0,1,2,3)$ are admissible pairs. Note that by Lemma 2.1, E_1 is never empty. Endowed with the metric

$$\text{dist}(\vec{u}, \ \vec{v}) = \sum_{j=1,2} \|u_1 - v_1\|_{L_T^{\gamma_j} L^{\rho_j}} + \sum_{j=0,1,3} \sum_{l=2,3} \|u_l - v_l\|_{L_T^{\gamma_j} L^{\rho_j}},$$

where $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$. E_1 is a complete metric space. Indeed, since $L^q(0,T;\dot{B}^s_{r,2})$ is reflexive, the closed ball of radius F_1 is weakly compact. We wish to find conditions on T and F_1 which imply that the map $M:\ (\epsilon,n_+,n_-)\to(M\epsilon,Mn_+,Mn_-)$, give by

$$M\epsilon=S(t)\epsilon_0-i\int_0^t S(t-t')\{(\frac{n_++n_-}{2})\epsilon+A|\epsilon|^p\epsilon\}dt',$$

$$Mn_\pm=S(\pm t)\tilde{n}_\pm\mp i\int_0^t S(\pm(t-t'))\{\frac{n_++n_-}{2}+B(\frac{n_++n_-}{2})^{K+1}+|\epsilon|^2\}dt',$$

is a strict contraction on E_1.

For $(\epsilon,n_+,n_-)\in E_1$, from Lemma 2.1 and Lemma 3.2 we have

$$\begin{aligned}&\sum_{j=1}^{2}\|M\epsilon\|_{L_T^{\gamma_j}\dot{B}^s_{\rho_j,2}}+\sum_{j=0,1,3}(\|Mn_+\|_{L_T^{\gamma_j}\dot{B}^s_{\rho_j,2}}+\|Mn_-\|_{L_T^{\gamma_j}\dot{B}^s_{\rho_j,2}})\\&\quad\le C\{\|\epsilon_0\|_{\dot{H}^s}+\|n_0\|_{\dot{H}^s}+\|\phi_0\|_{\dot{H}^s}\}+CT^{1-\frac{3}{\gamma_1}}(\|\epsilon\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}}+\sum_{j=+,-}\|n_j\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}})^2\\&\quad\ +CT(\sum_{j=+,-}\|n_j\|_{L_T^{\gamma_0}\dot{B}^s_{\rho_0,2}})+CT^{1-\frac{p+2}{\gamma_2}}\|\epsilon\|^{p+1}_{L_T^{\gamma_2}\dot{B}^s_{\rho_2,2}}+CT^{1-\frac{K+2}{\gamma_3}}(\sum_{j=+,-}\|n_j\|^{K+1}_{L_T^{\gamma_3}\dot{B}^s_{\rho_3,2}}).\end{aligned}$$

Let us take $F_1=2C\{\|\epsilon_0\|_{\dot{H}^s}+\|n_0\|_{\dot{H}^s}+\|\phi_0\|_{\dot{H}^s}\}$. Notice that $1-\frac{3}{\gamma_1}$, $1-\frac{p+2}{\gamma_2}$ and $1-\frac{K+2}{\gamma_3}\ge 0$ and $\lim_{T\to0}\|\cdot\|_{L_T^{\gamma_j}\dot{B}^s_{\rho_j,2}}=0$. Thus, there exists $T_1>0$, such that for any $T\in(0,T_1]$ and $(\epsilon,n_+,n_-)\in E_1$ we have

$$\begin{aligned}&CT^{1-\frac{3}{\gamma_1}}(\|\epsilon\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}}+\|n_+\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}}+\|n_-\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}})^2+CT(\|n_+\|_{L_T^{\gamma_0}\dot{B}^s_{\rho_0,2}}+\|n_-\|_{L_T^{\gamma_0}\dot{B}^s_{\rho_0,2}})\\&\quad+CT^{1-\frac{p+2}{\gamma_2}}\|\epsilon\|^{p+1}_{L_T^{\gamma_2}\dot{B}^s_{\rho_2,2}}+CT^{1-\frac{K+2}{\gamma_3}}(\|n_+\|^{K+1}_{L_T^{\gamma_3}\dot{B}^s_{\rho_3,2}}+\|n_-\|^{K+1}_{L_T^{\gamma_3}\dot{B}^s_{\rho_3,2}})\le\frac{1}{2}F_1.\end{aligned}$$

This proves $M:E_1\to E_1$.

For any (ϵ_1,u_+,u_-), $(\epsilon_2,v_+,v_-)\in E_1$, it follows from Lemma 3.2 that

$$\begin{aligned}&\sum_{j=0,1,3}(\|Mu_+-Mv_+\|_{L_T^{\gamma_j}L^{\rho_j}}+\|Mu_--Mv_-\|_{L_T^{\gamma_j}L^{\rho_j}})+\sum_{j=1}^{2}\|M\epsilon_1-M\epsilon_2\|_{L_T^{\gamma_j}L^{\rho_j}}\\&\quad\le CT\Big\{\sum_{j=+,-}\|u_j-v_j\|_{L_T^{\gamma_0}L^{\rho_0}}+T^{-\frac{p+2}{\gamma_2}}\Big(\sum_{j=1}^{2}\|\epsilon_j\|^p_{L_T^{\gamma_2}\dot{B}^s_{\rho_2,2}}\Big)\|\epsilon_1-\epsilon_2\|_{L_T^{\gamma_2}L^{\rho_2}}\\&\quad\quad+T^{-\frac{3}{\gamma_1}}\Big(\sum_{j=1}^{2}\|\epsilon_j\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}}+\sum_{j=+,-}\|u_j\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}}\\&\quad\quad+\sum_{j=+,-}\|v_j\|_{L_T^{\gamma_1}\dot{B}^s_{\rho_1,2}}\Big)\Big(\|\epsilon_1-\epsilon_2\|_{L_T^{\gamma_1}L^{\rho_1}}+\sum_{j=+,-}\|u_j-v_j\|_{L_T^{\gamma_1}L^{\rho_1}}\Big)\\&\quad\quad+T^{-\frac{K+2}{\gamma_3}}\Big(\sum_{j=+,-}\|u_j\|^K_{L_T^{\gamma_3}\dot{B}^s_{\rho_3,2}}+\sum_{j=+,-}\|v_j\|^K_{L_T^{\gamma_3}\dot{B}^s_{\rho_3,2}}\Big)\Big(\sum_{j=+,-}\|u_j-v_j\|_{L_T^{\gamma_3}L^{\rho_3}}\Big)\Big\}.\end{aligned}$$

Then there exists $T_2 \in (0, T_1]$, such that for any $T \in (0, T_2]$ we have

$$\sum_{j=0,1,3} (\|Mu_+ - Mv_+\|_{L_T^{\gamma_j} L^{\rho_j}} + \|Mu_- - Mv_-\|_{L_T^{\gamma_j} L^{\rho_j}}) + \sum_{j=1}^{2} \|M\epsilon_1 - M\epsilon_2\|_{L_T^{\gamma_j} L^{\rho_j}}$$

$$\leq \frac{1}{2} \sum_{j=0,1,3} (\|u_+ - v_+\|_{L_T^{\gamma_j} L^{\rho_j}} + \|u_- - v_-\|_{L_T^{\gamma_j} L^{\rho_j}}) + \frac{1}{2} \sum_{j=1}^{2} \|\epsilon_1 - \epsilon_2\|_{L_T^{\gamma_j} L^{\rho_j}}. \quad (3.13)$$

So $M: \ E_1 \to E_1$ is strict contraction; there exists a unique fixed point (ϵ, n_+, n_-) (of M) $\in E_1$. From Lemma 2.1, ϵ and $n_\pm \in C([0,T]; \dot{H}^s)$. For any admissible pair (q, r), from Lemma 2.1 and Lemma 3.2, it follows that

$$\|\epsilon_\pm\|_{L_T^q \dot{B}_{r,2}^s} + \|n_\pm\|_{L_T^q \dot{B}_{r,2}^s} \leq C\{\|\epsilon_0\|_{\dot{H}^s} + \|n_0\|_{\dot{H}^s} + \|\phi_0\|_{\dot{H}^s}\} + CT\{\sum_{j=+,-} \|n_j\|_{L_T^{\gamma_0} \dot{B}_{\rho_0,2}^s}\}$$

$$+CT^{1-\frac{3}{\gamma_1}} (\|\epsilon\|_{L_T^{\gamma_1} \dot{B}_{\rho_1,2}^s} + \sum_{j=+,-} \|n_j\|_{L_T^{\gamma_1} \dot{B}_{\rho_1,2}^s})^2 + CT^{1-\frac{p+2}{\gamma_2}} \|\epsilon\|_{L_T^{\gamma_2} \dot{B}_{\rho_2,2}^s}^{p+1}$$

$$+CT^{1-\frac{K+2}{\gamma_3}} \{\sum_{j=+,-} \|n_j\|_{L_T^{\gamma_3} \dot{B}_{\rho_3,2}^s}^{K+1}\} \leq F_1, \quad \forall T \in (0, T_2]. \quad (3.14)$$

From $(\epsilon_{0j}, n_{0j}, \phi_{0j}) \to (\epsilon_0, n_0, \phi_0)$ $(j \to \infty)$ in $H^s(R^d) \times H^s(R^d) \times H^s(R^d)$, it follows that $\forall \alpha \in (0,1)$, $\exists J(\alpha) > 0$, such that if $j > J(\alpha)$, then $\|\epsilon_{0j} - \epsilon_0\|_{H^s} + \|n_{0j} - n_0\|_{H^s} + \|\phi_{0j} - \phi_0\|_{H^s} < \alpha$. Now, we take $F_1 = 2C\{\|\epsilon_0\|_{\dot{H}^s} + \|n_0\|_{\dot{H}^s} + \|\phi_0\|_{\dot{H}^s} + 3\}$ in the definition of E_1; and define $M_j: \ (\epsilon, n_+, n_-) \longrightarrow (M_j\epsilon, M_j n_+, M_j n_-)$ as follows.

$$M_j \epsilon = S(t)\epsilon_{0j} - i\int_0^t S(t-t')\{(\frac{n_+ + n_-}{2})\epsilon + A|\epsilon|^p \epsilon\} dt',$$

$$M_j n_\pm = S(\pm t)\tilde{n}_{\pm,j} \mp i\int_0^t S(\pm(t-t'))\{\frac{n_+ + n_-}{2} + B(\frac{n_+ + n_-}{2})^{K+1} + |\epsilon|^2\} dt',$$

where $\tilde{n}_{\pm,j} = n_{0j} \mp i\phi_{0j}$. $\forall j > J(\alpha)$, using the same argument, there exists $\widehat{T}_2 \in (0, T_2)$, $\widehat{T}_2$ is independent of j, and a unique solution ϵ_j, $n_{\pm,j} \in L^q(0, \widehat{T}_2; B_{r,2}^s) \cap C([0, \widehat{T}_2]; H^s)$ for IVP (1.1)–(1.3) with $(\epsilon_0, n_0, \phi_0)$ replaced by $(\epsilon_{0j}, n_{0j}, \phi_{0j})$. By the same arguments as in the proof of (3.13), there exists $\tilde{T} = \tilde{T}(n_0, \phi_0) \in (0, \widehat{T}_2]$ such that

$$\sum_{l=1}^{2} \|\epsilon_j - \epsilon\|_{L_{\tilde{T}}^{\gamma_l} L^{\rho_l}} + \sum_{l=0,1,3} (\|n_{+,j} - n_+\|_{L_{\tilde{T}}^{\gamma_l} L^{\rho_l}} + \|n_{-,j} - n_-\|_{L_{\tilde{T}}^{\gamma_l} L^{\rho_l}})$$

$$\leq C\{\|\epsilon_{0j} - \epsilon_0\|_{L^2} + \|n_{0j} - n_0\|_{L^2} + \|\phi_{0j} - \phi_0\|_{L^2}\} \longrightarrow 0, \quad j \longrightarrow \infty. \quad (3.15)$$

For any admissible pair (q, r), by using the same arguments as in the proof of (3.14)(3.15), let $j \to \infty$, we have

$$\|\epsilon_j - \epsilon\|_{L_{\tilde{T}}^q L^r} + \sum_{l=+,-} \|n_{l,j} - n_l\|_{L_{\tilde{T}}^q L^r} \leq C\{\|\epsilon_{0j} - \epsilon_0\|_{L^2} + \|n_{0j} - n_0\|_{L^2} + \|\phi_{0j} - \phi_0\|_{L^2}\} \to 0,$$

$$\|\epsilon_j-\epsilon\|_{L^q_T\dot{B}^s_{r,2}}+\sum_{l=+,-}\|n_{l,j}-n_l\|_{L^q_T\dot{B}^s_{r,2}}\leq C\{\|\epsilon_{0j}-\epsilon_0\|_{\dot{H}^s}+\|n_{0j}-n_0\|_{\dot{H}^s}+\|\phi_{0j}-\phi_0\|_{\dot{H}^s}\}\to 0.$$

Since $n=\frac{n_++n_-}{2}$ and $(-\Delta)^{-1}n_t=\frac{n_+-n_-}{2i}$, this is the proof of part (I) for the theorem.

Using the same arguments as in the proof of Theorem 1.1 in [19], part (II) can be proved.

Proof of Theorem 2 Using the same arguments of the proof of Theorem 1 with homogeneous Sobolev spaces and homogeneous Besov spaces replaced by Sobolev spaces and Besov spaces respectively (for example, $\dot{H}^s_r$ repaced by H^s_r, $\dot{B}^s_{r,2}$ repaced by $B^s_{r,2}$), this theorem can be proved.

Proof of Theorem 3 Using the same arguments of the proof of Theorem 1 and Theorem2 with Besov spaces replaced by Sobolev spaces, this theorem can be proved.

4. Global Solution

In this section, we discuss the existence and uniqueness of global solution for IVP (1.1)–(1.3).

Lemma 4.1 *For IVP (1.1)–(1.3), suppose that the solutions ϵ, n, $(-\Delta)^{-1}n_t\in C([0,T];\,H^2)$, then*

$$\|\epsilon(\cdot,t)\|^2_{L^2}=\|\epsilon_0\|^2_{L^2},\quad \forall t\in[0,T]. \tag{4.1}$$

$$\begin{aligned}E(t)&=\int_{R^d}\{|\nabla\epsilon|^2+n|\epsilon|^2+\frac{2A}{p+2}|\epsilon|^{p+2}+\frac{1}{2}(|\Lambda^{-1}n_t|^2+n^2+|\nabla n|^2+\frac{2B}{K+2}n^{K+2})\}dx\\&=E(0),\quad \forall t\in[0,T].\end{aligned} \tag{4.2}$$

Moreover, if ϵ_0, n_0, $\phi_0\in H^1(R^d)$, and d, B, K, p, A satisfy the conditions (1.4)(1.5), then, $\forall t\in[0,T]$ we have

$$\|\epsilon(\cdot,t)\|_{H^1}+\|\Lambda^{-2}n_t(\cdot,t)\|_{H^1}+\|n(\cdot,t)\|_{H^1}\leq C(T,\ \|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},\ \|\phi_0\|_{H^1}), \tag{4.3}$$

where $\widehat{\Lambda^s\varphi}=|\xi|^s\hat{\varphi}(\xi)$, $\forall\varphi\in S(R^d)$, $s\in R$.

Proof The proof of (4.1)(4.2) follows the same lines as the proof of Lemma 1 and Lemma 2 in [11]. We write equations (1.1)(1.2) as the system of equations

$$i\epsilon_t+\Delta\epsilon-n\epsilon-A|\epsilon|^p\epsilon=0, \tag{4.4}$$

$$n_t-\Delta v=0, \tag{4.5}$$

$$v_t-n-Bn^{K+1}+\Delta n-|\epsilon|^2=0. \tag{4.6}$$

Thus, we have

$$E(t)=\int_{R^d}\{|\nabla\epsilon|^2+n|\epsilon|^2+\frac{2A}{p+2}|\epsilon|^{p+2}+\frac{1}{2}(|\nabla v|^2+n^2+|\nabla n|^2+\frac{2B}{K+2}n^{K+2})\}dx.$$

Using (4.4)–(4.6), it follows from straightforward calculation that $\frac{d}{dt}E(t) = 0$. From the hypotheses and the embedding theorems of Sobolev spaces, one has that

$$\|\epsilon_0\|_{L^{p+2}} \le C\|\epsilon_0\|_{H^1}, \quad \|n_0\|_{L^{K+2}} \le C\|n_0\|_{H^1},$$

$$\Big|\int n_0|\epsilon_0|^2 \, dx\Big| \le \|n_0\|_{L^2}\|\epsilon_0\|_{L^4}^2 \le C\|n_0\|_{L^2}\|\epsilon_0\|_{H^1}^2$$

Thus, we have

$$|E(0)| \le C\left(\|\epsilon_0\|_{H^1}^2 + \|n_0\|_{H^1}^2 + \|\phi_0\|_{H^1}^2 + \|n_0\|_{H^1}^{K+2} + \|\epsilon_0\|_{H^1}^{p+2} + \|n_0\|_{H^1}\|\epsilon_0\|_{H^1}^2\right). \quad (4.7)$$

From Cauchy's inequality and Gagliardo-Nirenberg's inequality it follows that

$$\begin{aligned}
\Big|\int_{R^d} n|\epsilon|^2 dx\Big| &\le \frac{B}{2(K+2)}\|n(\cdot,t)\|_{L^{K+2}}^{K+2} + C\|\epsilon(\cdot,t)\|_{L^{2(K+2)/(K+1)}}^{2(K+2)/(K+1)} \\
&\le \frac{B}{2(K+2)}\|n(\cdot,t)\|_{L^{K+2}}^{K+2} + C\|\nabla\epsilon(\cdot,t)\|_{L^2}^{d/(K+1)}\|\epsilon(\cdot,t)\|_{L^2}^{(2K+4-d)/(K+1)} \\
&\le \frac{B}{2(K+2)}\|n(\cdot,t)\|_{L^{K+2}}^{K+2} + \frac{1}{4}\|\nabla\epsilon(\cdot,t)\|_{L^2}^2 + C. \quad (4.8)
\end{aligned}$$

If $0 < p < \frac{4}{d}$, by the same arguments we see that

$$\frac{2|A|}{p+2}\int_{R^d}|\epsilon|^{p+2}dx \le C\|\nabla\epsilon(\cdot,t)\|_{L^2}^{pd/2}\|\epsilon(\cdot,t)\|_{L^2}^{(2p+4-pd)/2} \le \frac{1}{4}\|\nabla\epsilon(\cdot,t)\|_{L^2}^2 + C. \quad (4.9)$$

From the hypotheses and (4.7)–(4.9), we have

$$\begin{aligned}
&\|\nabla\epsilon(\cdot,t)\|_{L^2}^2 + \|\nabla n(\cdot,t)\|_{L^2}^2 + \|n(\cdot,t)\|_{L^2}^2 + \|\nabla v(\cdot,t)\|_{L^2}^2 + \|n(\cdot,t)\|_{L^{K+2}}^{K+2} \\
&\quad \le C(\|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},\ \|\phi_0\|_{H^1}), \quad \forall t \in [0,T]. \quad (4.10)
\end{aligned}$$

Take inner product of (4.6) with $2v$, we see that

$$\begin{aligned}
\frac{d}{dt}\|v(\cdot,t)\|_{L^2}^2 &= 2\int_{R^d}(n + Bn^{K+1} - \Delta n + |\epsilon|^2)v \, dx \\
&\le C\Big\{\|n(\cdot,t)\|_{L^2}^2 + \|n(\cdot,t)\|_{L^{2(K+1)}}^{2(K+1)} + \|\nabla n(\cdot,t)\|_{L^2}\|\nabla v(\cdot,t)\|_{L^2} \\
&\quad + \|\epsilon(\cdot,t)\|_{L^4}^4\Big\} + \|v(\cdot,t)\|_{L^2}^2.
\end{aligned}$$

From (4.10) and Gagliardo-Nirenberg's inequality we have

$$\|\epsilon(\cdot,t)\|_{L^4}^4 \le C\|\nabla\epsilon(\cdot,t)\|_{L^2}^d\|\epsilon(\cdot,t)\|_{L^2}^{4-d} \le C,$$

$$\|n(\cdot,t)\|_{L^{2(K+1)}} \le C\|\nabla n(\cdot,t)\|_{L^2}^{\theta}\|n(\cdot,t)\|_{L^{K+2}}^{1-\theta} \le C,$$

where $\frac{1}{2K+2} = \left(\frac{1}{2} - \frac{1}{d}\right)\theta + (1-\theta)\frac{1}{K+2}$, $0 < \theta \le 1$. Thus, we get

$$\frac{d}{dt}\|v(\cdot,t)\|_{L^2}^2 \le C + 2\|v(\cdot,t)\|_{L^2}^2, \quad \forall t \in [0,T].$$

Using Gronwall's inequality we have

$$\|v(\cdot,t)\|_{L^2}^2 \le Ce^{2T}, \ \ t \in [0,T]. \tag{4.11}$$

From (4.10)(4.11), (4.3) is verified. This completes the proof of this lemma.

Lemma 4.2 *Suppose that* $1 \le d \le 3$, *positive integer* $m \ge 2$, $J < \begin{cases} \infty, & d = 1,2 \\ 4, & d = 3 \end{cases}$. *Let*

$$\rho = \rho(J), \qquad \frac{1}{\rho(J)} = \begin{cases} \frac{d+J}{d(J+2)}, & 1 < \frac{d}{2} \\ \frac{1}{2}\{\frac{1}{2} + max(\frac{1}{2} - \frac{1}{d}, \frac{1}{2} - \frac{2}{d(J+2)}, \frac{1}{J+2})\}, & 1 \ge \frac{d}{2} \end{cases}. \tag{4.12}$$

Then for any $v \in H_\rho^m$ *we have*

$$\|D^m v^{J+1}\|_{L^{\rho'}} \le C\|v\|_{H_\rho^{m-1}}^J \|D^m v\|_{L^\rho}. \tag{4.13}$$

Proof Let $\bar{m} = \min\{m, J+1\}$. It follows from straightforward calculation that

$$|D^m v^{J+1}| \le C\sum_{k=1}^{\bar{m}} |v|^{J+1-k} \sum_{j\in\Omega(m,k)} \prod_{l=1}^{k} \left|D^{j_l} v\right|,$$

where $\Omega(m,k) = \{j = (j_1, \cdots, j_k) | j_1 + \cdots + j_k = m, \ 1 \le j_1 \le \cdots \le j_k\}$. Let us take $\frac{1}{b_l} = \frac{j_l}{m}$, $\frac{1}{\rho_l} = \frac{1}{\rho_\star} + \frac{1}{b_l}\left(\frac{1}{\rho} - \frac{1}{\rho_\star}\right)$, $\frac{1}{\rho_\star} = \frac{1}{J}\left(1 - \frac{2}{\rho}\right)$, $l = 1, \cdots, k$. Since $\frac{1}{\rho_\star} \in \left[\frac{1}{\rho} - \frac{1}{d}, \frac{1}{\rho}\right]$, we have $\frac{1}{\rho_l} \in \left(\frac{1}{\rho} - \frac{1}{d}, \frac{1}{\rho}\right]$ $(l = 1, \cdots, k)$. It follows from Cauchy's inequality and the embedding results for Sobolev spaces that

$$\begin{aligned}\|D^m v^{J+1}\|_{L^{\rho'}} &\le C\sum_{k=1}^{\bar{m}} \|v\|_{L^{\rho_\star}}^{J+1-k} \sum_{j\in\Omega(m,k)} \prod_{l=1}^{k} \left\|D^{j_l} v\right\|_{L^{\rho_l}} \\ &\le C\|v\|_{H_{\rho_1}^1}^J \|D^m v\|_{L^\rho} + \sum_{k=2}^{\bar{m}} \|v\|_{H_\rho^1}^{J+1-k} \sum_{j\in\Omega(m,k)} \prod_{l=1}^{k-1} \left\|D^{j_l} v\right\|_{H_\rho^1} \left\|D^{j_k} v\right\|_{H_\rho^1} \\ &\le C\|u\|_{H_\rho^{m-1}}^J \|D^m v\|_{L^\rho}.\end{aligned}$$

Lemma 4.3 *Suppose that* d, B, K, p, A *satisfy the conditions (1.4)(1.5). Let integer* $m \ge 2$, p *satisfies the condition* $P(m)$, ϵ_0, n_0, $\phi_0 \in H^m$ *and* (ϵ, n) *be the solution of IVP (1.1)–(1.3). If there exists* $0 < T < \infty$ *such that* ϵ, n, $(-\Delta)^{-1}n_t \in C([0,T]; H^{m-1}) \cap L^q(0,T; H_r^{m-1})$, *then* ϵ, n, $(-\Delta)^{-1}n_t \in C([0,T]; H^m) \cap L^q(0,T; H_r^m)$, *where* (q,r) *is any admissible pair.*

Proof From Theorem 3, ϵ_0, n_0, $\phi_0 \in H^m$, we see that there exists $T_m > 0$ and a unique solution (ϵ, n) of IVP (1.1)–(1.3) such that ϵ, n, $(-\Delta)^{-1}n_t \in C([0,T_m]; H^m) \cap$

$L^q(0,T_m;H_r^m)$. We denote by T_m^* the supremum of all above $T_m>0$. Using the same argument as in the proof of part (II) of Theorem 1, it follows that if $T_m^*<\infty$, then there is no solution of IVP (1.1)–(1.3) in $C([0,T_m^*];H^m)\cap L^q(0,T_m^*;H_r^m)$. We claim that $T_m^*>T$. Thus, this lemma is verified.

In fact, if $T_m^*\le T$, we write (1.1)–(1.3) as the integral equations

$$u=S(t)\epsilon(T_\theta)-iS_I(uv+A|u|^pu), \tag{4.14}$$

$$v=B_c(t)n(T_\theta)+B_s(t)n_t(T_\theta)+B_{SI}(v+Bv^{K+1}+|u|^2), \tag{4.15}$$

where ϵ_0, n_0 and $\Delta\phi_0$ are replaced by $\epsilon(T_\theta)$, $n(T_\theta)$ and $n_t(T_\theta)$ respectively; $T_\theta=T_m^*-\theta$, constant $\theta>0$ to be selected later.

Take $\rho_0=\rho(0)=2$, $\rho_1=\rho(1)$, $\rho_2=\rho(p)$, $\rho_3=\rho(K)$ and $\frac{2}{\gamma_j}=d\left(\frac{1}{2}-\frac{1}{\rho_j}\right)$ $(j=0,1,2,3)$, where $\rho(k)$ is defined by (4.12). It is obvious that (γ_j,ρ_j) $(j=0,1,2,3)$ are admissible pairs. Note that there exists a solution u, v, $(-\Delta)^{-1}v_t\in C([0,T-T_\theta];H^{m-1})\cap L^q(0,T-T_\theta;H_r^{m-1})$ for (4.14)(4.15). Indeed, $u(t)=\epsilon(T_\theta+t)$, $v(t)=n(T_\theta+t)$. Thus, $\forall T_1\in(0,T-T_\theta]$ we have

$$\|D^mu\|_{L^q_{T_1}L^r}\le C\|D^m\epsilon(T_\theta)\|_{L^2}+C\{\|D^m(uv)\|_{L^{\gamma_1'}_{T_1}L^{\rho_1'}}+\|D^m(|u|^pu)\|_{L^{\gamma_2'}_{T_1}L^{\rho_2'}}\},$$

$$\|D^mv\|_{L^q_{T_1}L^r)}\le C(\|D^mn(T_\theta)\|_{L^2}+\|D^{m-2}n_t(T_\theta)\|_{L^2})+CT_1\|D^mv\|_{L^{\gamma_0}_{T_1}L^{\rho_0}}$$
$$+C\{\|D^mv^{K+1}\|_{L^{\gamma_3'}_{T_1}L^{\rho_3'}}+\|D^m(|u|^2)\|_{L^{\gamma_1'}_{T_1}L^{\rho_1'}}\}.$$

Let (q,r) be equal to (γ_j,ρ_j) $(j=0,1,2,3)$ respectively, using Lemma 4.2 or the same arguments of the proof of Lemma 4.2, and employing Hölder's inequality, we obtain that

$$\sum_{j=1,2}\|D^mu\|_{L^{\gamma_j}_{T_1}L^{\rho_j}}+\sum_{j=0,1,3}\|D^mv\|_{L^{\gamma_j}_{T_1}L^{\rho_j}}\le C(1+\|D^m\epsilon(T_\theta)\|_{L^2}+\|D^mn(T_\theta)\|_{L^2}$$
$$+\|D^{m-2}n_t(T_\theta)\|_{L^2})+CT_1\|D^mv\|_{L^{\gamma_0}_{T_1}L^{\rho_0}}+CT_1^{1-\frac{3}{\gamma_1}}\left(\|D^mv\|_{L^{\gamma_1}_{T_1}L^{\rho_1}}+\|D^mu\|_{L^{\gamma_1}_{T_1}L^{\rho_1}}\right)$$
$$+CT_1^{1-\frac{p+2}{\gamma_2}}\|D^mu\|_{L^{\gamma_2}_{T_1}L^{\rho_2}}+CT_1^{1-\frac{K+2}{\gamma_3}}\|D^mv\|_{L^{\gamma_3}_{T_1}L^{\rho_3}},\quad \forall T_1\le T-T_\theta,$$

where constant C is only dependent on T, d, m, A, B, p, K, $\|\epsilon\|_{L^{\gamma_j}_TH^{m-1}_{\rho_j}}$ $(j=1,2)$, $\|n\|_{L^{\gamma_j}_TH^{m-1}_{\rho_j}}$ $(j=0,1,3)$; independent of $T_1,T_m^*,\|D^m\epsilon(T_\theta)\|_{L^2}$, $\|D^mn(T_\theta)\|_{L^2}$, $\|D^{m-2}n_t(T_\theta)\|_{L^2}$. Then there exists $\bar{T}_1>0$, independent of T_m^*, $\|D^m\epsilon(T_\theta)\|_{L^2}$, $\|D^mn(T_\theta)\|_{L^2}$, $\|D^{m-2}n_t(T_\theta)\|_{L^2}$, such that $C\bar{T}_1\le\frac{1}{2}$, $C\bar{T}_1^{1-\frac{3}{\gamma_1}}\le\frac{1}{2}$, $C\bar{T}_1^{1-\frac{p+2}{\gamma_2}}\le\frac{1}{2}$, $C\bar{T}_1^{1-\frac{K+2}{\gamma_3}}\le\frac{1}{2}$.

In the case $T_m^*=T$, let $\theta=\frac{\bar{T}_1}{2}$ and $T_1=\frac{\bar{T}_1}{2}$, we have

$$\sum_{j=1,2}\|D^m u\|_{L_{T_1}^{\gamma_j}L^{\rho_j}}+\sum_{j=0,1,3}\|D^m v\|_{L_{T_1}^{\gamma_j}L^{\rho_j}}\le C_1,\qquad \|D^m u\|_{L_{T_1}^{q}L^{r}}+\|D^m v\|_{L_{T_1}^{q}L^{r}}\le C_1.$$

Thus, there exists a solution $\tilde{\epsilon},\ \tilde{n}\in C([0,T_m^*];H^m)\cap L^q(0,T_m^*;H_r^m)$ for IVP (1.1)–(1.3). Indeed,

$$(\tilde{\epsilon}(t),\tilde{n}(t))=\begin{cases}(\epsilon(t),n(t)), & 0\le t\le T_m^*-\frac{\bar{T}_1}{2}\\ (u(t-T_m^*+\frac{\bar{T}_1}{2}),v(t-T_m^*+\frac{\bar{T}_1}{2})), & T_m^*-\frac{\bar{T}_1}{2}<t\le T_m^*\end{cases}.$$

It is contradictory.

In the case $T_m^*<T$, let $T_1=\min\{\bar{T}_1,T-T_m^*\}$ and $\theta=\frac{T_1}{2}$, we have

$$\sum_{j=1,2}\|D^m u\|_{L_{T_1}^{\gamma_j}L^{\rho_j}}+\sum_{j=0,1,3}\|D^m v\|_{L_{T_1}^{\gamma_j}L^{\rho_j}}\le C_2,\qquad \|D^m u\|_{L_{T_1}^{q}L^{r}}+\|D^m v\|_{L_{T_1}^{q}L^{r}}\le C_2.$$

Thus, there exists a solution $\tilde{\epsilon},\ \tilde{n}\in C([0,T_m^*+\frac{T_1}{2}];H^m)\cap L^q(0,T_m^*+\frac{T_1}{2};H_r^m)$ for IVP (1.1)–(1.3). Indeed,

$$(\tilde{\epsilon}(t),\tilde{n}(t))=\begin{cases}(\epsilon(t),n(t)), & 0\le t\le T_m^*-\frac{T_1}{2}\\ (u(t-T_m^*+\frac{T_1}{2}),v(t-T_m^*+\frac{T_1}{2})), & T_m^*-\frac{T_1}{2}<t\le T_m^*+\frac{T_1}{2}\end{cases}.$$

It is also contradictory. This completes the proof of the lemma.

Proof of Theorem 4 First, we consider the case $m=1$. Let series $\{\epsilon_{0j}\}_{j=1}^\infty$, $\{n_{0j}\}_{j=1}^\infty$ and $\{\phi_{0j}\}_{j=1}^\infty\subset H^2$ with $\epsilon_{0j}\to\epsilon_0,\ n_{0j}\to n_0,\ \phi_{0j}\to\phi_0\ (j\to\infty)$ in H^1, then from Theorem 3 and Lemma 4.3 it follows that there exists $T_1=T_1(\epsilon_0,\ n_0,\ \phi_0)>0$ and solutions

$$\epsilon_j,\ \ n_j,\ \ (-\Delta)^{-1}n_{jt}\in C([0,T_1];H^2)\cap L^q(0,T_1;H_r^2)$$

for IVP (1.1)–(1.3) with (ϵ_0,n_0,ϕ_0) replaced by $(\epsilon_{0j},n_{0j},\phi_{0j})$, such that

$$\begin{aligned}&\epsilon_j\to\epsilon,\quad n_j\to n,\quad (-\Delta)^{-1}n_{jt}\to(-\Delta)^{-1}n_t\quad (j\to\infty)\\ &\text{in }\ C([0,T_1];H^1(R^d))\cap L^q(0,T_1;H_r^1(R^d)),\end{aligned}$$

where (q,r) is any admissible pair and (ϵ,n) is the solution of IVP (1.1)–(1.3). For any $T\in(0,\infty)$, let $T_\star=\min\{T_1,T\}$. From Lemma 4.1, it follows that $\forall t\in[0,T_\star]$ we have

$$\begin{aligned}&\int_{R^d}\{|\nabla\epsilon_j|^2+n_j|\epsilon_j|^2+\frac{2A}{p+2}|\epsilon_j|^{p+2}+\frac{1}{2}(|\Lambda^{-1}n_{jt}|^2+n_j^2+|\nabla n_j|^2+\frac{2B}{K+2}n_j^{K+2})\}dx\\ &\quad=\int_{R^d}\{|\nabla\epsilon_{0j}|^2+n_{0j}|\epsilon_{0j}|^2+\frac{2A}{p+2}|\epsilon_{0j}|^{p+2}\}dx\\ &\qquad+\frac{1}{2}\int_{R^d}(|\nabla\phi_{0j}|^2+n_{0j}^2+|\nabla n_{0j}|^2+\frac{2B}{K+2}n_{0j}^{K+2})dx;\end{aligned}$$

$$\|\epsilon_j(\cdot,t)\|_{H^1}+\|\Lambda^{-2}n_{jt}(\cdot,t)\|_{H^1}+\|n_j(\cdot,t)\|_{H^1}\le C(T_\star,\ \|\epsilon_{0j}\|_{H^1},\ \|n_{0j}\|_{H^1},\ \|\phi_{0j}\|_{H^1}).$$

Let $j \to \infty$, $\forall t \in [0, T_\star]$ we have

$$E(t) = \int_{R^d} \{|\nabla \epsilon|^2 + n|\epsilon|^2 + \frac{2A}{p+2}|\epsilon|^{p+2} + \frac{1}{2}(|\Lambda^{-1}n_t|^2 + n^2 + |\nabla n|^2 + \frac{2B}{K+2}n^{K+2})\}dx$$
$$= E(0),$$

$$\|\epsilon(\cdot,t)\|_{H^1} + \|\Lambda^{-2}n_t(\cdot,t)\|_{H^1} + \|n(\cdot,t)\|_{H^1} \le C(T_\star, \|\epsilon_0\|_{H^1}, \|n_0\|_{H^1}, \|\phi_0\|_{H^1}).$$

In the case $T_1 \ge T$, the theorem is verified. In the case $T_1 < T$, from the proof of Lemma 4.1 we have

$$\|\epsilon(\cdot,t)\|_{H^1} + \|\Lambda^{-2}n_t(\cdot,t)\|_{H^1} + \|n(\cdot,t)\|_{H^1} \le C(T_\star,\ \|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},\ \|\phi_0\|_{H^1})$$
$$\le C(T,\ \|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},\ \|\phi_0\|_{H^1}), \quad \forall t \in [0, T_\star].$$

Then we can repeat the above arguments with $(\epsilon(x,T_1), n(x,T_1), n_t(x,T_1))$ instead of $(\epsilon_0, n_0, \Delta\phi_0)$. So there exists $\Delta T_1 > 0$, dependent on the constant $C(T,\ \|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},$ $\|\phi_0\|_{H^1})$, independent of T_1, such that $\epsilon,\ n,\ (-\Delta)^{-1}n_t \in C([0, T_1 + \Delta T_1]; H^1) \cap L^q(0, T_1 + \Delta T_1; H^1_r)$ and

$$\|\epsilon(\cdot,t)\|_{H^1} + \|\Lambda^{-2}n_t(\cdot,t)\|_{H^1} + \|n(\cdot,t)\|_{H^1} \le C(T_\star,\ \|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},\ \|\phi_0\|_{H^1})$$
$$\le C(T,\ \|\epsilon_0\|_{H^1},\ \|n_0\|_{H^1},\ \|\phi_0\|_{H^1}), \quad \forall 0 \le t \le T_\star = \min\{T, T_1 + \Delta T_1\}.$$

It must follows from n-times extensions that $T_1 + n\Delta T_1 > T$. So the theorem is verified in this case.

Finally, we consider the case $m \ge 2$. From Lemma 4.3 and the results which have been proved in the previous case, the theorem is also verified in this case.

Lemma 4.4 *Suppose that* $1 \le d \le 3$, *real number* $s \ge 2$, *positive integer* $J < \begin{cases} \infty, & d = 1,2 \\ 4, & d = 3 \end{cases}$. *Let*

$$\rho = \rho(J), \qquad \frac{1}{\rho(J)} = \frac{1}{2}\{\frac{1}{2} + \max(\frac{1}{2} - \frac{2}{d(J+2)}, \frac{1}{J+2})\}, \tag{4.16}$$

$$0 < \delta \le \delta_0 = \begin{cases} \frac{2}{J+2}, & d = 1,2 \\ \frac{3}{J+2} - \frac{1}{2}, & d = 3 \end{cases}, \tag{4.17}$$

and $[s] = [s+\delta]$. *Then for any* $v \in \dot{B}^{s+\delta}_{\rho,2}$ *we have*

$$\|v^{J+1}\|_{\dot{B}^{s+\delta}_{\rho',2}} \le C\|v\|^J_{B^s_{\rho,2}}(\|v\|_{L^\rho} + \|v\|_{\dot{B}^{s+\delta}_{\rho,2}}). \tag{4.18}$$

Proof Set $m = [s] + 1$, $\bar{m} = \min\{m, J+1\}$, by the proof of Theorem 3.1 in [19] we have

$$\|v^{J+1}\|_{\dot{B}^{s+\delta}_{\rho',2}} \le C\sum_{k=1}^{\bar{m}} \|v\|^{J+1-k}_{L^{\rho\star}} \sum_{j\in\Omega(m,k)} \prod_{q=1}^{k} \|v\|_{\dot{B}^{(s+\delta)/bq}_{\rho q, 2bq}},$$

where $\Omega(m,k)$ is defined in the proof of Lemma 4.2; $\frac{1}{b_q}=\frac{j_q}{m}$ $(q=1,\cdots,k)$, $\frac{1}{\rho_l}=\frac{1}{\rho_\star}$ $(l=1,\cdots,k-1)$, $\frac{1}{\rho_k}=\frac{1}{\rho}$, $\frac{1}{\rho_\star}=\frac{1}{J}\left(1-\frac{2}{\rho}\right)$. Since $\delta\le\delta_0$, we have $\frac{1}{\rho_q}-\frac{s+\delta}{db_q}>\frac{1}{\rho}-\frac{s}{d}$ $(q=1,\cdots,k-1)$. In fact, note that

$$\frac{1}{m}\le\frac{1}{b_q}\le\begin{cases}\frac{1}{2} & m \text{ is an even integer}\\ \frac{1}{3} & m \text{ is an odd integer}\end{cases}\quad(1\le q\le k-1),\quad s\ge\begin{cases}3 & m \text{ is an even integer}\\ 2 & m \text{ is an odd integer}\end{cases},$$

we have

$$\frac{1}{\rho}-\frac{s}{d}+\frac{s+\delta}{db_q}\le\begin{cases}\frac{1}{\rho}-\frac{s}{d}+\frac{s+\delta}{2d} & m \text{ is an even integer}\\ \frac{1}{\rho}-\frac{s}{d}+\frac{s+\delta}{3d} & m \text{ is an odd integer}\end{cases}$$

$$\le\begin{cases}\frac{1}{\rho}-\frac{3}{2d}+\frac{\delta_0}{2d} & m \text{ is an even integer}\\ \frac{1}{\rho}-\frac{4}{3d}+\frac{\delta_0}{3d} & m \text{ is an odd integer}\end{cases}\le 0\quad(1\le q\le k-1).$$

From $\frac{1}{\rho_\star}\le\frac{1}{\rho}$ and the embedding theorems of Besov spaces, it follows that $B^s_{\rho,2}\subset L^{\rho_\star}$, $B^s_{\rho,2}\subset B^{(s+\delta)/b_q}_{\rho_q,2b_q}$ $(q=1,\cdots,k-1)$, $B^{s+\delta}_{\rho,2}\subset B^{(s+\delta)/b_k}_{\rho_k,2b_k}$. Thus, we have (4.18). The lemma is proved.

Lemma 4.5 *Suppose that* $d,\ B,\ K,\ p,\ A$ *satisfy the conditions (1.4)(1.5). Let real number* $s\ge 2$, *p satisfies the condition* $P([s]+1)$, $0<\delta\le\delta_0=\begin{cases}\min\{\frac{2}{p+2},\frac{2}{K+2}\}, & d=1,2\\ \frac{3}{p+2}-\frac{1}{2}, & d=3\end{cases}$, $[s]=[s+\delta]$, $\epsilon_0,n_0,\phi_0\in H^{s+\delta}\subset H^s$ *and* (ϵ,n) *be the solution of IVP (1.1)–(1.3). If there exists* $T_s\in(0,\infty)$, *such that* $\epsilon,\ n,\ (-\Delta)^{-1}n_t\in L^q(0,T_s;B^s_{r,2})\cap C([0,T_s];H^s)$, *then* $\epsilon,\ n,\ (-\Delta)^{-1}n_t\in L^q(0,T_s;B^{s+\delta}_{r,2})\cap C([0,T_s];H^{s+\delta})$, *where* (q,r) *is any admissible pair.*

Proof From Theorem 2, ϵ_0, n_0 and $\phi_0\in H^{s+\delta}$, we see that there exists $T_{s+\delta}>0$ and a unique solution (ϵ,n) of IVP (1.1)–(1.3) such that $\epsilon,\ n,\ (-\Delta)^{-1}n_t\in C([0,T_{s+\delta}];H^{s+\delta})\cap L^q(0,T_{s+\delta};B^{s+\delta}_{r,2})$. We denote by $T^*_{s+\delta}$ the supremum of all above $T_{s+\delta}>0$. Using the same arguments as in the proof of part (II) of Theorem 1, it follows that if $T^*_{s+\delta}<\infty$, then there is no solution of IVP (1.1)–(1.3) in $C([0,T^*_{s+\delta}];H^{s+\delta})\cap L^q(0,T^*_{s+\delta};B^{s+\delta}_{r,2})$. We claim that $T^*_{s+\delta}>T_s$. Thus, this lemma is verified.

In fact, if $T^*_{s+\delta}\le T_s$, we write (1.1)–(1.3) as the integral equation

$$u=S(t)\epsilon(T_\theta)-iS_I(uv+A|u|^pu),\tag{4.19}$$

$$v=B_c(t)n(T_\theta)+B_s(t)n_t(T_\theta)+B_{SI}(v+Bv^{K+1}+|u|^2),\tag{4.20}$$

where ϵ_0, n_0 and $\Delta\phi_0$ are replaced by $\epsilon(T_\theta)$, $n(T_\theta)$ and $n_t(T_\theta)$ respectively; $T_\theta=T^*_{s+\delta}-\theta$, constant $\theta>0$ to be selected later.

Let us take $\rho_0=\rho(0)=2$, $\rho_1=\rho(1)$, $\rho_2=\rho(p)$, $\rho_3=\rho(K)$ and $\frac{2}{\gamma_j}=d\left(\frac{1}{2}-\frac{1}{\rho_j}\right)$, $(j=0,1,2,3)$, where $\rho(k)$ is defined by (4.16). It is obvious that (γ_j,ρ_j) $(j=0,1)$ are admissible pairs. Note that there exists a solution $u,\ v\in C([0,T_s-T_\theta];H^s)\cap$

$L^q(0,T_s-T_\theta;B^s_{r,2})$ for (4.19)(4.20). Indeed, $u(t)=\epsilon(T_\theta+t)$, $v(t)=n(T_\theta+t)$. Thus, $\forall T_1\in(0,T_s-T_\theta]$ we have

$$\|u\|_{L^q_{T_1}\dot B^{s+\delta}_{r,2}}\le C\|\epsilon(\cdot,T_\theta)\|_{\dot H^{s+\delta}}+C\{\|uv\|_{L^{\gamma_1'}_{T_1}\dot B^{s+\delta}_{\rho_1',2})}+\||u|^pu\|_{L^{\gamma_2'}_{T_1}\dot B^{s+\delta}_{\rho_2',2})}\},$$

$$\|v\|_{L^q_{T_1}\dot B^{s+\delta}_{r,2}}\le C\left(\|n(\cdot,T_\theta)\|_{\dot H^{s+\delta}}+\|n_t(\cdot,T_\theta)\|_{\dot H^{s+\delta-2}}\right)$$

$$+C\{T_1\|v\|_{L^{\gamma_0}_{T_1}\dot B^{s+\delta}_{\rho_0,2}}+\|v^{K+1}\|_{L^{\gamma_3'}_{T_1}\dot B^{s+\delta}_{\rho_3',2}}+\||u|^2\|_{L^{\gamma_1'}_{T_1}\dot B^{s+\delta}_{\rho_1',2}}\}.$$

Using Lemma 4.4 or the same arguments of the proof of Lemma 4.4, and employing Hölder's inequality, $\forall T_1\in[0,T_s-T_\theta]$ we obtain that

$$\|u\|_{L^q_{T_1}\dot B^{s+\delta}_{r,2}}\le C\left(1+\|\epsilon(\cdot,T_\theta)\|_{\dot H^{s+\delta}}\right)+CT_1^{1-\frac{3}{\gamma_1}}\left(\|u\|_{L^{\gamma_1}_{T_1}\dot B^{s+\delta}_{\rho_1,2}}+\|v\|_{L^{\gamma_1}_{T_1}\dot B^{s+\delta}_{\rho_1,2}}\right)$$

$$+CT_1^{1-\frac{p+2}{\gamma_2}}\|u\|_{L^{\gamma_2}_{T_1}\dot B^{s+\delta}_{\rho_2,2}},$$

$$\|v\|_{L^q_{T_1}\dot B^{s+\delta}_{r,2}}\le C\left(1+\|n(\cdot,T_\theta)\|_{\dot H^{s+\delta}}+\|n_t(\cdot,T_\theta)\|_{\dot H^{s+\delta-2}}\right)$$

$$+CT_1\|v\|_{L^{\gamma_0}_{T_1}\dot B^{s+\delta}_{\rho_0,2}}+CT_1^{1-\frac{3}{\gamma_1}}\|v\|_{L^{\gamma_1}_{T_1}\dot B^{s+\delta}_{\rho_1,2}}+CT_1^{1-\frac{K+2}{\gamma_3}}\|v\|_{L^{\gamma_3}_{T_1}\dot B^{s+\delta}_{\rho_3,2}},$$

where constant C is only dependent on T_s, d,s, A, B, p, K, $\|\epsilon\|_{L^{\gamma_j}_{T_s}B^s_{\rho_j,2}}(j=1,2)$, $\|n\|_{L^{\gamma_j}_{T_s}B^s_{\rho_j,2}}(j=0,1,3)$; independent of T_1, $T^*_{s+\delta}$, $\|\epsilon(\cdot,T_\theta)\|_{\dot H^{s+\delta}}$, $\|n(\cdot,T_\theta)\|_{\dot H^{s+\delta}}$, $\|n_t(\cdot,T_\theta)\|_{\dot H^{s+\delta-2}}$. So far using the same arguments of the proof of Lemma 4.3, this lemma can be verified.

Lemma 4.6 *Suppose that d, B, K, p, A satisfy the conditions (1.4)(1.5), real number $s\in[1,2)$, p satisfies the condition $P([s]+1)$. Let $0<\delta<\delta_0=\min\left(\frac12,\frac1K,\frac1p\right)$, $s+\delta\in(1,2)$, ϵ_0, n_0, $\phi_0\in H^{s+\delta}\subset H^s$ and (ϵ,n) be the solution of IVP (1.1)–(1.3). If there exists $T_s\in(0,\infty)$, such that ϵ, n, $(-\Delta)^{-1}n_t\in L^q(0,T_s;B^s_{r,2})\cap C([0,T_s];H^s)$, then ϵ, n, $(-\Delta)^{-1}n_t\in L^q(0,T_s;B^{s+\delta}_{r,2})\cap C([0,T_s];H^{s+\delta})$, where (q,r) is any admissible pair.*

Proof To prove this lemma by using the same arguments of the proof of Lemma 4.5, it is sufficient to establish some estimates of nonlinear terms which is similar to (4.18). Note that $m=[s]+1=2$, for any $v\in B^{s+\delta}_{\rho,2}$ and positive integer $J<\begin{cases}\infty, & d=1,2\\ 4, & d=3\end{cases}$, we have

$$\|v^{J+1}\|_{\dot B^{s+\delta}_{\rho',2}}\le C\Big(\int_0^\infty\{t^{-s-\delta}\sup_{|y|\le t}\|\sum_{k=0}^{2}\binom{2}{k}(-1)^kv^{J+1}(\cdot+ky)\|_{L^{\rho'}}\}^2\frac{dt}{t}\Big)^{\frac12}$$

$$\le C\|v\|^J_{L^{\rho^\star}}\|v\|_{\dot B^{s+\delta}_{\rho,2}} + C\|v\|^{J-1}_{L^{\rho^\star}}\|v\|_{\dot B^{(s+\delta)/2}_{\rho_{11},4}}\|v\|_{\dot B^{(s+\delta)/2}_{\rho_{12},4}},$$

where

$$\rho=\rho(J), \frac{1}{\rho(J)} = \begin{cases} \frac{d+sJ}{d(J+2)}, & 0\le s<\frac{d}{2}, \\ \frac{1}{2}\left(\frac{1}{2}+\max\left\{\frac{1}{2}-\frac{1}{d}, \frac{1}{2}-\frac{2}{(J+2)d}, \frac{1}{J+2}\right\}\right), & s\ge\frac{d}{2}, \end{cases};$$

$$\frac{1}{\rho_{11}} = \begin{cases} \frac{1}{2}\left(\frac{1}{\rho}+\frac{1}{\rho_*}\right), & d=1,2 \\ \frac{1}{\rho_*}+\frac{s+\delta}{2d}, & d=3, s\in[1,\frac{3}{2}) \\ \frac{1}{2}\left(\frac{1}{\rho}+\frac{1}{\rho_*}\right), & d=3, s\in[\frac{3}{2},2) \end{cases}, \frac{1}{\rho_{12}} = \begin{cases} \frac{1}{2}\left(\frac{1}{\rho}+\frac{1}{\rho_*}\right), & d=1,2 \\ \frac{1}{\rho}-\frac{s+\delta}{2d}, & d=3, s\in[1,\frac{3}{2}) \\ \frac{1}{2}\left(\frac{1}{\rho}+\frac{1}{\rho_*}\right), & d=3, s\in[\frac{3}{2},2) \end{cases};$$

$\frac{1}{\rho_*} = \frac{1}{J}\left(1-\frac{2}{\rho}\right)$. Since $\delta<\min\{\frac{1}{2},\frac{1}{J}\}$, we have $\frac{1}{\rho_{11}}-\frac{s+\delta}{2d}\ge\frac{1}{\rho}-\frac{s}{d}$. From $\frac{1}{\rho_*}\le\frac{1}{\rho}$, $\frac{1}{\rho_{11}}\le\frac{1}{\rho}$, $0<\frac{1}{\rho_{12}}\le\frac{1}{\rho}$ and the embedding theorems of Besov spaces, it follows that $B^s_{\rho,2}\subset L^{\rho_*}$, $B^s_{\rho,2}\subset B^{(s+\delta)/2}_{\rho_{11},4}$, $\dot B^{s+\delta}_{\rho,2}\subset\dot B^{(s+\delta)/2}_{\rho_{12},4}$. Thus, we have $\|v^{J+1}\|_{\dot B^{s+\delta}_{\rho',2}}\le C\|v\|^J_{B^s_{\rho,2}}\left(\|v\|_{B^s_{\rho,2}}+\|u\|_{\dot B^{s+\delta}_{\rho,2}}\right)$. So far using the same arguments of the proof of Lemma 4.5, the lemma is proved.

Proof of Theorem 5 Using Theorem 1–4, Lemma 4.5 and lemma 4.6, Theorem 5 is proved.

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