J. Partial Diff. Eqs. **19**(2006), 26–47
©International Academic Publishers Vol.19 No.1

BESOV SPACES AND SELF-SIMILAR SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS*

Miao Changxing
(Institute of Applied Physics and Computational Mathematics
PO Box 8009, Beijing 100088, China)
(E-mail: miao_changxing@mail.iapcm.ac.cn)
Zhang Bo
(Institute of Applied Mathematics, Chinese Academy of Sciences Beijing 100080, China;
School of MIS, Coventry University Coventry CV1 5FB, UK)
(E-mail: b.zhang@amt.ac.cn b.zhang@coventry.ac.uk)

Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

(Received Apr. 20, 2005)

Abstract In this paper, we establish the existence of global self-similar solutions for the heat and convection-diffusion equations. This we do in some homogeneous Besov spaces using the theory of Besov spaces and the Strichartz estimates. Further, the structure of the self-similar solutions has also been established by using an equivalent norm for Besov spaces.

Key Words Strichartz estimates; admissible triplet; self-similar solution; Besov spaces; evolution equations, well-posedness.

2000 MR Subject Classification 35K15, 35K20.

Chinese Library Classification O175.23, O175.26, O175.29.

1. Introduction

In this paper we study the existence and regularity of global self-similar solutions of the Cauchy problem for the semi-linear heat equation

$$u_t - \triangle u = \mu u^{\alpha+1}, \quad u(0,x) = f(x) \tag{1.1}$$

and the Cauchy problem for the convection-diffusion equation

$$\partial_t u - \triangle u = \vec{a} \cdot \nabla(|u|^\alpha u), \qquad u(0,x) = f(x), \tag{1.2}$$

where $\mu \in \mathbb{R}$, $\vec{a} \in \mathbb{R}^n \setminus \{0\}$, $\alpha > 0$, $u = u(t,x)$ is a real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^n$ and the initial data f is a real-valued function.

*The first author (CM) was supported by the NNSF of China and NSF of China Academy of Engineering Physics. The second author (BZ) was supported by the Academy of Mathematics and Systems Science through the Hundred Talent Program of the Chinese Academy of Sciences.

Self-similar solutions have been studied for other semilinear evolution equations such as the semilinear wave equation [1-4], the Navier-Stokes equations [5, 6] and the Schroedinger equations [7-10]. They often describe the large time behavior of general global solutions to the evolution equations under certain conditions. For example, it was shown in [6] that self-similar solutions for the Navier-Stokes equations constructed by Cannone [5] provide the large time asymptotic behavior of the global solutions.

A solution $u(t,x)$ of (1.1) or (1.2) is called a self-similar solution if for $\lambda > 0$,

$$u(t,x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x).$$

It is easy to verify that u is a self-similar solution if and only if

$$u(t,x) = t^{-\frac{1}{\alpha}} u\left(1, \frac{x}{\sqrt{t}}\right) = t^{-\frac{1}{\alpha}} V\left(\frac{x}{\sqrt{t}}\right)$$

for some function $V(x)$ called the profile of the self-similar solution u. Thus the Self-similar solution to nonlinear evolution equations can be studied through the study of the associated semi-linear elliptic equations for $V(x)$. However, it is usually very difficult to solve such nonlinear elliptic equations. On the other hand, the initial data for self-similar solutions must satisfy, for $\lambda > 0$,

$$f(x) = \lambda^{\frac{2}{\alpha}} f(\lambda x). \tag{1.3}$$

This leads to another way of looking for self-similar solutions of (1.1) or (1.2) by the study of small global well-posedness in some suitable function spaces of the Cauchy problem (1.1) or (1.2) with initial data f satisfying (1.3). These new global solutions admit a class of self-similar solutions. However, the condition (1.3) means that f is homogeneous degree $-2/\alpha$. Such homogeneous functions, in general, do not belong to the usual spaces such as the usual Sobolev space $H^{s,p}$, where the global well-posedness of the Cauchy problem holds. Thus, in order to construct self-similar solutions for evolution equations such as (1.1) or (1.2) we have to choose a suitable homogeneous Banach space X of degree $-2/\alpha$ and to show that the problem generates a global flow in X.

The well-posedness of the Cauchy problem for the heat equation (1.1) has been studied by many authors. For example, the existence and uniqueness of solutions have been studied in [7, 11-16] for the case when the initial data is in Sobolev spaces and in [17] for the case when the initial data is in Besov spaces. Self-similar solutions have also been dealt with for the heat equation (1.1) in [18, 14] by the study of the associated elliptic problem and in, e.g. [7] by studying the Cauchy problem. In [19, 20], the global solutions of the nonlinear heat equation have been shown to be asymptotically close to its self-similar solution. On the other hand, the global well-posedness including the large time behavior of the solution has been proved for the convection-diffusion (1.2) in [21], whilst the existence of positive self-similar solutions for (1.2) has been established in [22] in the case when $\alpha = 1/n$ through the study of the associated elliptic problem.

The remaining part of this paper is organized as follows. In Section 2, we introduce some Besov spaces and Strichartz estimates needed in this paper. Section 3 is devoted to the study of self-similar solutions for the semilinear heat and convection-diffusion equations. In the appendix we give a proof of an equivalent norm for Besov spaces (Proposition 2.1 below), which was given previously in [23] without proof and is used in this paper to study the structure of self-similar solutions for evolution equations.

We conclude this section with introducing several definitions and notations. Denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the Schwartz space and the space of Schwartz distribution functions respectively. For integer m, $C^m(\mathbb{R}^n)$ denotes the usual space of m-times continuously differentiable functions on $\mathbb{R}^n$, and for $1 \le r \le \infty$, $L^r(\mathbb{R}^n)$ denotes the usual Lebesgue space on $\mathbb{R}^n$ with the norm $\|\cdot\|_r$. For $s \in \mathbb{R}$ and $1 < r < \infty$, let $H^{s,r}(\mathbb{R}^n) = (1-\triangle)^{-\frac{s}{2}} L^r(\mathbb{R}^n)$, the inhomogeneous Sobolev space in terms of Bessel potentials, let $\dot{H}^{s,r}(\mathbb{R}^n) = (-\triangle)^{-\frac{s}{2}} L^r(\mathbb{R}^n)$, the homogeneous Sobolev space in terms of Riesz potentials, and write $H^s(\mathbb{R}^n) = H^{s,2}(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n) = \dot{H}^{s,2}(\mathbb{R}^n)$. For the detailed definitions of the above function spaces see, e.g. [24-28]. We shall omit $\mathbb{R}^n$ from spaces and norms. For any interval $I \in \mathbb{R}$ and any Banach space X we denote by $C(I;X)$ the space of strongly continuous functions from I to X, by $L^q(I;X)$ the space of strongly measurable functions from I to X with $\|u(\cdot);X\| \in L^q(I)$, and by $C_*(I;X)$ the space of functions in $L^\infty(I;X)$ that are continuous in the distributional sense. For a given function space X we denote by $\dot{X}$ its homogeneous space. Finally, for any $q > 0$, q' stands for the dual to q, i.e., $1/q + 1/q' = 1$.

2. Preliminaries

2.1 Besov spaces

In this subsection we introduce some equivalent definitions and norms for Besov spaces needed in this paper. The reader is referred to the well-known books [23, 24, 27, 28] for details.

Let $s > 0$, $1 \le p \le \infty$, $1 \le m \le \infty$. We first introduce the following equivalent norms for the Besov spaces $\dot{B}^s_{p,m}$ and $B^s_{p,m}$:

$$|v\|_{\dot{B}^s_{p,m}} \simeq \sum_{|\alpha|=N} \left(\int_0^\infty t^{-m\sigma} \sup_{|y|\le t} \| \triangle_y^2 \, \partial^\alpha v\|_p^m \frac{dt}{t} \right)^{\frac{1}{m}}, \tag{2.1}$$

$$\|v\|_{B^s_{p,m}} \simeq \|v\|_p + |v\|_{\dot{B}^s_{p,m}}, \tag{2.2}$$

where

$$\triangle_y^2 \, v \triangleq \tau_y v + \tau_{-y} v - 2v, \qquad \tau_{\pm y} v(\cdot) = v(\cdot \pm y),$$
$$\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, \cdots n,$$

$\alpha = (\alpha_1, \cdots, \alpha_n)$ and $s = N + \sigma$ with a nonnegative integer N and $0 < \sigma < 2$. In the special case when s is not an integer, (2.1) is also equivalent to the following norm:

$$\|v\|_{\dot{B}^s_{p,m}} \simeq \sum_{|\alpha|=[s]} \left(\int_0^\infty t^{-m(s-[s])} \sup_{|y|\le t} \| \triangle_y \partial^\alpha v\|_p^m \frac{dt}{t} \right)^{\frac{1}{m}}, \tag{2.3}$$

where $\triangle_{\pm y} v(\cdot) = \tau_{\pm y} v - v$ and $[s]$ denotes the largest integer not larger than s. In the case when $m = \infty$, the norm $\|v\|_{\dot{B}^s_{p,\infty}}$ in the above definition should be modified as follows:

$$\|v\|_{\dot{B}^s_{p,\infty}} \simeq \sum_{|\alpha|=N} \sup_{t>0} \sup_{|y|\le t} t^{-\sigma} \| \triangle_y^2 \partial^\alpha v\|_p, \quad s \in \mathbb{R} \tag{2.4}$$

$$\|v\|_{\dot{B}^s_{p,\infty}} \simeq \sum_{|\alpha|=[s]} \sup_{t>0} \sup_{|y|\le t} t^{-s-[s]} \| \triangle_y \partial^\alpha v\|_p, \quad s \notin \mathbb{Z}. \tag{2.5}$$

We now introduce the Littlewood-Paley characterization of Besov spaces. Let $\hat{\varphi}_0 \in C_c^\infty(\mathbb{R})$ with

$$\hat{\varphi}_0(\xi) = \begin{cases} 1, & |\xi| \le 1, \\ 0, & |\xi| \ge 2 \end{cases} \tag{2.6}$$

be the real-valued radial Bump function. It is easy to see that

$$\begin{aligned} \hat{\varphi}_j(\xi) &= \hat{\varphi}_0(2^{-j}\xi), \qquad j \in \mathbb{Z}, \\ \hat{\psi}_j(\xi) &= \hat{\varphi}_0(2^{-j}\xi) - \hat{\varphi}_0(2^{-j+1}\xi), \qquad j \in \mathbb{Z} \end{aligned} \tag{2.7}$$

are also real-valued radial Bump functions satisfying that

$$\sup_{\xi\in\mathbb{R}^n} 2^{j|\alpha|} |\partial^\alpha \hat{\psi}_j(\xi)| < \infty, \quad j \in \mathbb{Z},$$
$$\sup_{\xi\in\mathbb{R}^n} 2^{j|\alpha|} |\partial^\alpha \hat{\varphi}_j(\xi)| < \infty, \quad j \in \mathbb{Z}.$$

We have the Littlewood–Paley decomposition:

$$\hat{\varphi}_0(\xi) + \sum_{j=0}^\infty \hat{\psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n, \tag{2.8}$$

$$\sum_{j\in\mathbb{Z}} \hat{\psi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.9}$$

$$\lim_{j\to+\infty} \hat{\varphi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n. \tag{2.10}$$

For convenience, we introduce the following notations:

$$\triangle_j f = \mathcal{F}^{-1} \hat{\psi}_j \mathcal{F} f = \psi_j * f, \quad j \in \mathbb{Z}, \tag{2.11}$$

$$S_j f = \mathcal{F}^{-1} \hat{\varphi}_j \mathcal{F} f = \varphi_j * f, \quad j \in \mathbb{Z}. \tag{2.12}$$

Then we have the following Littlewood-Paley definition of Besov spaces:

$$B^s_{p,q}=\Big\{f\in\mathcal{S}'(\mathbb{R}^n)\Big|\|f\|_{B^s_{p,q}}=\|S_0f\|_p+\Big(\sum_{j=1}^{\infty}2^{jsq}\|\triangle_jf\|_p^q\Big)^{\frac{1}{q}}=\|\varphi_0*f\|_p+\Big(\sum_{j=1}^{\infty}2^{jsq}\|\psi_j*f\|_p^q\Big)^{\frac{1}{q}}<\infty\Big\},$$

$$\dot{B}^s_{p,q}=\Big\{f\in\mathcal{S}'(\mathbb{R}^n)\Big|\|f\|_{\dot{B}^s_{p,q}}=\Big(\sum_{j\in\mathbb{Z}}2^{jsq}\|\triangle_jf\|_p^q\Big)^{\frac{1}{q}}=\Big(\sum_{j\in\mathbb{Z}}2^{jsq}\|\psi_j*f\|_p^q\Big)^{\frac{1}{q}}<\infty\Big\},$$

$$\dot{B}^s_{p,\infty}=\Big\{f\in\mathcal{S}'(\mathbb{R}^n)\Big|\|f\|_{\dot{B}^s_{p,\infty}}=\sup_{j\in\mathbb{Z}}2^{js}\|\triangle_jf\|_p=\sup_{j\in\mathbb{Z}}2^{js}\|\psi_j*f\|_p<\infty\Big\},$$

$$\dot{B}^{-\sigma}_{p,\infty}=\Big\{f\in\mathcal{S}'(\mathbb{R}^n)\Big|\|f\|_{\dot{B}^{-\sigma}_{p,\infty}}=\sup_{t>0}t^{\frac{\sigma}{2}}\|H(t)f\|_p<\infty\},\quad \sigma>0,$$

where $1\le p\le\infty$, $1\le q<\infty$, $s\in\mathbb{R}$ and $H(t)=e^{t\Delta}$ is the heat semigroup. From the Littlewood-Paley characterization of Besov spaces, one easily sees that

$$L^3(\mathbb{R}^3)\subset L^{3,\infty}(\mathbb{R}^3)\subset\dot{B}^{-1+\frac{3}{p_1}}_{p_1,\infty}\subset\dot{B}^{-1+\frac{3}{p_2}}_{p_2,\infty}\subset\dot{B}^{-1}_{\infty,\infty}$$

for $3\le p_1\le p_2\le\infty$, where $L^{p,q}$ denotes the Lorentz space (see [29, 25] for details).

Proposition 2.1 *For $\mu\in\mathbb{R}^+$ let $\hat{\psi}_\mu(\xi)=\hat{\psi}_0(\mu^{-1}\xi)$ and $\triangle_\mu f=\mathcal{F}^{-1}(\hat{\psi}_\mu(\xi)\hat{f})=\psi_\mu*f$ with $\hat{\psi}_0$ being defined as in (2.7). Then*

$$\|f\|_{\dot{B}^s_{p,q}}\cong\Big(\int_0^\infty(\mu^s\|\triangle_\mu f\|_p)^q\frac{d\mu}{\mu}\Big)^{\frac{1}{q}}. \tag{2.13}$$

Note that $\triangle_\mu$ with Greek letter such as μ has different definition from $\triangle_j$ with English letter such as j. This equivalent norm of Besov spaces was introduced in [23]. However, no proof was given there. The case with $q=\infty$ has been used by Planchon [6, 30, 2] without a proof. We will give a proof in the appendix.

Besides the classical Besov spaces, we also need the so-called generalized Besov spaces.

Definition 2.1 *For $s\in\mathbb{R}$, $1\le q\le\infty$ and any Banach space E, define $\dot{B}^{s,q}_E$ as*

$$\dot{B}^{s,q}_E=\{f\in E|\ \|f;\dot{B}^{s,q}_E\|=(\sum_{j\in\mathbb{Z}}2^{jsq}\|\triangle_jf\|_E^q)^{\frac{1}{q}}<\infty\} \tag{2.14}$$

where $\triangle_j$ is the Littlewood-Paley operator on $\mathbb{R}^n$ defined in (2.11) and (2.12).

Remark 2.1 (i) One easily verifies that $\dot{B}_E^{s,q}$ can be characterized equivalently for $s<0$ by

$$\dot{B}_E^{s,q}=\{f\in E|\ \|f;\dot{B}_E^{s,q}\|=(\sum_{j\in\mathbb{Z}}2^{jsq}\|S_jf\|_E^q)^{\frac{1}{q}}<\infty\}.$$

(ii) In the study of self-similar solutions we usually use the Besov spaces defined by replacing L^p with the Lorentz space $L^{p,r}$.

(iii) Let $E=L^q(I;L^r)$ with $I=\mathbb{R}$ or $I\subset\mathbb{R}$ being an interval. Then we have

$$\begin{aligned}\mathcal{L}^q(I;\dot{B}^s_{r,\rho})&\triangleq\dot{B}^{s,\rho}_{L^q(I;L^r)}\\&=\{f\in L^q(I;L^r)|\ \|f;\dot{B}^{s,\rho}_{L^q(I;L^r)}\|=(\sum_{j\in\mathbb{Z}}2^{js\rho}\|\triangle_jf\|^\rho_{L^q(I;L^r)})^{\frac{1}{\rho}}<\infty\}\end{aligned}$$

(iv) In Proposition 2.1, if L^p is replaced with the Banach space E, then we have

$$\|f\|_{\dot{B}_E^{s,\rho}}\cong\left(\int_0^\infty\left(\mu^s\|\triangle_\mu f\|_E\right)^\rho\frac{d\mu}{\mu}\right)^{\frac{1}{\rho}},$$

where $\triangle_\mu$ is defined as in Proposition 2.1. In particular, letting $E=L^q(\mathbb{R},L^r)$ gives

$$\begin{aligned}\|f;\dot{B}^{s,\rho}_{L^q(\mathbb{R},L^r)}\|&\cong\left(\int_0^\infty\left(\mu^s\|\triangle_\mu f\|_{L^q(\mathbb{R},L^r)}\right)^\rho\frac{d\mu}{\mu}\right)^{\frac{1}{\rho}},\\\|f;\dot{B}^{s,\infty}_{L^q(\mathbb{R},L^r)}\|&\cong\sup_{\mu>0}\mu^s\|\triangle_\mu f\|_{L^q(\mathbb{R},L^r)}.\end{aligned}$$

2.2 Strichartz estimates

We introduce the so-called admissible and generalized admissible triplets needed in this paper.

Definition 2.2 *The triplet (q,r,p) is said to be admissible if*

$$\frac{1}{q}=\frac{n}{2}(\frac{1}{p}-\frac{1}{r}),$$

where

$$1<p\le r<\begin{cases}\dfrac{np}{n-2}, & n>2,\\ \infty, & n\le 2.\end{cases}$$

Definition 2.3 *The triplet* (q,r,p) *is called as a generalized admissible triplet with respect to the heat operator if*

$$\frac{1}{q}=\frac{n}{2}(\frac{1}{p}-\frac{1}{r}),$$

where

$$1<p\le r<\begin{cases}\dfrac{np}{n-2p}, & n>2p,\\ \infty, & n\le 2p.\end{cases}$$

Remark 2.2 (i) It is easy to see that if (q,r,p) is an admissible or a generalized admissible triplet, then q is uniquely determined by r and p so we may write $q=q(r,p)$.

(ii) Clearly, $r<q\le\infty$ if (q,r,p) is an admissible triplet, and $1<q\le\infty$ if (q,r,p) is a generalized admissible triplet.

Let B be a Banach space and let, for some $T>0$, $I=[0,T)$ and $\dot{I}=(0,T)$. For $\sigma>0$ define $\mathcal{C}_\sigma(I;B)$ and its homogeneous space $\dot{\mathcal{C}}_\sigma(I;B)$ by

$$\mathcal{C}_\sigma(I;B):=\{f\in C(\dot{I};B)\mid \|f;\mathcal{C}_\sigma(I;B)\|=\sup_{t\in I}t^{\frac{1}{\sigma}}\|f\|_B<\infty\},$$
$$\dot{\mathcal{C}}_\sigma(I;B):=\{f\in \mathcal{C}_\sigma(I;B)\mid \lim_{t\to 0^+}t^{\frac{1}{\sigma}}\|f\|_B=0\}.$$

Define also $C_*(I;B):=\{v\in L^\infty(I;B)\mid v$ is continuous in the distributional sense $\}$. Then it is easy to see that:

(i) $f\in\mathcal{C}_\sigma(I;L^r)$ if and only if $t^{\frac{1}{\sigma}}f\in C_b(I;L^r)$;

(ii) if (q,r,p) is a generalized admissible triplet, then

$$\mathcal{C}_{q(r,p)}(I;L^r)=\{f\in C(\dot{I};L^r)\mid \|f\|_{\mathcal{C}_{q(r,p)}(I;L^r)}=\sup_{t\in I}t^{\frac{1}{q}}\|f\|_r<\infty\},$$
$$\dot{\mathcal{C}}_{q(r,p)}(I;L^r)=\{f\in \mathcal{C}_{q(r,p)}(I;L^r)\mid \lim_{t\to 0^+}t^{\frac{1}{q}}\|f\|_r=0\};$$

in particular, $\mathcal{C}_{q(r,p)}(I;L^r)=C_b(I;L^p)$ if $r=p$.

(iii) $v\in C_*(I;B)$ means that v is a bounded flow in B.

It is well-known that the heat semigroup

$$H(t)=e^{\triangle t}=\mathcal{F}^{-1}(e^{-|\xi|^2t})\mathcal{F}\cdot=G(t,\cdot)*$$

generates an analytic semigroup in L^p with $1<p<\infty$, where

$$G(t,x)=(4\pi t)^{-n/2}\exp\left(-\frac{|x|^2}{4t}\right),\quad t>0,\quad x\in\mathbb{R}^n$$

is the fundamental solution of the heat operator $\partial_t - \Delta$ in $\mathbb{R}^+ \times \mathbb{R}^n$. By Hörmander–Mihlin multiplier theory and regularity estimates of the heat semigroup (see e.g. [17, 26]), one easily sees that for $t > 0$

$$\|H(t)\varphi\|_r \le C|t|^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|\varphi\|_p, \quad 1 \le p \le r \le \infty, \tag{2.15}$$

$$\|(-\triangle)^{\frac{d}{2}}H(t)\varphi\|_r \le Ct^{-\frac{d}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|\varphi\|_p, \quad d \ge 0, \quad 1 \le p \le r \le \infty, \quad p \ne \infty, \tag{2.16}$$

$$\|H(t)\varphi\|_{\dot{H}^{s+\theta,r}} \le Ct^{-\frac{\theta}{2}}\|\varphi\|_{\dot{H}^{s,r}}, \quad 1 < r < \infty, \quad \theta \ge 0, \tag{2.17}$$

$$\|H(t)\varphi\|_{H^{s+\theta,r}} \le C(T)t^{-\frac{\theta}{2}}\|\varphi\|_{H^{s,r}}, \quad t \in (0,T), \quad 1 < r < \infty, \quad \theta \ge 0. \tag{2.18}$$

By making use of (2.15)-(2.18) and Marcinkiewicz's interpolation theorem [26] one can easily obtain the following space-time estimates (Strichartz estimates).

Proposition 2.2 *Let $I = [0, \infty)$ or $I = [0, T)$ with $T > 0$.*

(ii) *Let (q, r, p) be any generalized admissible triplet and let $\varphi \in L^p$. Then $H(\cdot)\varphi \in \mathcal{C}_{q(r,p)}(I; L^r) \cap C_b(I; L^p)$ and*

$$\|H(\cdot)\varphi\|_{\mathcal{C}_{q(r,p)}(I;L^r)} \le C\|\varphi\|_p, \tag{2.19}$$

where C is constant independent of φ and T. Moreover, if $r > p$, then $H(\cdot)\varphi \in \dot{\mathcal{C}}_{q(r,p)}(I; L^r)$, that is

$$\lim_{t\to 0} t^{\frac{1}{q(r,p)}}\|H(t)\varphi\|_r = 0, \quad \frac{1}{q} = \frac{n}{2}(\frac{1}{p} - \frac{1}{r}). \tag{2.20}$$

(ii) *Let $r \ge p \ge p_0 := n\alpha/2 > 1$ and let (q, r, p_0) be a generalized admissible triplet. Then for $\varphi \in \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}$ we have $H(\cdot)\varphi \in \mathcal{C}_{q(r,p_0)}(I; L^r) \cap C_b(I; \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})$ and*

$$\|H(\cdot)\varphi\|_{\mathcal{C}_{q(r,p_0)}(I;L^r)} \le C\|\varphi; \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}\|, \tag{2.21}$$

$$\|H(\cdot)\varphi\|_{\mathcal{C}_*(I;\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})} \le C\|\varphi; \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}\|, \tag{2.22}$$

where C is constant independent of φ and T.

Proof For the proof of (i) one can see [31, 32, 17]. The estimate (2.22) is obvious. So we only prove (2.21).

$$\begin{aligned}
\sup_{t\in I} t^{\frac{1}{q(r,p_0)}}\|H(t)\varphi\|_r &\lesssim \sup_{t\in I} t^{\frac{1}{q(r,p_0)}}\left(\frac{t}{2}\right)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|H(\frac{t}{2})\varphi\|_p \\
&\lesssim \sup_{t\in I}(t/2)^{\frac{1}{2}(\frac{n}{p}-\frac{n}{p_0})}\|H(\frac{t}{2})\varphi\|_p \\
&\lesssim \sup_{t\in\mathbb{R}^+} t^{\frac{1}{2}(\frac{n}{p}-\frac{2}{\alpha})}\|H(t)\varphi\|_p = C\|\varphi; \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}\|,
\end{aligned}$$

which implies (2.22). The proof is thus completed.

Remark 2.3 (i) The estimate (2.19) easily follows from (2.16) and the definition of $\mathcal{C}_{q(p,r)}(I;L^r)$. The limit (2.20) can be easily shown by using the Banach-Steinhaus theorem together with the fact that L^p is a separable Banach space (see [12] for details).

(ii) If (q,r,p) is a generalized admissible triplet, then $H(\cdot)\varphi \notin L^q(I;L^r)$ since the condition $q>p$ is needed to use Marcinkiewicz's interpolation theorem.

3. Self-similar Solutions

In this section we shall study the self-similar solutions for the semi-linear heat and convection-diffusion equations. We first consider the heat equation (1.1).

Proposition 3.1 *Let $f \in \dot{B}^{s_\alpha}_{p,\infty}$ satisfy (1.3) and let (q,r,p) be any admissible triplet with $r>p$ and $s_\alpha = \frac{n}{p}-\frac{2}{\alpha}$. Then the self-similar solution u of (1.1) satisfies that*

$$u(t,x)=t^{-\frac{1}{\alpha}}U(\frac{x}{\sqrt{t}}) \in \mathcal{L}^q(\mathbb{R};\dot{B}^{s_\alpha}_{r,\infty}) = \dot{B}^{s_\alpha,\infty}_{L^q_t L^r_x}$$

if and only if its profile $U \in \dot{B}^{s_\alpha}_{q,r}$.

Proof By using Fourier transforms one easily verifies that

$$\begin{aligned}
\triangle_\mu u &\cong \int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\psi}_\mu(\xi)\hat{u}(\xi,t)d\xi \cong t^{-\frac{1}{\alpha}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\psi}_\mu(\xi)\widehat{U(\frac{x}{\sqrt{t}})}(\xi)d\xi \\
&\cong t^{-\frac{1}{\alpha}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{\psi}_\mu(\xi)\widehat{U}(\sqrt{t}\xi)d\sqrt{t}\xi \\
&\cong t^{-\frac{1}{\alpha}}\int_{\mathbb{R}^n} e^{i\frac{x}{\sqrt{t}}\cdot\sqrt{t}\xi}\hat{\psi}_{\sqrt{t}\mu}(\sqrt{t}\xi)\widehat{U}(\sqrt{t}\xi)d\sqrt{t}\xi \\
&\cong t^{-\frac{1}{\alpha}}\int_{\mathbb{R}^n} e^{i\frac{x}{\sqrt{t}}\cdot\xi}\hat{\psi}_{\sqrt{t}\mu}(\xi)\widehat{U}(\xi)d\xi \\
&= t^{-\frac{1}{\alpha}}(\triangle_{\sqrt{t}\mu}U)(\frac{x}{\sqrt{t}})
\end{aligned}$$

$$\begin{aligned}
\|\triangle_\mu u\|_{L^q_t L^r_x} &= \left(\int_0^\infty \left(\sqrt{t}^{-\frac{2}{\alpha}}\|(\triangle_{\sqrt{t}\mu}U)(\frac{x}{\sqrt{t}})\|_r\right)^q dt\right)^{\frac{1}{q}} \\
&= \left(\int_0^\infty \left(\sqrt{t}^{\frac{n}{r}-\frac{2}{\alpha}}\|\triangle_{\sqrt{t}\mu}U\|_r\right)^q dt\right)^{\frac{1}{q}} \\
&= \mu^{\frac{2}{\alpha}-\frac{n}{r}-\frac{2}{q}}\left(\int_0^\infty \left((\sqrt{t}\mu)^{\frac{n}{r}-\frac{2}{\alpha}}\|\triangle_{\sqrt{t}\mu}U\|_r\right)^q d(t\mu^2)\right)^{\frac{1}{q}} \\
&= 2^{\frac{1}{q}}\mu^{\frac{2}{\alpha}-\frac{n}{r}-\frac{2}{q}}\left(\int_0^\infty \left(\eta^{\frac{n}{r}-\frac{2}{\alpha}}\|\triangle_\eta U\|_r\right)^q \eta d\eta\right)^{\frac{1}{q}} \\
&= 2^{\frac{1}{q}}\mu^{\frac{2}{\alpha}-\frac{n}{r}-\frac{2}{q}}\left(\int_0^\infty \left(\eta^{\frac{n}{r}-\frac{2}{\alpha}+\frac{2}{q}}\|\triangle_\eta U\|_r\right)^q \frac{d\eta}{\eta}\right)^{\frac{1}{q}}.
\end{aligned}$$

Since for any admissible pair with $r > p$ we have

$$s_\alpha = \frac{n}{p} - \frac{2}{\alpha} = \frac{n}{r} + \frac{2}{q} - \frac{2}{\alpha},$$

then by the equivalent norm (2.13) of Besov spaces it follows that

$$\begin{aligned}\sup_{\mu>0} \mu^{s_\alpha} \|\triangle_\mu u\|_{L^q_t L^r_x} &= 2^{\frac{1}{q}} \sup_{\mu>0} \mu^{\frac{2}{\alpha}-\frac{n}{r}-\frac{2}{q}+s_\alpha} \left(\int_0^\infty \left(\eta^{\frac{n}{r}-\frac{2}{\alpha}+\frac{2}{q}} \|\triangle_\eta U\|_r\right)^q \frac{d\eta}{\eta}\right)^{\frac{1}{q}} \\ &= 2^{\frac{1}{q}} \left(\int_0^\infty \left(\eta^{s_\alpha} \|\triangle_\eta U\|_r\right)^q \frac{d\eta}{\eta}\right)^{\frac{1}{q}} \cong \|U\|_{\dot{B}^{s_\alpha}_{r,q}}.\end{aligned}$$

Proposition 3.2 *Assume that* $f \in \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}$ *satisfies* (1.3). *Let* $r \ge p \ge p_0 = \dfrac{n\alpha}{2} > 1$ *and let* (q, r, p_0) *be any generalized admissible triplet. Assume that the self-similar solution* u *of* (1.1) *satisfies that*

$$u(t,x) = t^{-\frac{1}{\alpha}} U\Big(\frac{x}{\sqrt{t}}\Big) = U_{\sqrt{t}}(x) \in C_*([0,\infty); \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}) \cap \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r).$$

Then its profile $U(x)$ *satisfies that*

$$\|u; C_*(\mathbb{R}^+; \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\| \cong \|H(1)U\|_p, \tag{3.1}$$

$$\|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\| \cong \|U\|_r. \tag{3.2}$$

Proof It is easy to verify that

$$\begin{aligned}\|u; C_*(\mathbb{R}^+, \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\| &= \sup_{t\in\mathbb{R}^+} t^{-\frac{1}{2}(\frac{n}{p}-\frac{2}{\alpha})} \|H(t)U_{\sqrt{t}}\|_p \\ &= \sup_{t\in\mathbb{R}^+} t^{-\frac{1}{2}(\frac{n}{p}-\frac{2}{\alpha})} t^{-\frac{1}{\alpha}} \|(H(1)U)(\frac{x}{\sqrt{t}})\|_p \\ &= \sup_{t\in\mathbb{R}^+} t^{-\frac{1}{2}(\frac{n}{p}-\frac{2}{\alpha})-\frac{1}{\alpha}+\frac{n}{p}} \|H(1)U\|_p \\ &= \|H(1)U\|_p, \\ \|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+, L^r)\| &= \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q}} \|U_{\sqrt{t}}\|_r = \sup_{t\in\mathbb{R}^+} \sqrt{t}^{\frac{2}{q}-\frac{2}{\alpha}-\frac{n}{r}} \|U(x)\|_r = \|U\|_r.\end{aligned}$$

The proof is completed.

Proposition 3.3 (i) *Let* $f \in L^p$. *Then* $\|\triangle_0 f\|_p \le C\|H(1)f\|_p$.

(ii) *Let* $f \in \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}$ *satisfy* (1.3) *and let* $\triangle_0 f \in L^1 \cap L^p$. *Then* $H(1)f \in L^p$ *and*

$$\|f; \dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}\| := \|H(1)f\|_p \lesssim \|\triangle_0 f\|_1 + \|\triangle_0 f\|_p.$$

Proof One easily sees that for any initial data f satisfying (1.3),

$$\begin{aligned}\|H(t)f;C_*(\mathbb{R}^+,\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\| &= \|H(t)f_\lambda;C_*(\mathbb{R}^+,\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\| \\ &\le \sup_{t\in\mathbb{R}^+} t^{-\frac{1}{2}(\frac{n}{p}-\frac{2}{\alpha})}\|H(t)(H(1)f)_{\sqrt{t}}\|_p \\ &\le \sup_{t\in\mathbb{R}^+} t^{-\frac{1}{2}(\frac{n}{p}-\frac{2}{\alpha})}t^{-\frac{1}{\alpha}}\|H(1)(H(1)f)(\frac{x}{\sqrt{t}})\|_p \\ &\le \|H(1)f\|_p.\end{aligned}$$

In particular, we have

$$\|f;C_*(\mathbb{R}^+,\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\| = \sup_{t\in\mathbb{R}^+} t^{-\frac{1}{2}(\frac{n}{p}-\frac{2}{\alpha})}t^{-\frac{1}{\alpha}}\|H(1)f(\frac{x}{\sqrt{t}})\|_p = \|H(1)f\|_p.$$

We now establish the relationship between $\|H(1)f\|_p$ and $\|\triangle_0 f\|_p$ for $1<p<\infty$. First let $\hat{h}(\xi)=\hat{h}(|\xi|)\in C_c^\infty$ satisfy that $\hat{h}(\xi)=1$ for $2^{-1}<|\xi|<2$ and $\operatorname{supp}(\hat{h})\subset\{\xi\mid 2^{-2}\le|\xi|\le 2^2\}$. Then one easily sees that

$$\begin{aligned}\|\triangle_0 f\|_p &= \|\mathcal{F}^{-1}(\hat{\psi}(\xi)\hat{f}(\xi))\|_p = \|\mathcal{F}^{-1}(\hat{\psi}(\xi)h(\xi)\hat{f}(\xi))\|_p \\ &\le \|\mathcal{F}^{-1}(e^{|\xi|^2}h(\xi)\hat{\psi}(\xi))*\mathcal{F}^{-1}(e^{-|\xi|^2}\hat{f}(\xi))\|_p \\ &\le \|\mathcal{F}^{-1}(e^{|\xi|^2}h(\xi)\hat{\psi}(\xi))\|_1\|\mathcal{F}^{-1}(e^{-|\xi|^2}\hat{f}(\xi))\|_p \\ &\lesssim \|H(1)f\|_p.\end{aligned}$$

Next decompose $F=H(1)f$ as follows:

$$\begin{aligned}H(1)f &= H(1)((1-\varphi_0)*f)+H(1)\varphi_0*f \\ &= \mathcal{F}^{-1}(e^{-|\xi|^2}(1-\hat{\varphi}_0)\hat{f})+\mathcal{F}^{-1}(e^{-|\xi|^2}\hat{\varphi}_0\hat{f}) \\ &\stackrel{\triangle}{=} F_1+F_2.\end{aligned}$$

One easily verifies from the definition of $\hat{\varphi}_0$ (see (2.6)) that

$$\operatorname{supp}(\hat{F}_1)\subset\{\xi\mid|\xi|\ge 1\},\qquad \triangle_j(F_1)=0,\quad j\le -1,$$
$$\operatorname{supp}(\hat{F}_2)\subset\{\xi\mid|\xi|\le 2\},\qquad \triangle_j(F_2)=0,\quad j\ge 2.$$

Since f satisfies the scaling condition (1.3), it follows that

$$\begin{aligned}\triangle_j(f) &= \mathcal{F}^{-1}(\hat{\psi}_0(\frac{\xi}{2^j})\hat{f})\cong\int_{\mathbb{R}^n}e^{i2^jx\cdot\xi}\hat{\psi}_0(\xi)2^{jn}\hat{f}(2^j\xi)d\xi \\ &\cong \int_{\mathbb{R}^n}e^{i2^jx\cdot\xi}\hat{\psi}_0(\xi)\widehat{f(\frac{x}{2^j})}(\xi)d\xi \\ &\cong \int_{\mathbb{R}^n}e^{i2^jx\cdot\xi}\hat{\psi}_0(\xi)2^{j\frac{2}{\alpha}}\hat{f}(\xi)d\xi \\ &\cong 2^{j\frac{2}{\alpha}}\triangle_0(f)(2^jx).\end{aligned}\tag{3.3}$$

Noting that for $\hat{\tilde{\psi}}_j = \sum_{\ell=-2}^{2} \hat{\psi}_{j+\ell}$ and $\hat{\tilde{\varphi}}(\xi) = e^{-|\xi|^2}\hat{\varphi}(\xi)$,

$$\begin{aligned}\triangle_j(F_2) &= \mathcal{F}^{-1}\left(\hat{\psi}_j(\xi)\hat{F}_2\right) = \mathcal{F}^{-1}\left(\hat{\psi}_j(\xi)\hat{\tilde{\psi}}_j(\xi)\hat{F}_2\right)\\ &= \mathcal{F}^{-1}\left(\hat{\psi}_j(\xi)\hat{\tilde{\psi}}_j(\xi)\hat{\varphi}(\xi)\hat{f}e^{-|\xi|^2}\right)\\ &= \triangle_j(f) * \tilde{\triangle}_j(\tilde{\varphi}),\end{aligned}$$

we obtain that

$$\|\triangle_j F_2\|_p \lesssim \|\tilde{\triangle}_j(\tilde{\varphi})\|_1 \|\triangle_j(f)\|_p \lesssim 2^{j(\frac{2}{\alpha}-\frac{n}{p})}\|\triangle_0 f\|_p, \qquad j \le 1,$$

where use has been made of the fact that $\|\tilde{\triangle}_j(\tilde{\varphi})\|_1 \le C\|\tilde{\varphi}\|_1$. Now letting $f_1 = (1-\hat{\varphi})\hat{f}$ and using (3.3) and the regularity of the analytic semi-group $H(t)$ we have for $j \ge 0$,

$$\|\triangle_j(F_1)\|_p = \|H(1)\triangle_j(f_1)\|_p \lesssim \|\triangle_j(f_1)\|_1 \lesssim \|\triangle_j(f)\|_1 \lesssim 2^{(\frac{2}{\alpha}-n)j}\|\triangle_0 f\|_1.$$

This implies the proposition and completes the proof.

Theorem 3.1 *Let $r \ge p > p_0 = \dfrac{n\alpha}{2} > 1$ and let (q, r, p_0) be any generalized admissible triplet satisfying that*

$$\max(p_0, 1+\alpha) < r < p_0(1+\alpha). \tag{3.4}$$

Let $f \in \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}$ satisfy (1.3). Then there exist η, $\beta > 0$ depending on p such that if $\|f\|_{\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}} < \eta$, then the problem (1.1) has a unique self-similar solution

$$u \in C_*\left([0,\infty); \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}\right) \cap \mathcal{C}_{q(r,p_0)}([0,\infty); L^r)$$

with

$$u(t,x) = H(t)f(x) + w(t,x) = H(t)f(x) + t^{-\frac{1}{\alpha}}W\left(\frac{x}{\sqrt{t}}\right) = t^{-\frac{1}{\alpha}}U\left(\frac{x}{\sqrt{t}}\right), \tag{3.5}$$

where

$$\left\|w(t,x); L^\infty(\mathbb{R}^+; L^{p_0})\right\| < \infty, \tag{3.6}$$

$$\sup_{t>0} t^{\frac{1}{q(r,p_0)}}\|u(t,\cdot)\|_r < \beta. \tag{3.7}$$

Moreover,

$$U(\cdot) = H(1)f(\cdot) + W(\cdot) \in L^r. \tag{3.8}$$

Proof From the equivalent norm of Besov spaces and Proposition 2.2, we obtain that for any generalized admissible triplet (q,r,p_0),

$$\|H(t)f;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|+\|H(t)f;C_*(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\|\le C\|f;\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\|, \tag{3.9}$$

where $C>0$ is a constant independent of φ and (q,r,p_0).

Denote by Λ the set of all generalized admissible triplets (q,r,p_0) satisfying (3.4), and define

$$X:=\{u\in C_*(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}))\cap\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r),\quad (q,r,p_0)\in\Lambda\}$$

with the norm

$$\|u;X\|:=\sup_{(q,r,p_0)\in\Lambda}\sup_{t\in\mathbb{R}^+}t^{\frac{1}{q}}\|u\|_r+\sup_{t\in\mathbb{R}^+}\|u;\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty}\|.$$

Let us introduce the complete metric space

$$\mathcal{X}=\{u\in X\mid \|u;X\|\le M\}$$

with the metric $d(u,v)=\|u-v;X\|$, where M is a positive constant to be determined later, and consider, in the metric space $\mathcal{X}$, the operator $\mathcal{T}$ defined by

$$\mathcal{T}u:=H(t)f+\int_0^t H(t-\tau)F(u(\tau,\cdot))d\tau=H(t)f+\mathcal{G}F(u),\quad u\in\mathcal{X}, \tag{3.10}$$

where $F(u)=\mu u^{\alpha+1}$. It can be shown that $\mathcal{T}$ is a contractive mapping from $\mathcal{X}$ into itself. In fact, from the L^p–L^r estimates (2.15)–(2.18) of the heat semi-group $H(t)$ it follows that for any $(q,r,p_0)\in\Lambda$ and $u\in\mathcal{X}$,

$$\begin{aligned}\|\mathcal{G}F(u);\mathcal{C}_{q(r,p_0)(\mathbb{R}^+;L^r)}\| &\le \sup_{t\in\mathbb{R}^+}t^{\frac{1}{q}}\int_0^t|t-\tau|^{-\frac{n}{2}(\frac{1+\alpha}{r}-\frac{1}{r})}\|u\|_r^{\alpha+1}d\tau\\ &\le \int_0^1|1-\tau|^{-\frac{n\alpha}{2r}}\tau^{-\frac{\alpha+1}{q}}d\tau\|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha+1}\\ &\le C\|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha+1},\end{aligned} \tag{3.11}$$

where use has been made of the fact that $n\alpha/(2r)<1$ and $q>1+\alpha$. Further, noting that $L^{p_0}\subset\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})$ we obtain again from (2.15)–(2.18) that for any $(q,r,p_0)\in\Lambda$ and $u\in\mathcal{X}$,

$$\begin{aligned}&\|\mathcal{G}F(u);C_*(\mathbb{R}^+;\dot{B}^{\frac{n}{p}-\frac{2}{\alpha}}_{p,\infty})\|\le C\|\mathcal{G}F(u);C_*(\mathbb{R}^+;L^{p_0})\|\\ &\qquad\le\sup_{t\in\mathbb{R}^+}\int_0^t|t-\tau|^{-\frac{n}{2}(\frac{1+\alpha}{r}-\frac{1}{p_0})}\|u\|_r^{\alpha+1}d\tau\\ &\qquad\le C\sup_{t\in\mathbb{R}^+}\int_0^t|t-\tau|^{-\frac{n}{2}(\frac{\alpha+1}{r}-\frac{1}{p_0})}\tau^{-\frac{\alpha+1}{q}}d\tau\|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha+1}\\ &\qquad\le C\int_0^1|1-\tau|^{-\frac{n}{2}(\frac{\alpha+1}{r}-\frac{1}{p_0})}\tau^{-\frac{\alpha+1}{q}}d\tau\|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha+1}\\ &\qquad\le C\|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha+1}.\end{aligned} \tag{3.12}$$

Combining (3.9), (3.11) with (3.12) yields that for $u \in \mathcal{X}$,

$$\|\mathcal{T}u; X\| \leq C\|f; \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}\| + 2C\|u; \mathcal{C}_{q(r,p_0)}(I; L^r)\|^{\alpha+1}. \tag{3.13}$$

Similarly, it can be derived that for $u, v \in \mathcal{X}$

$$d(\mathcal{T}u, \mathcal{T}v) \leq C[\|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\|^{\alpha} + \|v; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\|^{\alpha}]d(u, v). \tag{3.14}$$

Let $M = 2C\|f; \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}\|$. Then it is easy to see from (3.13)-(3.14) that there exist $\eta > 0$ and $\beta > 0$ such that if $\|f\|_{\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}} < \eta$, then $\mathcal{T}$ is a contractive mapping from $\mathcal{X}$ into itself so, by the Banach contraction mapping principle, the problem (1.1) has a unique solution $u \in \mathcal{X}$ satisfying (3.7). The results (3.5)-(3.8) follow from Propositions 2.2, 3.2 and 3.3. The theorem is thus proved.

Now consider the Cauchy problem for the convection-diffusion equation (1.2). we have the following result.

Theorem 3.2 *Let $r \geq p > p_0 = n\alpha > 1$ and let (q, r, p_0) be any generalized admissible triplet satisfying that*

$$\max(p_0, 1+\alpha) < r \leq p_0(1+\alpha). \tag{3.15}$$

Let $f \in \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}$ satisfy (1.3). Then there exist $\eta,\ \beta > 0$ depending on p such that if $\|f\|_{\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}} < \eta$, then the problem (1.2) has a unique self-similar solution

$$u \in C_*\left(\mathbb{R}^+; \dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}\right) \cap \mathcal{C}_{q(r,p_0)}\left(\mathbb{R}^+; L^r\right)$$

with

$$u(t,x) = H(t)f(x) + w(t,x) = H(t)f(x) + t^{-\frac{1}{\alpha}}W\left(\frac{x}{\sqrt{t}}\right) = t^{-\frac{1}{\alpha}}U\left(\frac{x}{\sqrt{t}}\right), \tag{3.16}$$

where

$$\left\|w(t,x); L^{\infty}(\mathbb{R}^+; L^{p_0})\right\| < \infty, \tag{3.17}$$

$$\sup_{t>0} t^{\frac{1}{q(r,p_0)}}\|u(t,\cdot)\|_r < \beta. \tag{3.18}$$

Moreover,

$$U(\cdot) = H(1)f(\cdot) + W(\cdot) \in L^r. \tag{3.19}$$

Proof In view of the equivalent norm of Besov spaces and Proposition 2.2, it can be seen that for any generalized admissible triplet (q, r, p_0),

$$\|H(t)f; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\| + \|H(t)f; C_*(\mathbb{R}^+; \dot{B}^{\frac{n}{p}-\frac{1}{\alpha}}_{p,\infty})\| \leq C\|f; \dot{B}^{\frac{n}{p}-\frac{1}{\alpha}}_{p,\infty})\|, \tag{3.20}$$

where $C > 0$ is a constant independent of φ and (q, r, p_0).

Denote by Λ^* the set of all generalized admissible triplets (q, r, p_0) satisfying (3.15). Define

$$Y := \{u \in C_*(\mathbb{R}^+; \dot{B}^{\frac{n}{p}-\frac{1}{\alpha}}_{p,\infty})) \cap \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r), \quad (q, r, p_0) \in \Lambda^*\}$$

with the norm

$$\|u; Y\| := \sup_{(q,r,p_0)\in\Lambda^*} \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q}} \|u\|_r + \sup_{t\in\mathbb{R}^+} \|u; \dot{B}^{\frac{n}{p}-\frac{1}{\alpha}}_{p,\infty}\|.$$

Now introduce the complete metric space

$$\mathcal{Y} := \{u \in Y \mid \|u; Y\| \leq M\}$$

with the metric $d(u, v) = \|u - v; Y\|$, $u, v \in \mathcal{Y}$, where $M > 0$ is a constant to be determined later, and consider, in the metric space $\mathcal{Y}$, the operator $\mathcal{T}$ defined by (3.10) with

$$F(u) = \vec{a} \cdot \nabla(|u|^\alpha u).$$

We now prove that $\mathcal{T}$ is a contractive mapping from $\mathcal{Y}$ into itself. First, the L^p–L^r estimates (2.15)–(2.18) of the heat semi-group $H(t)$ imply that for any $(q, r, p_0) \in \Lambda^*$ and $u \in \mathcal{Y}$,

$$\begin{aligned}\|\mathcal{G}F(u); \mathcal{C}_{q(r,p_0)(\mathbb{R}^+;L^r)}\| &\leq \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q}} \int_0^t |t-\tau|^{-\frac{1}{2}-\frac{n\alpha}{2r}} \|u\|_r^{\alpha+1} d\tau \\ &\leq \int_0^1 |1-\tau|^{-(\frac{1}{2}+\frac{n\alpha}{2r})} \tau^{-\frac{\alpha+1}{q}} d\tau \|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\|^{\alpha+1} \\ &\leq C\|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\|^{\alpha+1},\end{aligned} \tag{3.21}$$

where use has been made of the fact that $1/2 + n\alpha/(2r) < 1$ and $q > 1 + \alpha$. Next, from the estimates (2.15)-(2.18) again, and since $L^{p_0} \subset \dot{B}^{\frac{n}{p}-\frac{1}{\alpha}}_{p,\infty}$, it follows that for any $(q, r, p_0) \in \Lambda^*$ and $u \in \mathcal{Y}$,

$$\begin{aligned}&\|\mathcal{G}F(u); C_*(\mathbb{R}^+; \dot{B}^{\frac{n}{p}-\frac{1}{\alpha}}_{p,\infty})\| \leq C\|\mathcal{G}F(u); C_*(\mathbb{R}^+; L^{p_0})\| \\ &\leq \sup_{t\in\mathbb{R}^+} \int_0^t |t-\tau|^{-\frac{1}{2}-\frac{n}{2}(\frac{1+\alpha}{r}-\frac{1}{p_0})} \|u\|_r^{\alpha+1} d\tau \\ &\leq C \sup_{t\in\mathbb{R}^+} \int_0^t |t-\tau|^{-\frac{1}{2}-\frac{n}{2}(\frac{\alpha+1}{r}-\frac{1}{p_0})} \tau^{-\frac{\alpha+1}{q}} d\tau \|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\|^{\alpha+1} \\ &\leq C \int_0^1 |1-\tau|^{-\frac{1}{2}-\frac{n}{2}(\frac{\alpha+1}{r}-\frac{1}{p_0})} \tau^{-\frac{\alpha+1}{q}} d\tau \|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\|^{\alpha+1} \\ &\leq C\|u; \mathcal{C}_{q(r,p_0)}(\mathbb{R}^+; L^r)\|^{\alpha+1}.\end{aligned} \tag{3.22}$$

From (3.20)-(3.22) it follows that

$$\|\mathcal{T}u;Y\| \le C\|f;\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}\| + 2C\|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha+1}, \quad u\in\mathcal{Y}, \tag{3.23}$$

$$d(\mathcal{T}u,\mathcal{T}v) \le C[\|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha} + \|u;\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)\|^{\alpha}]d(u,v), \quad u,v\in\mathcal{Y}. \tag{3.24}$$

Let $M = 2C\|f;\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}\|$. Then (3.23) and (3.24) imply that there exist $\eta>0$ and $\beta>0$ such that if $\|f\|_{\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}} < \eta$, then $\mathcal{T}$ is a contractive mapping from $\mathcal{Y}$ into itself so, by the Banach contraction mapping principle, the problem (1.2) has a unique solution $u\in\mathcal{Y}$ satisfying (3.18). The results (3.16)-(3.19) follow from Propositions 2.2, 3.2 and 3.3. The proof is thus completed.

The following corollary means that Theorems 3.1 and 3.2 remain true without the restrictions (3.4) and (3.15) on r, α and p_0.

Corollary 3.1 (i) *Let $r\ge p>p_0=n\alpha/2>1$ and let (q,r,p_0) be any generalized admissible triplet. Assume that $f\in\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}$ satisfies* (1.3). *Then there exist η, $\beta>0$ depending on p such that if $\|f\|_{\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}}<\eta$, then the problem* (1.1) *has a unique self-similar solution*

$$u\in C_*(\mathbb{R}^+;\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}})\cap\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)$$

satisfying (3.5) − (3.8).

(ii) *Let $\alpha\ge1$, $r\ge p>p_0=n\alpha>1$ and let (q,r,p_0) be any generalized admissible triplet. Let $f\in\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}$ satisfy* (1.3). *Then there exist η, $\beta>0$ depending on p such that if $\|f\|_{\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}}<\eta$, then the problem* (1.2) *has a unique self-similar solution*

$$u\in C_*(\mathbb{R}^+;\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}})\cap\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)$$

satisfying (3.16) − (3.19).

Proof It is enough to prove that for any generalized admissible triplet (q,r,p_0), the solution u, obtained in Theorem 3.1 in the case when $(q,r,p_0)\in\Lambda$ (for (i)) or in Theorem 3.2 in the case when $(q,r,p_0)\in\Lambda^*$ (for (ii)), satisfies that $u\in\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)$.

Consider first the case $r\le1+\alpha$. By interpolation between $C_*(\mathbb{R}^+;L^{p_0})$ and any space $\mathcal{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})$ with $(\tilde{q},\tilde{r},p_0)\in\Lambda$ (or with $(\tilde{q},\tilde{r},p_0)\in\Lambda^*$), we have $u\in\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)$.

Consider now the case $r\ge p_0(\alpha+1)$ (for (i)) or the case $r>p_0(\alpha+1)$ (for (ii)). Let $\tilde{r}=p_0(1+\alpha)-\epsilon$, with $\epsilon>0$ being chosen so small that

$$\frac{n}{2}\left(\frac{\alpha+1}{\tilde{r}}-\frac{1}{r}\right)<1 \tag{3.25}$$

in the case of (i) or

$$\frac{1}{2}+\frac{n}{2}\left(\frac{\alpha+1}{\tilde{r}}-\frac{1}{r}\right)<1 \tag{3.26}$$

in the case of (ii). Note that (3.25) is guaranteed by the fact that $r < np_0/(n-2p_0)$ if $n > 2p_0$ and $r < \infty$ if $n \le 2p_0$ in the case of (i), whilst (3.26) is true since $p_0 = n\alpha \ge n$ and $r < \infty$ in the case of (ii). Let

$$\frac{1}{\tilde{q}}=\frac{n}{2}(\frac{1}{p_0}-\frac{1}{\tilde{r}}).$$

Then $(\tilde{q},\tilde{r},p_0)\in\Lambda,\ \Lambda^*$ and $\tilde{q} > 1+\alpha$. A direct calculation yields that

$$\begin{aligned}\|\mathcal{G}F(u)\|_{\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)} &\le \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q}}\int_0^t |t-\tau|^{-\frac{n}{2}(\frac{\alpha+1}{\tilde{r}}-\frac{1}{r})}\|u\|_{\tilde{r}}^{\alpha+1}d\tau\\ &\le \int_0^1 |1-\tau|^{-\frac{n}{2}(\frac{\alpha+1}{\tilde{r}}-\frac{1}{r})}\tau^{-\frac{1+\alpha}{\tilde{q}}}d\tau\cdot\|u;\mathcal{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})\|^{\alpha+1}\\ &\le C\|u;\mathcal{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})\|^{\alpha+1},\end{aligned}$$

where use has been made of (3.25) in the case of (i). For (ii) we have similarly as above that

$$\begin{aligned}\|\mathcal{G}F(u)\|_{\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)} &\le \sup_{t\in\mathbb{R}^+} t^{\frac{1}{q}}\int_0^t |t-\tau|^{-\frac{1}{2}-\frac{n}{2}(\frac{\alpha+1}{\tilde{r}}-\frac{1}{r})}\|u\|_{\tilde{r}}^{\alpha+1}d\tau\\ &\le \int_0^1 |1-\tau|^{-\frac{1}{2}-\frac{n}{2}(\frac{\alpha+1}{\tilde{r}}-\frac{1}{r})}\tau^{-\frac{1+\alpha}{\tilde{q}}}d\tau\cdot\|u;\mathcal{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})\|^{\alpha+1}\\ &\le C\|u;\mathcal{C}_{\tilde{q}(\tilde{r},p_0)}(\mathbb{R}^+;L^{\tilde{r}})\|^{\alpha+1},\end{aligned}$$

where use has been made of (3.26). These estimates together with (3.9) and (3.10) imply that $u\in\mathcal{C}_{q(r,p_0)}(\mathbb{R}^+;L^r)$. The proof is thus complete.

Remark 3.1 (i) Denote by Σ_{n-1} the unit sphere in $\mathbb{R}^n$. Let $\Omega\in C^k(\Sigma_{n-1})$ with $k>0$ and define

$$f(x)=\frac{\Omega(x')}{|x|^d},\quad x'=\frac{x}{|x|},\quad 0<d<n.$$

Then Lemma 4 in [10] assures that

$$|\triangle_0(f)(x)|\le C\|\Omega\|_{C^k}(1+|x|)^{-k-d}.$$

Thus, we only need to choose k with $k\ge n$ so that $f\in\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}$.

(ii) Theorems 3.1 and 3.2 and Corollary 3.1 generalize the previously known results for the semilinear heat equation and for the Navier-Stokes equations:

$$\partial_t u + Au = \mathcal{P}\nabla(u\otimes u),\quad A=-\mathcal{P}\triangle,\quad u(0,x)=f(x)\in E_p,$$

where $\mathcal{P}$ denotes the orthogonal projection of the vector space $[L^p]^n$ into the subspace of the divergence-free vector space $E_p := \{u \in [L^p]^n \mid \mathrm{div} u = 0\}$. In particular, the profile U here belongs to the function spaces of a wider class compared with the previous results. See [20, 6] for details.

(iii) Similar results in Theorems 3.1 and 3.2 and Corollary 3.1 remain valid if $\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{2}{\alpha}}$ (or $\dot{B}_{p,\infty}^{\frac{n}{p}-\frac{1}{\alpha}}$) is replaced by the Lorentz space $L^{p_0,\infty}$ with $p_0 = n\alpha/2 > 1$ (or $p_0 = n\alpha > 1$).

(iv) Our method can be used to study self-similar solutions for other semi-linear evolution equations such as the complex Ginzburg-Landau equation:

$$\partial_t u = (a + ib)\triangle u + f(u), \quad a > 0, \quad b \in R,$$

the Burgers viscous equation:

$$\partial_t u - \partial_x^2 u = \mu \partial_x (|u|^{\alpha+1}), \quad \mu \in \mathbb{R},$$

and the more general semi-linear parabolic equation:

$$u_t - \Delta u = Q(D) f(u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \quad u(0, x) = \varphi(x), \quad x \in \mathbb{R}^n,$$

where $Q(D)$ is a homogeneous pseudo-differential operator of order $d \in [0, 2)$ and $f(u)$ is a nonlinear function which behaves like $|u|^\alpha u$ with $\alpha > 0$.

(v) Under suitable conditions, the global solution $u(t, x)$ to the semilinear evolution equation converges to a self-similar solution (see [20] for details).

Remark 3.2 In the study of the self-similar solutions of nonlinear Schrödinger and wave equations, the Schrödinger-type semigroup $S(t) = e^{i\triangle t}$ does not provide an equivalent norm for the Besov space $\dot{B}^\sigma_{p,\infty}$. This is different from the study of the self-similar solutions of parabolic equations where the heat semigroup $H(t) = e^{t\triangle}$ does provide an equivalent norm for the Besov space $\dot{B}^\sigma_{p,\infty}$ which was used in [5, 33, 20, 6]. However, to study self-similar solutions for the Schrödinger and wave equations, Cazenave and Weissler [8] introduced the new function space

$$E_{\sigma,p} := \left\{ f \in \mathcal{S}(\mathbb{R}^n) \mid \sup_t t^{\frac{1}{\sigma}} \|S(t)f\|_p < \infty, \right.$$
$$\left. \sigma = \frac{2(\alpha-1)p}{2p - n(\alpha-1)}, \quad 2 < p < 2^* = \frac{2n}{n-2} \right\}$$

(see also [10]). Recently in [30, 2, 9] the self-similar solutions have been studied for the Schrödinger equations in the space $C_*(\mathbb{R}; \dot{B}^{s_c}_{2,\infty})$.

Appendix: Proof of Proposition (2.1)

From the definition of $\hat{\psi}_j$ and $\hat{\psi}_\mu$ it is easy to see that

$$\begin{aligned}
&\operatorname{supp}(\hat{\psi}_{j-1}(\xi)) \subset \{2^{j-2} \le |\xi| \le 2^j\},\\
&\operatorname{supp}(\hat{\psi}_{j+2}(\xi)) \subset \{2^{j+1} \le |\xi| \le 2^{j+3}\},\\
&\quad \operatorname{supp}(\widehat{\triangle_\mu f}) = \operatorname{supp}(\hat{\psi}_\mu(\xi)\hat{f}) = \operatorname{supp}(\hat{\psi}_0(\frac{\xi}{\mu})\hat{f}) \subset \{2^{j-1} \le |\xi| \le 2^{j+2}\}
\end{aligned}$$

for $\mu \in (2^j, 2^{j+1})$. Thus it follows that

$$\begin{aligned}
\triangle_\mu f &= \sum_{j\in\mathbb{Z}} \triangle_j \triangle_\mu f = \sum_{k=j-1}^{j+2} \triangle_k \triangle_\mu f, \quad \mu \in (2^j, 2^{j+1}),\\
\|\triangle_\mu f\|_p &\lesssim \sum_{k=j-1}^{j+2} \|\triangle_k f\|_p, \quad \mu \in (2^j, 2^{j+1}),
\end{aligned}$$

where use has been made of the fact that $\psi_k(x) = 2^k\psi_0(2^k x)$ and $\|\psi_k\|_1 = \|\psi_0\|_1 < 1$. Further, it can be easily derived that

$$\begin{aligned}
\left(\int_0^\infty (\mu^s\|\triangle_\mu f\|_p)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}} &= \left(\sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \left(\mu^s\|\triangle_\mu f\|_p\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}\\
&\lesssim \left(\sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \left(\mu^s \sum_{k=j-1}^{j+2} \|\triangle_k f\|_p\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}\\
&\lesssim \left(\sum_{j\in\mathbb{Z}} \sum_{k=j-1}^{j+2} \int_{2^j}^{2^{j+1}} \left(\mu^s\|\triangle_k f\|_p\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}\\
&\lesssim \left(\sum_{j\in\mathbb{Z}} \sum_{k=j-1}^{j+2} \int_{2^j}^{2^{j+1}} \left(2^{(j+1)s}\|\triangle_k f\|_p\right)^q \frac{d\mu}{\mu}\right)^{\frac{1}{q}}\\
&\lesssim \left(\sum_{j\in\mathbb{Z}} \sum_{k=j-1}^{j+2} \left(2^{(j+1)s}\|\triangle_k f\|_p\right)^q\right)^{\frac{1}{q}}\\
&\lesssim \left(\sum_{j\in\mathbb{Z}} \left(2^{js}\|\triangle_j f\|_p\right)^q\right)^{\frac{1}{q}} \cong \|f\|_{\dot{B}^s_{p,q}},
\end{aligned}$$

where $\lesssim$ denotes the presence of a constant.

On the other hand, since $\hat{\psi}_0(\xi) = \hat{\psi}_0(|\xi|)$ and

$$\int_{\mathbb{R}^n} \frac{\hat{\psi}_0(\xi)}{|\xi|^n} d\xi = \int_{\Sigma^n}\int_0^\infty \frac{\hat{\psi}_0(r)}{r^n} r^{n-1} dr d\sigma < \infty,$$

where Σ^n is the unit sphere in $\mathbb{R}^n$, it then follows that

$$\int_0^\infty \frac{\hat{\psi}_0(r)}{r} dr = C < \infty.$$

Thus a simple computation gives that for $\xi \neq 0$,

$$\int_0^\infty \frac{\hat{\psi}_\mu(\xi)}{\mu} d\mu = \int_0^\infty \frac{\hat{\psi}_0(\frac{|\xi|}{\mu})}{\mu} d\mu = \int_0^\infty \frac{\hat{\psi}_0(r)}{r} dr = C < \infty,$$

so

$$\begin{aligned} \hat{f}(\xi) &= \frac{1}{C} \int_0^\infty \hat{\psi}_\mu(\xi) \hat{f}(\xi) \frac{d\mu}{\mu}, \\ \hat{\psi}_j \hat{f}(\xi) &= \frac{1}{C} \int_0^\infty \hat{\psi}_j(\xi) \hat{\psi}_\mu(\xi) \hat{f}(\xi) \frac{d\mu}{\mu}. \end{aligned}$$

The last equation implies that

$$\triangle_j(\xi) f(x) = \frac{1}{C} \int_0^\infty \triangle_j \triangle_\mu f \frac{d\mu}{\mu}.$$

Noting that $supp(\hat{\psi}_j) \subset (2^{j-1}, 2^{j+1})$, we obtain that

$$\begin{aligned} \|\triangle_j f(x)\|_p &\lesssim \int_0^\infty \|\triangle_j \triangle_\mu f\|_p \frac{d\mu}{\mu} \lesssim \int_{2^{j-2}}^{2^{j+2}} \|\triangle_j \triangle_\mu f\|_p \frac{d\mu}{\mu} \\ &\lesssim \left(\int_{2^{j-2}}^{2^{j+2}} \|\triangle_j f\|_p^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}} \left(\int_{2^{j-2}}^{2^{j+2}} \frac{d\mu}{\mu} \right)^{\frac{1}{q'}} \\ &\lesssim \left(\int_{2^{j-2}}^{2^{j+2}} \|\triangle_j f\|_p^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}} \\ \left(\sum_{j \in \mathbb{Z}} 2^{js} \|\triangle_j f\|_p^q \right)^{\frac{1}{q}} &\leq \left(\sum_{j \in \mathbb{Z}} \int_{2^{j-2}}^{2^{j+2}} 2^{jsq} \|\triangle_j f\|_p^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} \int_{2^{j-2}}^{2^{j+2}} \mu^{qs} \|\triangle_j f\|_p^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} \left(\int_{2^{j-2}}^{2^{j-1}} + \int_{2^{j-1}}^{2^j} + \int_{2^j}^{2^{j+1}} + \int_{2^{j+1}}^{2^{j+2}} \right) \mu^{qs} \|\triangle_j f\|_p^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}} \\ &\lesssim \left(\sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \mu^{qs} \|\triangle_j f\|_p^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}} \\ &\cong \left(\int_0^\infty \left(\mu^s \|\triangle_\mu f\|_p \right)^q \frac{d\mu}{\mu} \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of Proposition 2.1.

References

[1] Pecher H. Self-similar and asymptotically self-similar solutions of nonlinear wave equations. *Math. Ann.*, 2000, **316**: 259-281.

[2] Planchon F. Self-similar solutions and Besov spaces for semilinear Schrödinger and wave equations. *Proc. Journees Equations aux Derivees Partielles*, St-Jean de Monts, 1999.

[3] Planchon F. Self-similar solutions and semi-linear wave equations in Besov spaces. *J Math Pures Appl.*, 2000, **79**: 809-820.

[4] Ribaud F, Youssfi A. Global solutions and self-similar solutions of semi-linear wave equations. *Math. Z.*, 2002, **239**: 231-262.

[5] Cannone M. Ondelettes, Paraproduits et Navier–Stokes. Diderot Editeur, N.D.de Nazareth, Pairs, 1995.

[6] Planchon F. Asympotic behavior of global solutions to the Naiver-Stokes equations in $\mathbb{R}^3$. *Rev. Mat. Iberoramericana*, 1998, **14**: 71-93.

[7] Cazenave T, Weissler F B. Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations. *Math. Z.* 1998, **228**: 83-120.

[8] Cazenave T, Weissler F B. More self-similar solutions of the nonlinear Schödinger equation. *Nonlinear Differential Equations Appl.*, 1998, **5**: 355-365.

[9] Miao C, Zhang B, Zhang X. Self-similar solutions for nonlinear Schrödinger equations. *Methods Applications Analysis*, 2003, **10**: 119-136.

[10] Ribaud F, Youssfi A. Regular and self-similar solutions of nonlinear Schödinger equations. *J. Math. Pure Appl.*, 1998, **77**: 1065–1079.

[11] Fujita H. On the blowing up of solutions of a Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Faculty Sci. Univ. Tokoyo*, 1966, **13**: 109-124.

[12] Giga Y. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Differ. Equations*, 1986, **61**: 186-212.

[13] Guo B, Miao C. Space–time means and solutions to a class of nonlinear parabolic equations. *Science in China*, 1998, **41**: 682-693.

[14] Haraux AWeissler F B. Non-uniqueness for a semilinear initial value problem. *Indiana Univ. Math. J.*, 1982, **31**: 167-189.

[15] Ribaud F. Cauchy problem for semilinear parabolic equation with data in $H_p^s(\mathbb{R}^n)$ spaces. *Rev. Mat. Iberoamericana*, 1998, **14**: 1-45.

[16] Terraneo H. Non-uniqueness for a critical nonlinear heat equation. *Comm. Partial Differential Equations*, 2002, **27**: 185-218.

[17] Miao C, Zhang B. The Cauchy problem for parabolic equations in Besov spaces. *Houston J. Math.*, 2004 **30**: 829-878.

[18] Escobedo M, Kavian O. Variational problems related to self-similar solutions of the heat equation. *Nonlinear Anal., TMA*, 1987, **11**: 1103-1133.

[19] Kavian O. Remarks on the large time behaviour of a nonlinear diffusion equation. *Ann. Inst. H. Poincaré, Analyse Nonlinéaire*, 1987, **4**: 423–452.

[20] Meyer Y. Large-time behavior and self-similar solutions of some semilinear diffusion equations. In: Harmonic Analysis and Partial Differential Equations (Edited by M. Christ, C.E. Kenig and C. Sadosky), The University of Chicago Press, 1999, pp. 241–261.

[21] Escobedo M, Zuazua E. Large time behavior for convection-diffusion equation in $\mathbb{R}^n$,. *J. Differ. Equations*, 1991, **100**: 119–161.

[22] Aguirre J, Escobedo M, Zuazua E. Self-similar solutions of a convection-diffusion equation and related elliptic problems. *Comm. Partial Differential Equations*, 1990, **15**: 139-157.

[23] Peetre J. New Thoughts on Besov Spaces. Duke Univ. Math. Series, Duke University, Durham, 1976.

[24] Bergh J. Löfström J. Interpolation Spaces. New York, Springer, 1976.

[25] Miao C. Harmonic Analysis and Applications to Partial Differential Equations. Beijing, Science Press, 1999. (in Chinese)

[26] Stein E M. Singular Integrals and Differentiability Property of Functions. Princeton Univ. Press, 1970.

[27] Triebel H. Theory of Function Spaces. New York, Springer, 1983.

[28] Triebel H. Theory of Function Spaces II. Birkhäuser, Basel, 1992.

[29] Keel M, Tao T. Endpoint Strichartz estimates *Amer. J. Math.*, 1998, **120**: 955–980.

[30] Planchon F, On the Cauchy problem in Besov spaces for a nonlinear Schrödinger equation. *Commun. Comtemp. Math.*, 2000, **2**: 243-254.

[31] Miao C. Time–space estimates of solutions to general semi-linear parabolic equations. *Tokyo J. Math.*, 2001, **24**: 247–278.

[32] Miao C, Gu Y. Space–time estimates for parabolic type operator and application for nonlinear parapolic equations. *J. Partial Differential Equations*, 1998, **11**: 301–312.

[33] Cannone M, Planchon F. Self-similar solutions for Navier-Stokes equations in $\mathbb{R}^3$. *Comm. in PDE*, 1996, **21**: 179–193.