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WELL–POSEDNESS OF A FREE BOUNDARY PROBLEM IN THE LIMIT OF SLOW–DIFFUSION FAST–REACTION SYSTEMS*

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Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

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Abstract We consider a free boundary problem obtained from the asymptotic limit of a FitzHugh–Nagumo system, or more precisely, a slow–diffusion, fast–reaction equation governing a phase indicator, coupled with an ordinary differential equation governing a control variable v. In the range $(-1, 1)$, the v value controls the speed of the propagation of phase boundaries (interfaces) and in the mean time changes with dynamics depending on the phases. A new feature included in our formulation and thus made our model different from most of the contemporary ones is the nucleation phenomenon: a phase switch occurs whenever v elevates to 1 or drops to -1. For this free boundary problem, we provide a weak formulation which allows the propagation, annihilation, and nucleation of interfaces, and excludes interfaces from having (space–time) interior points. We study, in the one space dimension setting, the existence, uniqueness, and non–uniqueness of weak solutions. A few illustrating examples are also included.

Key Words Well-posedness; FitzHugh-Nagumo system; free boundary problem.
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1. Introduction

Interfacial phenomena are commonplace in physics, chemistry, biology, and in various other fields. They occur whenever a continuum is present that can exist in at least two different “states” and there is some mechanism that generates or enforces a spatial separation between these two states. The common boundaries are called *interfaces* or *free boundaries*. These interfaces are observed to manifest various geometrical patterns, such as front of shock waves [1, 2], rotating spiral waves and expanding target patterns [3, 4]. From both physical and mathematical point of view, it is very important to know the shape and motion of these boundaries.

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One of the commonly used mathematical model in studying the interfacial phenomena is the following reaction diffusion system:

$$\begin{cases} u_t = \varepsilon\Delta u + \varepsilon^{-1} f(u,v), \\ v_t = D\Delta v + g(u,v) \end{cases} \tag{1.1}$$

with typical f and g given by

$$f(u,v) = F(u) - v, \quad F(u) = u(3/\sqrt[3]{2} - 2u^2), \quad g(u,v) = u - \gamma v - b, \tag{1.2}$$

where $D \geq 0$, $\gamma > 0$ and $b \in \mathbb{R}$ are constants, and $0 < \varepsilon \ll 1$ serves as a small parameter. This system models the propagation of chemical waves in excitable or bistable or oscillatory media, where u and v represent the propagator and controller respectively [5]. It also describes an activator-inhibitor model; see Ohta, Mimura and Kobayashi [6]. When $D = O(\varepsilon)$, (1.1) was used by Tyson and Fife to study the Belousov-Zhabotinskii reagent [7]. When $D = 0$, (1.1) is the well–known FitzHugh-Nagumo model for nerve impulse propagation; see [8, 9, 10, 11, and references therein].

In this paper, we shall study a free boundary problem obtained as the singular limit, as $\varepsilon \searrow 0$, of the FitzHugh–Nagumo model. For the reader's convenience, here we provide a formal derivation of this free boundary problem; for more details, see Fife [5, Chapter 4], X.Y. Chen [12], and X. Chen [13].

The local minimum and maximum of the cubic function $F(u)$ in (1.2) is -1 and 1. If $v \in (-1, 1)$, the equation $f(u,v) = F(u) - v = 0$, for u, has three real roots, $h_-(v)$, $h_0(v)$ and $h_+(v)$, where $h_-(v) < h_0(v) < h_+(v)$; see Figure 1 (c). The roots $h_-(v)$ and $h_+(v)$ are stable equilibria of the ode

$$u_t = \varepsilon^{-1} f(u,v), \tag{1.3}$$

with attraction domains $(-\infty, h_0(v))$ and $(h_0(v), \infty)$ respectively. If $\mp v > 1$, $u = h_\pm(v)$ is the only equilibrium of (1.3) and is globally stable; i.e., its attraction domain is $\mathbb{R}$.

Consider (1.1a), regarding v as a known function. Since ε is small, (1.1a) is often referred to as a *slow-diffusion fast-reaction* equation[5, 14]. For smooth initial data $u(x,0)$, the $\varepsilon\Delta u$ term can be neglected initially, and (1.1a) can be approximated by (1.3). Hence, at each point x in the spatial domain, u approaches quickly to either $h_+(v(x,0))$ or $h_-(v(x,0))$ depending on the sign of $u(x,0) - h_0(v(x,0))$ (extending $h_0(v) = \pm\infty$ for $\pm v > 1$). Consequently, two disjoint spatial regions Ω_+ and Ω_-, where $u \approx h_+(v)$ and $u \approx h_-(v)$ respectively, are formed. The remaining region $\Omega_0 = (\Omega_+ \cup \Omega_-)^{\mathrm{c}}$, located near the set where $u(x,0) - h_0(v(x,0)) = 0$, is very thin and can be regarded as a hypersurface called *interface*. This process is commonly referred to as *the generation of interface*. A rigorous verification of this process for the one space dimensional case was first carried out by Fife and Hsiao [15]. In the special case $v \equiv 0$ (and in general space dimension), de Mottoni and Schatzman [16] established a much finer result; see also a recent general result of Soner [17]. When v is not known a priori but satisfies (1.1b) with $D > 0$, both X. Y. Chen [12] and X. Chen [13] rigorously proved, that Ω_0 (defined as $\{x \mid \min\{|u - h_+(v)|, |u - h_-(v)|\} > \varepsilon\}$) is of thickness $O(\varepsilon|\ln\varepsilon|)$ at some time t of order $O(\varepsilon|\log\varepsilon|)$.

Similarly, for each $t > 0$, the spatial domain is divided into three regions: two phase regions $\Omega_\pm(t)$ where $u \approx h_\pm(v)$, and an interfacial region $\Omega_0(t) = (\Omega_+(t) \cup \Omega_-(t))^{\mathrm{c}}$, so thin that it can be regarded as an interface. The sharp transition of u across the interface incorporated with the geometric shape of the interface constitutes the driving force for the movement of the interface, referred to as *the propagation of interface*.

As $\varepsilon \to 0$, the transition layer $\Omega_0(t)$ tends to a hypersurface $\Gamma(t)$ and v to a limit function, which we still denote by v. Formally, the singular limit (v, Γ) satisfies the following free boundary problem (with $\varepsilon = 0$):

$$\begin{cases} v_t = D\Delta v + g(h_\pm(v), v) & \text{in } \Omega_\pm(t), \\ \frac{\partial\Gamma}{\partial t} = \{W(v) - \varepsilon\kappa\}\mathbf{N} & \text{on } \Gamma(t) = (\Omega_+(t) \cup \Omega_-(t))^{\mathrm{c}}, \quad t > 0, \end{cases} \tag{1.4}$$

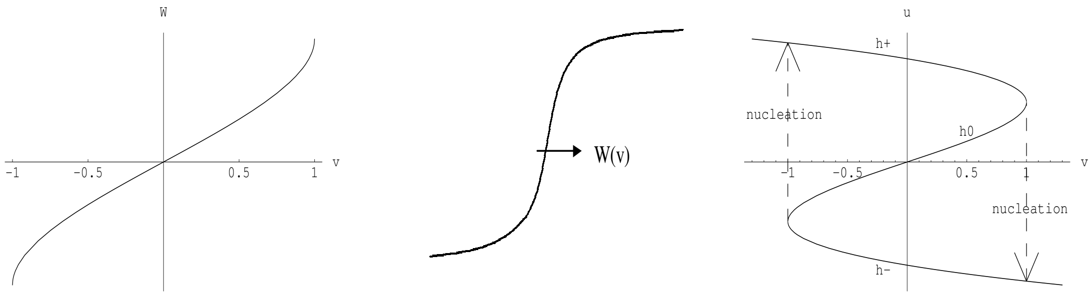
where κ is the mean curvature of the hypersurface $\Gamma(t)$ at point $x \in \Gamma(t)$, $\mathbf{N}$ is the unit normal to $\Gamma(t)$ pointing from $\Omega_-(t)$ to $\Omega_+(t)$, and $W(v)$ is the speed of the traveling wave $(W(v), U(\cdot; v))$ of

$$\begin{cases} U_{zz} + W\, U_z + f(U, v) = 0 & \forall z \in \mathbb{R}, \\ U(\pm\infty, v) = h_\pm(v),\ U(0, v) = h_0(v). \end{cases} \tag{1.5}$$

For $f(u, v)$ given in (1.1), we obtained an explicit formula

$$W(v) = \frac{3}{\sqrt[6]{2}} \cos\left(\frac{\pi}{3} + \frac{\arccos(v)}{3}\right), \quad v \in (-1, 1).$$

Figure 1 plots the function $W(v)$, a generic shape of the function $U(\cdot, v)$, and the



(a) The function $W(\cdot)$ (b) The function $U(\cdot, v)$ (c) Nullcline of $f(u, v) = 0$

Figure 1:

nullclines for $f(u, v) = 0$. The derivation from (1.1) to (1.4) can be obtained by substituting the formal expansion $u \approx U(\frac{d(x,t)}{\varepsilon}, v)$, where $d(x,t)$ is the signed distance to the interface, into (1.1a), expanding the resulting equation in ε power, and using $\kappa = \Delta d$.

Traveling wave problems of ODE type (1.5) have been extensively studied; see, for example, Aronson and Weinberger [18]. In connection with the PDE problem $u_t = u_{xx} + f(u, v)$ where $x \in \mathbb{R}$ and $v =$ constant, a classical result, given by Fife and McLeod [19], shows that the traveling wave of (1.5) are globally exponentially stable. Recently, X. Chen [20] proved a general existence, uniqueness, and global exponential stability result of traveling waves for integral–differential equations possessing a comparison principle. In particular, if (1.1b) is elliptic (i.e., drop the v_t term), then the result of [20] can be applied to the system (1.1) in some parameter ranges, by expressing v in term of u via a Green's formula. For the traveling wave problem of the original system (1.1), see Carpenter [21], Conley [22], Hastings [23, 24], Jones, Kopell and Langer [25], and references therein.

Note that solutions to (1.4) is indeed the $\varepsilon \to 0$ limit of (1.1) plus (an important part of) the $O(\varepsilon)$ order correction. If $v \equiv 0$ (discard the equation for v), this correction becomes the leading order term, and after changing the time scale $t \to \varepsilon t$, the interface motion equation becomes $\Gamma_t = \kappa$, the so called *motion by mean curvature* equation, extensively studied in [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, and the refereces therein].

When $v \equiv 0$, (1.1a) becomes, after a change of time scale, the Allen–Cahn [37] equation $u_t = \Delta u - \varepsilon^{-2} F(u)$. The connections between the Allen–Cahn equation and the motion by mean curvature equation have been rigorously established to an amazing depth; see [38, 39, 40, 41, 42, 16, 43, 44, 45, 5, 15, 46, 47, 14, 48, 17]. Among them, Evans, Soner and Souganidis [45] proved that, global in time, the union of the limit ($\varepsilon \to 0$) points of the zero level set of solutions to the Allen–Cahn equation is indeed the viscosity solution, developed in [28, 29], of the motion by mean curvature equation. Ilmanen [47], on the other hand, proved that the limit of the zero level set of a *convergent* subsequence of solutions of the Allen–Cahn equation is actually the (weak) varifold solution of the motion by mean curvature equation introduced and studied by Brakke [27]. A more descriptive convergence result, also global in time, can be found in Soner's work [17].

Rigorous verification of the connection between (1.1) and the limit problem (1.4) was carried out by X. Chen [13]. When v is a given known function and the space dimension is one, an earlier result was first given by Fife and Hsiao [15]. It is worth mentioning here that some results of Fife–Hsiao [15] do not apply to the case when initially $W(v) = 0$ on the interface. Here in this paper, we shall demonstrate that in such a case, the limit problem does not, in general, have a unique solution, and therefore, one cannot expect the convergence of solutions of (1.1) without specifying initial data more accurately than $u(x, 0) \approx h^{\pm}(v(x, 0))$ for $x \in \Omega^{\pm}(0)$. On the other hand, when $\varepsilon = 0$ and the space dimension is one, extremely delicate results are given by [39, 40, 41, 49, 50, 46].

Equation (1.1a) can be coupled with other equations to obtain models such as the phase field, the Cahn-Hilliard, and viscous Cahn–Hilliard. For the relevant asymptotic behavior as $\varepsilon \to 0$, see [51, 52, 53, and references therein].

When $D > 0$, the free-boundary problem (1.4) has been quite well studied. X.Y. Chen [12] established, for $\varepsilon > 0$, the *local* existence and uniqueness of a smooth solution with smooth initial value. X. Chen [13] extended X.Y. Chen's result to the case $\varepsilon = 0$, which is significantly different from the case $\varepsilon > 0$. Xin [54] studied a particular solution, related to a spiral wave, of (1.4) with $\varepsilon = 1$, $W(v) = v$, and $g(h_\pm, v) = \pm 1$. Also, for the case $\varepsilon = 0$, the space dimension is one and the interface is only a single point, Hilhorst, Nishiura, and Mimura [55] constructed a unique solution of (1.4). Remarkably, Giga, Goto, and Ishii [56] established the global existence of a weak solution to (1.4) for both the case $\varepsilon > 0$ and $\varepsilon = 0$.

In this paper, we investigate certain *qualitative behavior* of global in time solutions of (1.4). For the purpose of demonstrating certain special features of solutions of (1.4), we consider only the one space dimension case, and assume that $D = 0$, which corresponds to the FitzHugh–Nagumo system. More precisely, we consider the following problem:

$$\begin{cases} v_t &= G^\pm(v) \quad \text{in } \Omega_\pm(t), \\ \frac{\partial\Gamma}{\partial t} &= W(v) \quad \text{on } \Gamma(t) = \partial\Omega_\pm(t), t > 0, \end{cases} \tag{\textbf{P}}$$

where $G^\pm(v) = g(h_\pm(v), v)$ for $\pm v \le 1$.

The solutions of Hilhorst, Nishiura, and Mimura [55], X. Chen [13] and X. Y. Chen [12] are all classical solutions where interfaces are smooth and do not experience any topological changes. In general interfaces may collide and annihilate each other and therefore (global in time) classical solutions may not exist. Giga, Goto and Ishii [56] introduced viscosity (weak) solutions to (1.4) by defining the interface Γ as the zero level set of the viscosity solution ϕ to $\phi_t = W(v)\,|\nabla\phi| + \varepsilon|\nabla\phi|\mathrm{div}(\frac{\nabla\phi}{|\nabla\phi|})$ ($\varepsilon \ge 0$). This formulation takes care of topological changes such as the annihilation of interfaces. However, there is another phenomenon, the *nucleation*, needs to be considered.

A careful analysis of the original system (1.1) shows that, if $v(x,t) > 1$, then the phase state at x will immediately switch to the "$-$" phase (regardless of its neighbors' phase states). Similarly, if $v(x,t) < -1$, the phase state at x will switch to the "$+$" phase. We refer to this phenomenon as *nucleation* (see Figure 1)(c). This phenomenon was ignored in most of the past works. Giga, Goto and Ishii [56], as well as Hilhort, Nishiura and Mimura [55], assumed that $W(v)$ and $g(h_\pm(v), v)$ are well–defined and continuous for all v in $\mathbb{R}$, whereas X.Y. Chen [12] and X. Chen [13] considered the solution only for a small time interval where $v \in (-1, 1)$. In this paper, we will take into account the nucleation phenomenon.

In the next section we will provide a weak formulation for problem (**P**), taking into account of both annihilation and nucleation of interfaces. This weak formulation, based on the original dynamics of (1.1), was first introduced by X. Chen in [57] and is different from that of Giga, Goto and Ishii [56]. To help the reader to have an idea of what a general solution may look like, we provide several examples in Section 3, including a non–uniqueness example. In the rest of the paper, we prove our main result roughly stated as follows:

> *If the initial speeds are not zero on all initial interfacial points, then problem* (**P**) *admits a unique, global in time, weak solution.*

If the initial speed at an interfacial point *is* zero, there are, in general, countably many solutions; see our example given in Section 3.4. The non–uniqueness of (**P**) is not due to our deficiency in the definition of weak solutions, but due to the nature of the problem. That is to say, for each weak solution of (**P**), we expect that there exists a sequence of solutions of (1.1) converging to it as $\varepsilon \to 0$.

2. A Weak Formulation of (P) and the Main Result

In the sequel, we denote by $B(x, r)$ an open ball centered at x with radius r, and by $\bar{B}(x, r)$ a closed ball. If $r \le 0$, then $B(x, r) = \emptyset$. Also $M := \sup_{v\in(-1,1)} |W(v)|$.

We use the following weak formulation introduced in [57].

Definition 1 *Let $\mathcal{D}$ be a closed domain in $\mathbb{R} \times [0, \infty)$. We say that (v, Q^+, Q^-) is a (weak) solution to* (**P**) *in $\mathcal{D}$ if $v \in C^0(\mathcal{D})$, Q^+ and Q^- are disjoint and (relatively) open in $\mathcal{D}$, and the followings hold:*

(1) **(Dynamics)** $v_t \in L^\infty(\mathcal{D})$ *and* $v_t = G^\pm(v)$ *in* $Q^\pm$*;*

(2) **(Nucleation)** $\{(x, t) \in \mathcal{D} \mid \pm v > 1\} \subset Q^\mp$*;*

(3) **(Propagation)** *If* $B(x_0, r_0) \times \{t_0\} \subset Q^\pm$ *and* $\pm v < 1$ *in* $\bar{B}(x_0, r_0 + M\delta) \times [t_0, t_0 + \delta] \subset \mathcal{D}$ *for some* $\delta > 0$*, then* $B(x_0, r_0 + c^\pm\delta) \times \{t_0 + \delta\} \subset Q^\pm$*, where* $c^\pm = \min\{\mp W(v(x, t)) \,|\, x \in \bar{B}(x_0, r_0 + M\delta), t \in [t_0, t_0 + \delta]\}$*;*

(4) **(No Fattening)** $\mathrm{m}(\Gamma) = 0$*, where* $\Gamma = \mathcal{D}\setminus(Q^+ \cup Q^-)$ *and* m *denotes the Lebesgue measure in* $\mathbb{R}^2$*.*

Remark 2.1 The **nucleation** criterion implies $\pm v \le 1$ in $Q^\pm$. Suppose that $G^\pm(\pm1) \ne 0$. Then since $Q^\pm$ is open, we obtain from the **dynamics** criterion that $\pm v < 1$ in $Q^\pm \setminus \partial\mathcal{D}$ and that any point $(x, t) \in \mathcal{D} \setminus \partial\mathcal{D}$ where $v = \pm1$ cannot be an interior point of $\overline{Q^\pm}$. Thus, the **no fattening** criterion implies that $\{\, (x, t) \in \mathcal{D} \setminus \partial\mathcal{D} \mid v(x, t) = \pm1 \,\} \subset \overline{Q^\mp}$.

On the other hand, if one of $G^\pm(\pm1)$, say $G^+(1)$ vanishes, then interior points in $\{(x, t) \mid v(x, t) = 1\}$ can have choices of being in Q^+ or Q^-, thereby creating non–uniqueness. To avoid this situation, in the sequel we shall always assume that $G^\pm(\pm1) \ne 0$. Also, we shall work only on "compatible" initial conditions; namely,

$\pm v(\cdot,0) < 1$ in $\partial\mathcal{D} \cap Q^{\pm}$. The generation of interface indicates that initial conditions to (**P**) should always be compatible.

In the sequel, we need only the **dynamics**, **propagation**, and the following criteria (to replace the **nucleation** and **no fattening** criteria):

$$\{(x,t) \in \mathcal{D} \setminus \partial\mathcal{D} \mid \pm v(x,t) \geq 1\} \subset \overline{Q^{\mp}}.$$

Remark 2.2 To understand better the **propagation** criterion, we first note that if $(x_0,t_0) \in Q^{\pm}$, then $\pm v(x_0,t_0) < 1$ and consequently, $\pm v < 1$ in some neighborhood of (x_0,t_0). Hence, letting δ approach zero we see that $Q^{\pm}$ shrinks/expands with a velocity at most/least $W(v)$. The (necessary) introduction of $M, \delta, c^{\pm}$, etc. enables us to let (x_0,t_0) approach the boundary of $Q^{\pm}$ and thus to conclude that the boundary of $Q^{\pm}$ will shrink/expand with a speed no more/less than $W(v)$. In particular, if Q^+ and Q^- share a common boundary, then it moves with a speed $W(v)$, in the direction from the "$-$" phase region to "$+$" phase region. Thus, in the case of classical solutions, this condition is compatible with the equation $\Gamma_t = W(v)$. We remark that, due to the **nucleation** criterion and the assumption that $G^{\pm}(\pm 1) \neq 0$, the value of $W(v)$ for $|v| > 1$ and the value $G^{\pm}(v)$ for $\pm v \geq 1$ are not needed. Nevertheless, for $c^{\pm}$ to have a clear meaning, in the sequel, we assume that $W(v)$ has been extended for all $v \in \mathbb{R}$.

Remark 2.3 The **no fattening** criterion $\mathrm{m}(v) = 0$ is not required in the weak formulation of Giga, Goto, and Ishii [56]. When $\mathrm{m}(\Gamma) > 0$, the dynamics for v on Γ in the definition of [56] is $v_t \in [G^-(v), G^+(v)]$ on Γ. We suspect that this provides room for the existence of non–physical weak solutions of (**P**), i.e., weak solutions of (**P**) which is not a limit of any convergent subsequence of solutions of (1.1).

Typically, non–uniqueness occurs when solutions in the sense of [56] has fattened interface, i.e., $\mathrm{m}(\Gamma) > 0$. We believe that $\mathrm{m}(\Gamma) = 0$ for any $\varepsilon \to 0$ limit of a *convergent sub*sequence of solutions of (1.1). To get a one–to–one correspondence between limits of convergent subsequence of solutions of (1.1) and weak solutions of (**P**), more conditions may be needed, but for the moment, we would rather believe that (1)–(4) are sufficient. We shall provide more discussions on the uniqueness issue in Section 3.4 where a non–uniqueness example is given.

Throughout this paper, we always assume the followings:

(**A1**) $W \in C^1((-1,1))$, $W(0) = 0$, $W'(v) > 0$ *for all* $v \in (-1,1)$, *and*

$$M := \sup\{\, |W(v)|\, |v \in (-1,1)\} < \infty;$$

(**A2**) $G^+ \in C^0((-\infty,1])$, $G^- \in C^0([-1,\infty))$, $G^{\pm}(\pm 1) \neq 0$, and $\pm G^{\pm}(v) > 0$ if $\pm W(v) \leq 0$.

The condition that $\pm G^{\pm}(v) > 0$ if $\pm W(v) \leq 0$ (i.e., if $\pm v \leq 0$) is crucial in our subsequent analysis. It implies that any interface will propagate without changing direction, until it annihilates with another approaching interface or meets a nucleation point.

In the sequel, we say that a (not necessarily bounded) function $T(\cdot)$ on $\mathbb{R}$ is *Lipschitz* if there exists a constant $L > 0$ such that $|T(x_1)-T(x_2)| \le L\,|x_1-x_2|$ for any $x_1, x_2 \in \mathbb{R}$; we write

$$|T'(x)| := \limsup_{y\to x} |\frac{T(y)-T(x)}{y-x}|.$$

Also, $\{(x,t)|\, x \in \mathbb{R},\ t \ge T(x)\}$ is abbreviated as $\{t \ge T\}$. Our main result is as follows.

Theorem 1 (Existence and Uniqueness of Initial Value Problem) *Let $\Omega_\pm \subset \mathbb{R}$ and $v_0(x) : \mathbb{R} \to \mathbb{R}$ be given. Assume that Ω_+ and Ω_- are disjoint and open, that $\partial\Omega_+ = \partial\Omega_- =: \Gamma_0$ has finitely many points, and that $\Omega_+ \cup \Omega_- \cup \Gamma_0 = \mathbb{R}$. Also assume that $v_0(x)$ is bounded and Lipschitz continuous in $\mathbb{R}$ and that*

$$\pm v_0 < 1 \quad in\ \Omega^\pm, \qquad\qquad W(v_0) \ne 0 \quad on\ \Gamma_0. \tag{2.1}$$

Then problem $(\mathbf{P})$ *has a unique weak solution (v, Q^+, Q^-) in $\mathbb{R} \times [0,\infty)$ satisfying $v(x,0) = v_0(x)$ on $\mathbb{R}$ and $\{x \mid (x,0) \in Q^\pm\} = \Omega_\pm$.*

In order to prove Theorem 1, we consider a more general problem, the Cauchy problem, where the initial value of v and the location of the phase regions are specified on a curve in the space-time domain.

Definition 2 *Let $T : \mathbb{R} \to [0,\infty)$ and $\psi : \mathbb{R} \to \mathbb{R}$ be functions and Ω_+, Ω_- be sets in $\mathbb{R}$. We say that (v, Q^+, Q^-) has Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ if $v(x, T(x)) = \psi(x) \quad \forall x \in \mathbb{R}$ and $\{x \mid (x, T(x)) \in Q^\pm\} = \Omega_\pm$.*

To ensure the existence of a unique solution for the Cauchy problem, we provide, for the Cauchy data, a sufficient condition, which we call property **S**, defined as follows:

Definition 3 *A quadruple $(T, \psi, \Omega_+, \Omega_-)$ is said to have property* **S** *(solvable) and write $(T, \psi, \Omega_+, \Omega_-) \in$* **S** *if the followings hold:*

(**S**1) *Ω_+, $\Omega_- \subset \mathbb{R}$ are open and disjoint, $\partial\Omega_+ = \partial\Omega_- =: \Gamma_0$ consists of a finite number of points, and $\Omega_+ \cup \Omega_- \cup \Gamma_0 = \mathbb{R}$;*

(**S**2) *$\psi : \mathbb{R} \to \mathbb{R}$ is bounded, Lipschitz continuous, and $\pm\psi < 1$ in $\Omega_\pm$;*

(**S**3) *The function $T : \mathbb{R} \to [0,\infty)$ is Lipschitz continuous and satisfies $\pm W(\psi)\ |T'| < 1$ on $\bar{\Omega}_\pm$;*

(**S**4) *$W(\psi) \ne 0$ on Γ_0.*

Theorem 2 *Let $(T, \psi, \Omega_+, \Omega_-) \in$* **S**. *Then* $(\mathbf{P})$ *has a unique solution on $\{t \ge T\}$ with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$* .

Note that Theorem 1 is just a special case of Theorem 2 with $T \equiv 0$. Nevertheless, our introduction of the Cauchy problem is mainly for the proof of Theorem 1.

Remark 2.4 1. The condition (**S**1) (except the finiteness of Γ_0) is necessary to ensure the uniqueness of a solution. Here for simplicity, we assume that Γ_0 consists of finitely many points. We expect that this is general enough in real applications, and in the special case when Γ_0 does consist of infinitely many points, a unique solution can be obtained by taking the limit of the unique solution with Γ_0 finite.

2. As mentioned earlier, condition (**S**2) is only a compatibility condition for the existence of a solution.

3. Condition (**S**3) is simply a non–characteristic condition on the curve where Cauchy data is given for the pde $\Gamma_t = W(v)$ (regarding Γ as the zero level set of ϕ which solves $\phi_t = |\nabla\phi| W(v)$).

4. Finally condition (**S**4) is one of the keys in our uniqueness proof. Indeed, as can been from a non-uniqueness example given in Section 3.4, if (**S**4) does not hold, there exist, in general, infinitely many solutions.

The rest of the paper is organized as follows. In Section 3, we give several examples to illustrate the generic behavior of solutions to (**P**). Also, we give a non-uniqueness example demonstrating the necessity of (**S**4) for the uniqueness. Sections 4–6 are dedicated to the proof of Theorem 2. In Section 4, we study the dynamics of the interface, i.e., the shrinkage/expansion of the "+/−" phase region and the nucleation. Then we provide a local existence and uniqueness result in Section 5. Finally, we prove Theorem 2 in Section 6.

3. Examples of Solutions

There are three distinguished cases according to the combination of the signs of $G^{\pm}(\pm 1)$ [5, Chapter 4].

(1) $G^+(1) < 0$ and $G^-(-1) > 0$; see Figure (a).

This is referred to as a *Bistable* case, since there exists an equilibrium in each of the "±" phase. Also $G^{\pm}(1) < 0 < G^{\pm}(-1)$ and the equation $v_t = G^{\pm}(v)$ imply that v cannot reach ± 1, so that nucleation will not occur.

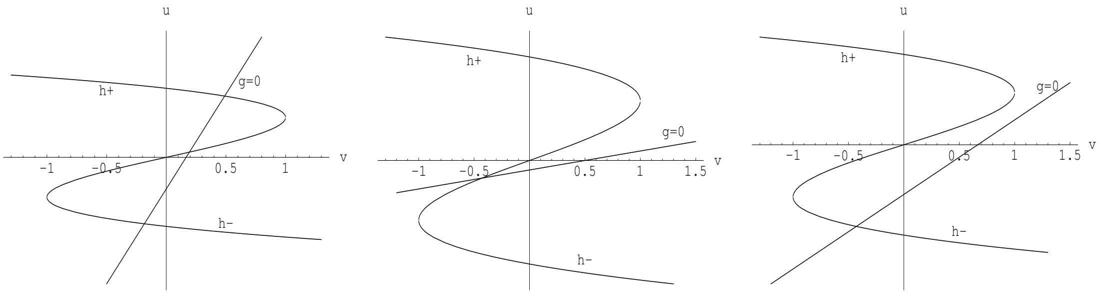
(2) $G^+ > 0$ in $(-\infty, 1]$ and $G^- < 0$ in $[-1, \infty)$; see Figure (b).

This case is called *Oscillatory* since the phase at any point switches between "+" and "−" phases infinitely many times.

(3) Neither (1) nor (2); see Figure (c).

We call this case *Excitable* since nucleation can occur, and at any fixed point x, the phase can change only finitely many times and the value of v eventually rests at one of the zeros of $G^{\pm}$.

We remark that the case considered by Hilhort, Nishiura and Mimura [55] is indeed a Bistable case.



(a)Bistable case (b)Oscillatory Case (c)Excitable Case

Figure 2: $G^{\pm}(v) = h^{\pm}(v) - \gamma v + b$

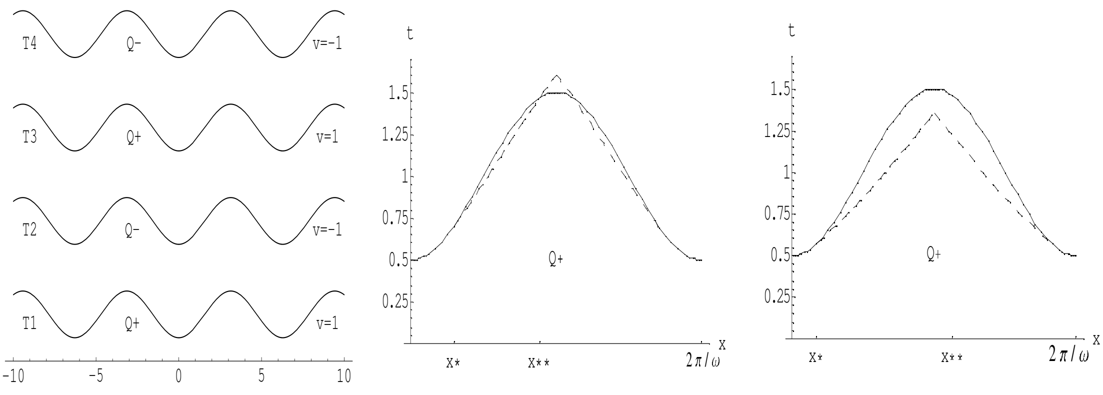
As mentioned before, the dynamics of the interface includes propagation, annihilation, and nucleation. To give the reader an idea of what interfaces may look like, we provide three examples of unique solutions, one for each of the Oscillatory, Bistable and Excitable cases. We also provide an example having countably many solutions.

For convenience, we use $\Phi^{\pm}(\alpha, t)$ to denote the solution of the following ode

$$\left\{\begin{array}{lcl} \Phi_t^{\pm} &=& G^{\pm}(\Phi^{\pm}), \\ \Phi^{\pm}|_{t=0} &=& \alpha \end{array}\right. \quad \Longleftrightarrow \quad t = \int_{\alpha}^{\Phi^{\pm}(\alpha,t)} \frac{ds}{G^{\pm}(s)}. \tag{3.1}$$

3.1 The Oscillatory Case

For simplicity, we assume $W(v) = v$, $G^{+} \equiv 1$, and $G^{-} \equiv -1$. Then $\Phi^{\pm}(\alpha, t) = \alpha \pm t$. We consider the initial value $v(x, 0) = \frac{1}{2}\cos(\omega x)$, $\Omega_{+} = \mathbb{R}$ and $\Omega_{-} = \emptyset$, where ω is a parameter.



(a) $\omega = 1$ (b) $\omega = 2.5$ (c) $\omega = 4$

Figure 3: Interfaces for the oscillatory case examples

Figure 3(a) shows the interface and the phase regions Q^+ and Q^- of the unique solution to (**P**) for $\omega = 1$, whereas Figure 3(b) and 3(c) show, for $\omega = 2.5$ and 4 respectively, the first layers of the interface represented by the minimum of the curves in the Figure. We remark that solutions are periodic in space with period $2\pi/\omega$.

When $\omega = 1$, the solution is also periodic in time, and is given by

$$v(x,t) = (-1)^j(1 - T_{j+1}(x) + t), \qquad \forall\, x \in \mathbb{R},\ t \in [T_j(x), T_{j+1}(x)],\ j = 0, 1, , \cdots,$$
$$Q^+ = \{(x,t)\,|x \in \mathbb{R},\, T_{2k}(x) < t < T_{2k+1}(x), k \geq 0\} \cup \mathbb{R} \times \{0\},$$
$$Q^- = \{(x,t)\,|\, x \in \mathbb{R},\, T_{2k+1}(x) < t < T_{2k+2}(x), k \geq 0\},$$

where $T_0 \equiv 0$ and $T_j(x) = 2j - 1 - \frac{1}{2}\cos x$ for all integer $j \geq 1$.

Notice that initially the system is uniformly in "+" phase state. At each $x \in \mathbb{R}$, the phase switches between the "+" phase and the "−" phase at time $t = T_j(x)$, $j = 1, 2, \cdots$; all of these phase changes are due to the nucleation. In this particular example, the effect of propagation of interface is totally suppressed by nucleation. Indeed, the speed of propagation of interface is $|W(v)|\,|_\Gamma = 1$, whereas the "speed" due to the nucleation is $|\frac{dx}{dt}| = |\frac{dx}{dT_j(x)}| = |\frac{2}{\sin(x)}|$.

If $\omega > 2$, then both nucleation and propagation play roles in the evolution of the interface. Consider a half period interval $[0, \pi/\omega]$. Let $x^* = \frac{1}{\omega}\arcsin(2/\omega)$. Then at each $x \in [0, x^*]$, the phase switches due to nucleation from "+" to "−" at time $T = 1 - v_0(x)$ at which $v = 1$. At each $x \in (x^*, \pi/\omega]$, the phase can change either by nucleation which occurs at time $1 - v_0(x)$, or by the propagation of interface from neighboring points, depending on which occurs earlier. Indeed, solving equation, for $t = \hat{T}(z)$,

$$\begin{cases} \frac{dz}{d\hat{T}(z)} = \hat{T} + v_0(z) = \hat{T} + \frac{1}{2}\cos(\omega z), \quad z > x^* \\ \hat{T}(x^*) = 1 - v_0(x^*), \end{cases}$$

we see that $\hat{T}(x) < 1 - v_0(x)$ for $x \in (x^*, x^{**})$ where $x^{**} > x^*$ is the point $\hat{T}(x^{**}) = 1 - v(x^{**})$. Hence, the first layer of interface (in $x \in [0, \pi/\omega]$) is given by $t = 1 - v_0(x)$ for $x \in [0, x^*]$, $t = \hat{T}(x)$ for $x \in [x^*, \min\{\pi/\omega, x^{**}\}]$ and $t = 1 - v_0(x)$ for $x \in (\min\{\pi/\omega, x^{**}\}, \pi/\omega]$ (if it is not empty). See Figure 3 (b) and (c).

For other layers of the interface, the idea is similar, but the computation is much more involved. We omit them here.

3.2 The Bistable Case

We assume that $W(v) = v$, $G^+(v) = \frac{1}{2} - v$, and $G^-(v) = -(\frac{1}{2} + v)$. Solving (3.1) gives $\Phi^\pm(\alpha, t) = \pm\frac{1}{2}(1 - e^{-t}) + \alpha e^{-t}$.

We consider initial value given by $\Omega_+ = (1,2) \cup (3,4) \cup (5,\infty)$, $\Omega_- = \mathbf{R}\backslash\bar{\Omega}_+$ and $v(x,0) = -\frac{1}{2}$ for $x \leq 4$, and $= \frac{1}{2}$ for $x > 5$, $= -\frac{1}{2} + (x-4)$ for $x \in (4,5]$.

The regions Q^+, Q^- and the interface of the solution are showed in Figure 4, and is obtained as follows.

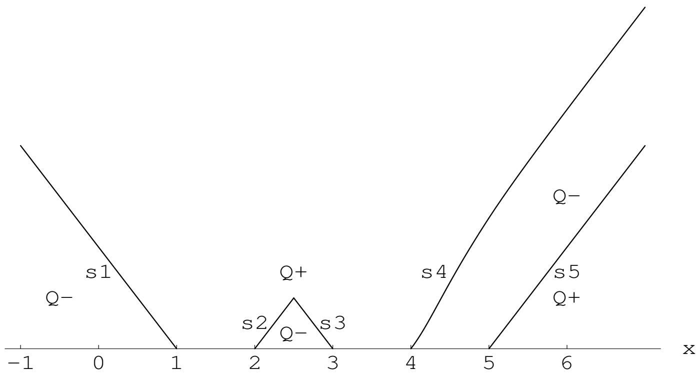


Figure 4: Interfaces for the bistable case example

Below and on $x = s_1(t)$, $v(x,t) = \Phi^-(v_0(x),t) = -\frac{1}{2}$. Hence solving $s_1' = W(v(s_1,t)) = -\frac{1}{2}$ gives $s_1(t) = -\frac{t}{2}+1$ for all $t \geq 0$. Similarly, $v(x,t) = \Phi^+(v_0(x),t) = \frac{1}{2}$ for $x \geq s_5(t) = 5 + \frac{1}{2}t$, $t \geq 0$.

Below and on s_2 and s_3, $v(x,t) = \Phi^-(v_0(x),t) = -\frac{1}{2}$, so that $s_2(t) = 2 + \frac{1}{2}t$ and $s_3(t) = 3 - \frac{1}{2}t$ for $0 \leq t \leq 1$. At $t = 1$, $s_2 = s_3 = \frac{5}{2}$ and the two interfaces annihilate.

Below $x = s_4(t)$ and above $x = s_5(t)$, $v(x,t) = \Phi^-(v_0(x),t)$ for $x \in (4,5)$ and $v(x,t) = \Phi^-(v(x,T_5(x)), t-T_5(x))$ for $x > 5$ where $t = T_5(x) = 2(x-5)$ is the inverse of $x = s_5(t) = 5+\frac{1}{2}t$. Hence, the inverse $t = T_4(x)$ of $x = s_4(t)$ solves $\frac{dx}{dT_4(x)} = -\Phi^-(v_0, T_4)$ for $x \in [4,5]$ and $\frac{dx}{dT_4(x)} = -\Phi^-(\Phi^+(v_0,T_5),T_4 - T_5) = \frac{1}{2} - e^{-T_4+2(x-5)}$ for $x > 5$. This equation has a unique monotonic solution $T_4(x)$ for all $x \geq 4$ and it satisfies $T_4(x) > T_5(x)$ for all $x > 5$.

Finally, the region above the curves $x = s_1, s_2, s_3$, and s_4 belongs Q^+ and v can be obtained by solving $v_t = G^+(v)$ together with known "initial" values on $x = s_1, s_2, s_3, s_4$.

It is easy to verify that such obtained (v, Q^+, Q^-) is a solution to the given initial value problem, and is the only solution by Theorem 1.

We remark that as $t \to \infty$, $s_4'(t) \to 1/2$ and $s_5(t) - s_4(t) = \frac{1}{2}\ln t + \frac{1}{2}\ln 2 + o(1)$ as $t \to \infty$. (If $G^-(v) = a - v$ with $a \in (-1,-1/2)$, then $s_5 - s_4$ approaches a constant and $x = s_5$ and $x = s_4$ approach the "front" and "back" of a pulse traveling wave.)

In this example, there are a few features shared in the general case:

(i) There is no nucleation;

(ii) If two neighboring interfaces move in opposite directions approaching each other at time $t = 0$ (such as $x = s_2(t)$ and $x = s_3(t)$ in the example), they will keep moving without changing their directions until they annihilate each other at a finite time;

(iii) If two neighboring interfaces move in the same direction initially (such as $x = s_4(t)$ and $x = s_5(t)$ in the example), they will keep moving in the same direction and they will never intersect.

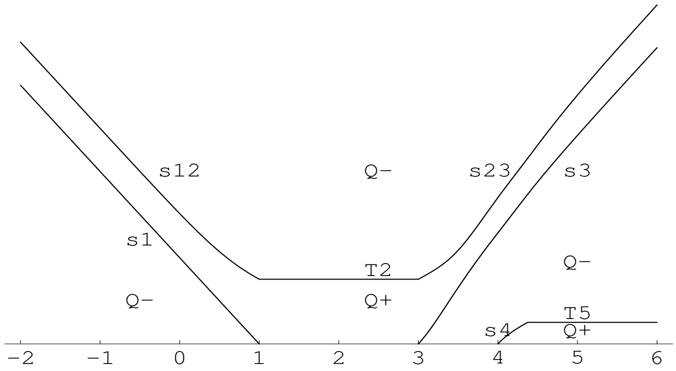


Figure 5: Interfaces for the excitable case example

These properties will be some of the keys in our subsequent analysis.

3.3 The Excitable Case

We take $W(v) = v$, $G^+ \equiv 1$, and $G^-(v) = -(\frac{1}{2} + v)$. Then $\Phi^+(\alpha, t) = \alpha + t$ and $\phi^-(\alpha, t) = -\frac{1}{2} + (\alpha + \frac{1}{2})e^{-t}$. We consider an initial data given by $\Omega_- = (-\infty, 1) \cup (3, 4)$, $\Omega_+ = \mathbf{R}\backslash\bar{\Omega}_-$, and $v(x, 0) = -\frac{1}{2}$ for $x \leq 3$, $= \frac{1}{2}$ for $x > 4$, and $= -\frac{1}{2} + (x - 3)$ for $x \in (3, 4)$.

Figure 5 shows the regions Q^+, Q^- and the interface of the solution to this initial value problem.

Below and on $x = s_1$, $v(x, t) = -\frac{1}{2}$. Consequently, $s_1(t) = 1 - \frac{1}{2}t$.

The interface $x = s_4(t) = 4 + \frac{1}{2}t + \frac{1}{2}t^2$ for $t \in [0, \frac{1}{2}]$ is due to the propagation, and the interface $t = T_5(x) \equiv \frac{1}{2}$ for $x > 4\frac{3}{8}$, on which $v = 1$, is due to the nucleation.

Below $x = s_3$ (and above $x = s_4, t = T_5$), v can be calculated by $v_t = G^-(v)$ and $s_3' = -W(v(s_3, t))$. One can show that $s_3' > 0$ for all $t \geq 0$ and $s_3'(t) \to \frac{1}{2}$ as $t \to \infty$.

For $x \in [1, 3]$, $v = \Phi^+(v_0, t) = -\frac{1}{2} + t$ for all $t < \frac{3}{2}$ and nucleation occurs at $t = T_2(x) \equiv \frac{3}{2}$.

For $s_1 < x < 1$ the interface at $\{x = 1, t = \frac{3}{2}\}$ will propagate, while nucleation may take a role. Calculation under the assumption of nucleation and propagation respectively tell us that only propagation takes a role. Hence below $x = s_{12}$ and above $x = s_1$, $v = \Phi^+(v(x, T_1(x)), t - T_1(x)) = -\frac{1}{2} + t - T_1(x)$ where $T_1(x) = 2(1 - x)$ is the inverse of $x = s_1(t)$. Solving $s_{12}' = -W(v(s_{12}, t))$ with initial value $s_{12}(\frac{3}{2}) = 1$ then gives $s_{12}(t) = \frac{3}{2} - \frac{1}{2}t + \frac{1}{4}e^{3-2t}$ for all $t \geq \frac{3}{2}$. Now we can check that on $x = s_{12}$, $v = \frac{1}{2}(1 + e^{3-2t}) < 1$ for all $t > \frac{3}{2}$, and hence the interface $x = s_{12}$ is indeed due to propagation. Similarly we can calculate s_{23}.

We remark that in a general situation, the calculation of s_{12}, T_2, and s_{23} is much more involved, and should be proceeded as follows:

(i) Pretend that $v_t = G^+(v)$ for the rest of the domain and find a curve $t = T^*(x)$ on which $v = 1$. Nucleation occurs only at points on the curve $t = T^*(x)$.

(ii) At every point $(y, T^*(y))$, calculate an interface $t = h(y, T^*(y); \cdot)$ based solely on propagation.

(iii) Take the infimum of $h(y, T^*(y); \cdot)$ for all y. This infimum is then the required interface.

In the example, the infimum is attained by $h(1, \frac{3}{2}; x)$ for all $x < 1$ which gives $x = s_{12}$, by $h(3, \frac{3}{2}; x)$ for $x \geq 3$ which gives s_{23}, and by $h(x, \frac{3}{2}; x) = \frac{3}{2}$ for all $x \in [1, 3]$.

Finally, we remark that, due to the excitable nature of the system, at every x, its phase will eventually rest on the "$-$" state. Also, as $t \to \infty$, $x = s_1, s_{12}$ approach a pulse traveling wave, and so do $x = s_{23}, s_3$.

3.4 A Non–uniqueness Example

We consider a bistable case where $W(v) = v$ and $G^{\pm}(v) = \pm\frac{1}{2} - v$. Then $\Phi^{\pm}(\alpha, t) = \pm\frac{1}{2}(1 - e^{-t}) + \alpha e^{-t}$. We consider the initial value $\Omega_+ = (0, \infty)$, $\Omega_- = (-\infty, 0)$, and $v(x, 0) = v_0(x) \equiv 0$. Note that $W(v(x, 0)) = 0$ on $\Gamma_0 = \{0\}$ so that Theorem 1 cannot be applied.

This initial value problem has infinitely many solutions. The phase regions of some of the solutions are shown in Figure 6 (a)–(d). We remind the reader that once we know the phase regions, then v is uniquely determined by its continuity and the equation $v_t = G^{\pm}(v)$ in $Q^{\pm}$.

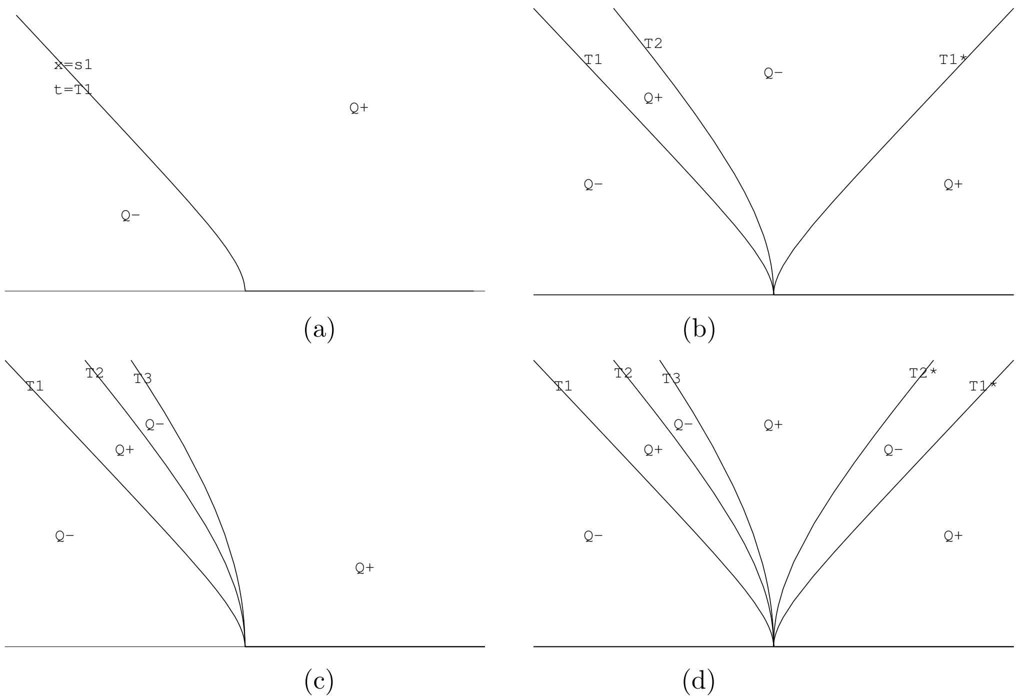


Figure 6: Some solutions to the non–uniqueness example

In Figure 6(a), the interface is given by $x = s_1(t) := -\frac{1}{2}(t + e^{-t} - 1)$, $Q^{\pm} = \{\pm(x - s_1) > 0\}$, $v = \Phi^-(v_0(x), t) = \frac{1}{2}(e^{-t} - 1)$ for $x < s_1$, $= \Phi^+(v_0, t) = \frac{1}{2}(1 - e^{-t})$ for

$x \geq 0$, and $= \Phi^+(v(x, T_1(x)), t - T_1(x)) = \frac{1}{2}(e^{-t} + 1 - 2e^{T_1(x)-t})$ for $s_1 < x < 0$, where $t = T_1(x)$ is the inverse function of $x = s_1(t)$. It is easy to verify that $s_1'(t) = W(v(s_1, t))$ and that (v, Q^+, Q^-) is a solution.

This solution can be obtained as the limit of unique solutions to a sequence of initial value problems of (**P**). Indeed, for any small positive ϵ, let $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$ be solution to (**P**) with initial data $\Omega^- = (-\infty, 0)$, $\Omega_+ = (0, \infty)$ and $v_\epsilon(x, 0) = -\epsilon$. By Theorem 1, $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^+)$ exists and is unique, and $Q^\pm$ is given by $Q_\epsilon^\pm = \{ \pm(x - s^\epsilon(t)) > 0 \}$ where $s^\epsilon(t) = s_1(t) + \epsilon(e^{-t} - 1)$. Hence, as $\epsilon \searrow 0$, $(v^\epsilon, Q_\epsilon^+, Q_\epsilon^-) \to (v, Q^+, Q^-)$.

The solution whose phase regions shown in Figure 6 (b) has three interfaces given by $t = T_1(x), T_2(x)$, and $T_{1*}(x)$, where $T_{1*}(x) = T_1(-x)$ for $x \geq 0$, and $T_2(x)$ solves

$$\frac{dx}{dT_2(x)} = -\frac{1}{2}(e^{-T_2} + 1 - 2e^{T_1 - T_2}) \text{ and } T_2(x) > T_1(x) \quad \forall x < 0, \quad \lim_{x \nearrow 0} T_2(x) = 0. \quad (3.2)$$

By considering T_1 as the independent variable and writing

$$\frac{dT_2}{dT_1} = \frac{dT_2}{dx}\frac{dx}{dT_1} = \frac{1 - e^{-T_1}}{1 + e^{-T_2} - 2e^{T_1 - T_2}},$$

we can show that (3.2) has a unique solution T_2 for all $x < 0$; we omit the details.

This solution, again, can be obtained as a limit of unique solutions of a sequence of initial value problems. Consider, for every small positive ϵ, the initial value $(v_0^\epsilon, \Omega_+^\epsilon, \Omega_-^\epsilon)$ given by $\Omega_-^\epsilon = (-\infty, -\epsilon) \cup (0, \epsilon)$, $\Omega_+^\epsilon = \mathbb{R} \setminus \bar{\Omega}_-^\epsilon$, and $v_0^\epsilon(x) = -\epsilon$ for $x < -\epsilon$, $= \epsilon + 2x$ for $x \in (-\epsilon, 0]$, and $= \epsilon$ for $x > 0$. Since $W(v_0^\epsilon) \neq 0$ on $\Gamma_0^\epsilon := \{-\epsilon, 0, \epsilon\}$, by Theorem 1, this initial value problem has a unique solution $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$. Simple calculation shows that this solution has three interfaces, given by $t = T_1^\epsilon(x), T_2^\epsilon(x)$ and $T_1^{\epsilon *}$, where $t = T_1^\epsilon(x)$ is the inverse of $x = s^\epsilon(t) := -\frac{1}{2}(t + e^{-t} - 1) + \epsilon(e^{-t} - 1)$, $T_1^{\epsilon *}(x) = T_1^\epsilon(-x)$ and $T_2^\epsilon(x)$ solves a differential equation analogous to (3.2) for $x \leq -\epsilon$ whereas for $x \in (-\epsilon, 0]$, T_2 is monotonically decreasing and $T_2(-\epsilon) = O(\sqrt{\epsilon})$. Sending $\epsilon \searrow 0$, we can show that $(v_\epsilon, Q_\epsilon^+, Q_\epsilon^-)$ approaches the solution shown in Figure 6 (b).

In a similar manner, we can obtain solutions with arbitrary odd number of interfaces. All these solutions are classical for $t > 0$. Figure 6 (c) (d) provides the phase regions of two of such examples.

Remark 3.1 Following the same idea as shown above, one can construct infinitely many solutions for a general initial data which does not satisfy (**S**4). We omit the details.

Remark 3.2 If we use the definition of Giga, Goto, and Ishii [56] (i.e., the interface Γ is the zero level set of the viscosity solution ϕ to $\phi_t = |\nabla\phi| W(v)$), then one can show that the interface is *uniquely* given by the "fat" set $\Gamma = \{(x, t) \mid |x| \leq \frac{1}{2}(t + e^{-t} - 1), t \geq 0\}$, the function v is uniquely given by $v = \Phi^\pm(0, t)$ in $Q^\pm := \{(x, t) \mid \pm x > \frac{1}{2}(t + e^{-t} - 1), t \geq 0\}$, whereas the value of v in the interior of Γ can be arbitrary as long as $v_t \in [G^-(v), G^+(v)]$. Thus, the weak solution of [56] is unique if it refers to $(v|_{Q^+ \cup Q^-}, Q^+, Q^-)$, and is not unique if it refers to (v, Q^+, Q^-).

All the weak solutions in our definition as well as in [56] are uniquely determined in the *"deterministic"* (by initial data) region$\{(x,t) \mid |x| > \frac{1}{2}(t + e^{-t} - 1)\}$. The difference between the weak solutions of [56] and ours is in the *"indeterministic"* region $U = \{(x,t) \mid |x| < \frac{1}{2}(t + e^{-t} - 1)\}$. The solutions of [56] are totally "indeterministic", i.e. arbitrary in U, whereas that of ours have certain restrictions.

As an important feature, our solutions are indeterministic *only at* $t = 0$; namely, any of our weak solutions is stable under small perturbations at any time $t = t_0 > 0$.

Remark 3.3 We believe that every weak solution in our definition is "physical" in the sense that it can be obtained as a limit of a sequence of solutions of (1.1) as $\varepsilon \to 0$. For example, consider the solution (v, Q^+, Q^-) depicted in 6(b), and also the solution $(v_\epsilon, Q^+_\epsilon, Q^-_\epsilon)$ we mentioned. Since for every fixed $\epsilon > 0$, $(v_\epsilon, Q^+_\epsilon, Q^-_\epsilon)$ is unique, one can shown, by the analysis in [42, 13, 57], that there exists a sequence $\{(u^\varepsilon_\epsilon(x,0), v^\varepsilon_\epsilon(x,0))\}_{\varepsilon>0}$ of initial values to (1.1) such that, as $\varepsilon \to 0$, the solutions $(u^\varepsilon_\epsilon, v^\varepsilon_\epsilon)$ to (1.1) with these initial values have the limit $(v_\epsilon, Q^+_\epsilon, Q^-_\epsilon)$ (namely, $v^\varepsilon_\epsilon \to v_\epsilon$ in $\mathbb{R} \times [0, \infty)$ and $u^\varepsilon_\epsilon \to h^\pm(v_\epsilon)$ in $Q^\pm_\epsilon$). Upon selecting a subsequence from the double indexes (ε, ϵ), we then can conclude that (v, Q^+, Q^-) in Figure 6(b) can be obtained as a limit of the solutions of (1.1) as $\varepsilon \to 0$.

Our definition excludes a solution defined by $Q^\pm = \{(x,t) \mid \pm x > 0\}$ and $v = \pm\frac{1}{2}(1 - e^{-t})$ in $Q^\pm$, since in our definition v is required to be continuous. On the other hand, this "solution" is not stable for small perturbations at any time $t = t_0 \geq 0$. We expect that a relaxation of the continuity requirement on v in our definition would lead to a one-to-one correspondence between limits of solutions of (1.1) (as $\varepsilon \to 0$) and weak solutions of (**P**).

4. Dynamics of Interfaces

In this section, we study the evolution of the interface according to the motion equation $\Gamma_t = W(v)$ and the nucleation mechanics. We investigate the shrinkage of the "+" phase region and the expansion of the "−" phase region. (The opposite phase change is analogous.) In particular, for any two neighboring interfacial points which initially approach each other, we shall find a unique curve, $t = H(x)$, such that it is a component of the interface connecting these two initial interfacial points. Hence, $H(x)$ is precisely the first time of phase change at x. Any local maximum of H can be regarded as the time of annihilation of two neighboring approaching interfaces, and any local minimum of H can be understood as the time of nucleation. If one uses the default that there are interfacial points at $\pm\infty$ that move inward, then the annihilation (at time $H(\pm\infty) = \infty$) of a finite interfacial point and an infinite one can be understood as the approaching of an interface towards infinity. This analysis will allow us to characterize all components of the interface and to construct, layer by layer in the space–time domain, unique solutions to (**P**) with Cauchy data in **S**.

4.1 Shrinkage of the "+" phase region

We denote by $\Phi^{\pm}(\alpha,t)$ the solution to (3.1). For convenience, we extend $G^{\pm}(v)$ by zero for $\pm v\ge 2$ and by a linear interpolation for $\pm v\in(1,2)$. Also, we extend $W(v)$ by the constant $W(\pm1)$ for all $\pm v>1$. Since the values of $G^{+}(v)$ for $v>1$, $G^{-}(v)$ for $v<-1$, and $W(v)$ for $|v|\ge 1$ are not used for any solution to (**P**), these extensions will not affect our final result.

Consider (**P**) with Cauchy data $(T,\psi,\Omega_+,\Omega_-)$ in the domain $\{t\ge T(x)\}:=\{(x,t)\mid x\in\mathbb{R},t\ge T(x)\}$. Let $(a,b)\subset\Omega_+$ be an interval such that $a,b\notin\Omega_+$, $W(v(a,T(a)))>0$, and $W(v(b,T(b)))>0$, so that "initially" (i.e., $t=T$) the "+" phase region is shrinking.

Propagation and annihilation of interfaces. Let's assume, for the moment, that there is no nucleation. Then interfaces started at $(a,T(a))$ and $(b,T(b))$ can be written as $x=s^{\mathrm{R}}(t)$ and $x=s^{\mathrm{L}}(t)$ respectively, where

$$\frac{ds^{\mathrm{R}}}{dt}=W(v(s^{\mathrm{R}},t)),\qquad -\frac{ds^{\mathrm{L}}}{dt}=W(v(s^{\mathrm{L}},t)).\tag{4.1}$$

The curve $t=T(\cdot)$, on which the Cauchy data is given, is "characteristic" to equations in (4.1) at points where $|\frac{dx}{dT}|=W(\psi)$. For this reason, we impose the "non–characteristic" condition $\pm W(\psi)|T'|<1$ on $\bar{\Omega}_{\pm}$.

Suppose we know a priori that s^{R} and s^{L} are monotonic. Then the region below $x=s^{\mathrm{R}}$ and $x=s^{\mathrm{L}}$ is in Q^{+} (since nucleation is ignored). Hence, solving $v_t=G^{+}(v)$ in this region gives $v(x,t)=\Phi^{+}(\psi(x),t-T(x))$. Consequently, (4.1) can be solved uniquely in terms of (T,ψ,a,b). As a part of a guess–and–check process, we shall show below in Lemma 4.1 that such uniquely obtained functions s^{R} and s^{L} are indeed strictly monotonic. For this we need the condition that $\pm G^{\pm}(v)>0$ for $\mp v\ge 0$. In such a manner, we obtain a whole component of the interface being the union of the curve $x=s^{\mathrm{R}}(t)$ for $t\in[T(a),t^*]$ and the curve $x=s^{\mathrm{L}}(t)$ for $t\in[T(b),t^*]$, where t^* is the time such that $s^{\mathrm{R}}(t^*)=s^{\mathrm{L}}(t^*)$, i.e., the time of annihilation of the two interfaces starting from $(a,T(a))$ and $(b,T(b))$ respectively.

Note that the union of the two curves $x=s^{\mathrm{R}}(t)$ and $x=s^{\mathrm{L}}(t)$ for $t\le t^*$ is a graph in x. Hence, it is convenient to use the inverse function of $x=s^{\mathrm{R,L}}$. We denote by $t=h(y,\mu;x)$ the inverse of $x=s^{\mathrm{R}}(y,\mu;t)$ for $x\ge y$ and $x=s^{\mathrm{L}}(y,\mu;t)$ for $x\le y$, where $s^{\mathrm{R,L}}(y,\mu;t)$ are solutions to (4.1) with initial data $s^{\mathrm{R,L}}(y,\mu;\mu)=y$. Then $h(y,\mu;\cdot)$ solves

$$\mathrm{sgn}(x-y)\frac{dx}{dh(y,\mu;x)}=W(\Phi^{+}(\psi(x),h-T(x)))\quad\text{for }\ x\in\mathbb{R}\setminus\{y\},\quad h(y,\mu;y)=\mu,\tag{4.2}$$

where $\mathrm{sgn}(z)=1$ if $z>0$ and $\mathrm{sgn}(z)=-1$ if $z<0$. The whole component of the interface mentioned earlier then can be written as $t=H(x)$ for $x\in(a,b)$, where

$H(x) = \min\{h(a, T(a); x),\ h(b, T(b); x)\}$. The lens shape region $\{T(x) \le t < H(x)\}$ is one component of Q^+.

Nucleation of phase regions. Next we take into account the nucleation. Let $y \in (a, b)$ be an arbitrary fixed point. If the phase at y is not affected by the expansion of neighboring "−" phase regions, then, due to the nucleation mechanics, it will change from the "+" phase to the "−" phase at time $T^*(y)$ at which $v = 1$. Once the phase at y is changed, the new "−" phase region $\{y\}$ will expand to change the phase of its neighboring points. Hence, at any point $x \in (a, b)$, the phase will be changed at a time no later than $h(y, T^*(y); x)$, or more precisely, no later than $H(T, \psi, a, b; x)$ defined by

$$H(T,\psi,a,b;x) := \begin{cases} \inf\{\ h(y, T^*(y); x) \mid y \in [a,b] \cap \mathbb{R}\ \} & \text{if } \ x \in (a,b), \\ T(x) & \text{if } \ x \in \mathbb{R} \setminus (a,b), \end{cases} \tag{4.3}$$

$$T^*(y) := \begin{cases} \sup\{t \ge T(y) \mid \Phi^+(\psi(y), \tau - T(y)) < 1\ \forall \tau \in [T(y), t)\} & \text{if } \ y \in (a,b), \\ T(x) & \text{if } \ x \in \mathbb{R} \setminus (a,b). \end{cases} \tag{4.4}$$

Here we have used the obvious notation $[a, b] \cap \mathbb{R}$ to include cases where $a = -\infty$ and/or $b = \infty$. We also use the extension $h(y, T^*(y); \cdot) \equiv \infty$ if $T^*(y) = \infty$. We shall prove in the next section that $t = H(T, \psi, a, b; x)$ is precisely the first time of phase change from "+" to "−" at point $x \in (a, b)$.

Here we establish the well–definedness and a few properties of $H(T, \psi, a, b; x)$. When there is no ambiguity, we write $H(T, \psi, a, b; x)$ as $H(x)$.

Lemma 4.1 *Let $\psi \in L^\infty(\mathbb{R} \to \mathbb{R})$ and $T : \mathbb{R} \to [0, \infty)$ be Lipschitz, and $(a, b) \subseteqq \mathbb{R}$ be an interval such that*

$$\psi < 1 \quad \text{in } (a,b), \qquad W(\psi)\,|T'| < 1 \ \text{ on } [a,b] \cap \mathbb{R}. \tag{4.5}$$

(1) *For any $y \in [a, b] \cap \mathbb{R}$ and $\mu \in [T(y), \infty)$ satisfying $W(\Phi^+(\psi(y), \mu - T(y))) > 0$, problem* (4.2) *admits a unique solution $h(y, \mu; x)$ for all $x \in [a, b] \cap \mathbb{R}$, and the solution satisfies*

$$T < h < \infty, \qquad \frac{\operatorname{sgn}(x-y)}{h'} = W(\Phi^+(\psi, h - T)) > 0 \quad \text{on} \ \ ([a,b] \setminus \{y\}) \cap \mathbb{R}. \tag{4.6}$$

(2) *Assume in addition to* (4.5) *that*

$$W(\psi(a)) > 0 \text{ if } \ a \in \mathbb{R}, \quad W(\psi(b)) > 0 \text{ if } \ b \in \mathbb{R}. \tag{4.7}$$

Define T^* as in (4.4) *and H as in* (4.3). *Then either $\{(a, b) = \mathbb{R}, T^* \equiv \infty, H \equiv \infty\}$ or $H < \infty$ on $\mathbb{R}$ and the followings hold:*

- **(a)** *For each $x \in [a, b] \cap \mathbb{R}$, there exists $y^x \in [a, b] \cap \mathbb{R}$ such that $H(\cdot) = h(y^x, T^*(y^x); \cdot)$ on the closed interval with end points x and y^x;*

(b) *$H > T$ on (a,b), $W(\Phi^+(\psi, H-T)) > 0$ on $[a,b]\cap\mathbb{R}$, and $\Phi^+(\psi, t-T) < 1$ on $\{(x,t) \mid x \in (a,b), T(x) \le t < H(x)\}$;*

(c) *For any $x_1 \in (a,b)$, there exists $\delta_0 = \delta_0(x_1) > 0$ such that for all $\delta \in (0,\delta_0)$,*

$$H(x_2) \ge H(x_1) - \delta \quad \forall\, x_2 \in B(x_1, c(\delta)\delta),$$

where $c(\delta) = \min_{\bar{B}(x_1, M\delta)\times[H(x_1)-\delta, H(x_1)]}\{W(\Phi^+(\psi, t-T))\} > 0$;

(d) *H is Lipschitz continuous on $\mathbb{R}$.*

Remark 4.1 An analogue of Lemma 4.1 for the phase change from "$-$" to "$+$" can be obtained by changing $\psi < 1$, $\Phi^+ < 1$, and $W(\cdot)$ to $\psi > -1$, $\Phi^- > -1$, and $-W(\cdot)$ respectively, since the nucleation from "$\pm$" phase to "$\mp$" phase occurs at $v = \pm 1$ and the shrinking velocity of the "$\pm$" phase is $\pm W(v)$. We omit the full statement.

Proof of Lemma 4.1 (1). First we prove assertion (1). We recall that $W(v) > 0$ if and only if $v > 0$. Also, $G^+ > 0$ on $(-\infty, 0]$. Define

$$T_0(x) := T(x) + \int_{\min\{0,\psi(x)\}}^{0} \frac{1}{G^+(z)}dz. \tag{4.8}$$

Then $T_0 = T$ if $\psi > 0$; $\Phi^+(\psi, T_0 - T) = 0$ if $\psi \le 0$; T_0 is Lipschitz; and $\Phi^+(\psi, t-T) > 0$ if $t > T_0$.

For simplicity, we write $h(y,\mu;x)$ as $h(x)$. Due to symmetry, we need only consider the case $x \in (y,b]\cap\mathbb{R}$.

Since $h'(y+) = 1/W(\Phi^+(\psi(y), \mu - T(y))) > 0$, (4.2) admits a unique solution near the right-hand side of $x = y$ and the solution can be uniquely extended as long as $h' > 0$, or sufficiently, as long as $h > T_0$. We denote by (y,c) the *maximal* interval in $[y,b)$ where h exists and $h > T_0$.

First we prove $c > y$ by showing that either $h(y) > T_0(y)$ or $h(y) = T_0(y)$ and $h_0'(y+) > |T_0'(y)|$.

When $\mu > T(y)$ and $\psi(y) > 0$, $h(y) = \mu > T(y) = T_0(y)$.

When $\mu > T(y)$ and $\psi(y) \le 0$, $\Phi^+(\psi(y), T_0(y) - T(y)) = 0 < \Phi^+(\psi(y), \mu - T(y))$, so that $T_0(y) < \mu = h(y)$.

When $\mu = T(y)$, $W(\psi(y)) = W(\Phi^+(\psi(y), \mu - T(y))) > 0$ so that $\psi(y) > 0$, $T_0(y) = T(y) = h(y)$, and $|T_0'(y)| = |T'(y)| < 1/W(\psi(y)) = h'(y+)$. In conclusion, $c > y$.

Now we show that $c = b$. Suppose for the contrary that $c < b$. Since $h' > 0$ on $[y,c)$, the limit $h(c-) := \lim_{x\nearrow c} h(x)$ exists. We claim that $h(c-) < \infty$. Indeed, if $h(c-) = \infty$, then $\liminf_{x\nearrow c}\Phi^+(\psi(x), h(x) - T(x)) > 0$ so that $\Phi^+(\psi, h-T)$ is uniformly positive and $h' = 1/W(\Phi^+(\psi, h-T))$ is uniformly bounded in $[y,c)$, which implies that $h(c-) < \infty$, a contradiction. Hence we must have $h(c-) < \infty$.

As (y, c) is the maximal interval where $h > T_0$ and the solution can always be extended as long as $h > T_0$, we must have $h(c-) = T_0(c)$. Consequently,

$$0 \le h'(c-) := \liminf_{x\nearrow c} h'(x) \le \liminf_{x\nearrow c} \frac{h(c-)-h(x)}{c-x}$$
$$\le \liminf_{x\nearrow c} \frac{T_0(c)-T_0(x)}{c-x} \le |T_0'(c)| < \infty.$$

Hence, from the equation $h' = 1/W(\Phi^+(\psi, h-T))$, we see that $\Phi^+(\psi(c), h(c-)-T(c)) = \psi(c) > 0$ and $h'(c-) = 1/W(\psi(c))$. It then follows that $T_0 = T$ in a neighborhood of c and $1/W(\psi(c)) = h'(c-) \le |T_0'(c)| = |T'(c)|$, contradicting to (4.5). This contradiction shows that $c = b$. If $b < \infty$, we can use a similar argument to show that $h(b-) > T(b)$ and $h'(b-) > 0$ exist and are finite, and the solution satisfies $h > T_0$ in $(y, b]$. Since $T_0 \ge T$ and since $W(\Phi^+(\psi, h-T)) = 1/h' > 0$ if $h > T_0$, the first assertion of the Lemma thus follows.

(2). Next we prove the second assertion. First we consider the case $(a, b) = \mathbb{R}$.

If $T^* \equiv \infty$, then $H \equiv \infty$ and there is nothing to prove. Hence, we assume that $T^*(y_0) < \infty$ for some $y_0 \in \mathbb{R}$. Consequently, $T(\cdot) \le H(\cdot) \le h(y_0, T^*(y_0); \cdot) < \infty$ since $h(y, T^*(y); \cdot) \ge T(\cdot)$ for all $y \in \mathbb{R}$.

(a). Let $x \in \mathbb{R}$ be any fixed point. Let $\{y_j\}_{j=1}^{\infty}$ be a sequence such that $h(y_j, T^*(y_j); x) \to H(x)$ as $j \to \infty$. Since $\operatorname{sgn}(x-y)\, h'(y, T^*(y); \cdot) = 1/W(\Phi^+) \ge 1/M$, $h(y, T^*(y); x) \ge T^*(y) + |y-x|/M \ge |y-x|/M$ for all $y \in \mathbb{R}$. It then follows that $\{y_j\}$ is uniformly bounded. Consequently, by taking a subsequence if necessary, we can assume that the whole sequence $\{y_j\}$ converges to a finite point y^x as $j \to \infty$. Hence, $T^*(y^x) = \lim_{j\to\infty} T^*(y_j) < \infty$ and $h(y_j, T^*(y_j); \cdot) \to h(y_x, T^*(y^x); \cdot)$ uniformly on any compact subset of $\mathbb{R}$. In particular, $H(x) = h(y^x, T^*(y^x); x)$.

Now we claim that $H(\cdot) = h(y^x, T^*(y^x); \cdot)$ on the closed interval bounded by x and y^x. Without loss of generality, we assume that $y^x \le x$. If the claim is not true, then there exists $z \in [y^x, x)$ such that $H(z) < h(y^x, T^*(y^x); z)$. Let y^z be the point such that $H(z) = h(y^z, T^*(y^z); z)$.

If $y^z \le z$, then since both $h(y^z, T^*(y^z); \cdot)$ and $h(y^x, T^*(y^x); \cdot)$ solve the same differential equation on $[z, x]$ and the "initial value" at z for $h(y^x, T^*(y^x); \cdot)$ is strictly bigger than that for $h(y^z, T^*(y^z); \cdot)$, we conclude by an ode comparison principle that $h(y^x, T^*(y^x); \cdot) > h(y^z, T^*(y^z); \cdot)$ on $[z, x]$; but this contradicts to the definition of $H(x)$ and y^x.

If $y^z \in (z, x]$, then $h(y^z, T^*(y^z); y^z) < h(y^z, T^*(y^z); z) < h(y^x, T^*(y^x); z) < h(y^x, T^*(y^x); y^z)$. Similar to the previous case, a comparison between $h^*(y^x, T^*(y^x); \cdot)$ and $h(y^z, T^*(y^z); \cdot)$ on the interval $[y^z, x]$ also gives a contradiction.

Finally, if $y^z > x$, then $h(y^z, T^*(y^z); x) < h(y^z, T^*(y^z); z) < h(y^x, T^*(y^x); z) < h(y^x, T^*(y^x); x)$, which still gives a contradiction. In conclusion, $H = h(y^x, T^*(y^x); \cdot)$ on the closed interval bounded by x and y^x. This proves the assertion (a).

(b). The assertion (b) follows from the assertion (a), (4.6), and $H(x) \le h(x, T^*(x); x) = T^*(x)$.

(c). Let $\delta_0 > 0$ be small enough such that $W(\Phi^+) > 0$ on $\bar{B}(x_1, M\delta_0) \times [H(x_1) - \delta_0, H(x_1)]$. Let $\delta \in (0, \delta_0]$ be any number, and $x_2 \in B(x_1, c(\delta)\delta)$ where $c(\delta) = \min_{\bar{B}(x_1,M\delta)\times[H(x_1)-\delta,H(x_1)]}\{W(\Phi^+(\psi, t - T))\}$. Without loss of generality we assume that $x_2 < x_1$. Let $y = y^{x_2}$ be the point such that $H(\cdot) = h(y, T^*(y); \cdot)$ in the interval bounded by y and x_2. Similar to the proof of the assertion (a), by considering separately the three cases $y \le x_2$, $y \in (x_2, x_1]$, and $y > x_1$, we can compare the function $H(x_2) + (x - x_2)/c(\delta)$ and the function $h(y, T^*(y); x)$ for $x \in [x_2, x_1]$ to conclude that $h(y, T^*(y); x) \le H(x_2) + (x - x_2)/c(\delta)$ for all $x \in [x_2, x_1]$. In particular, taking $x = x_1$ gives the assertion (c).

(d). The assertion (c) implies the local Lipschitz continuity of H; nevertheless, we need to show the uniform Lipschitz continuity of H.

Since $G^+ > 0$ on $(-\infty, 0]$, there is $m_0 \in (0, 1]$ such that $G^+ > 0$ on $(-\infty, m_0]$. We define

$$T_m(x) = T(x) + \int_{\min\{m,\psi(x)\}}^{m} \frac{ds}{G^+(s)} \qquad \forall\, x \in \mathbb{R}, m \in [0, m_0].$$

As T and ψ are Lipschitz and ψ is uniformly bounded, $\sup_{m\in[0,m_0]} \|T'_m\|_{L^\infty(\mathbb{R})} < \infty$. Let $m^* \in (0, m_0)$ be any fixed constant such that

$$0 < W(m^*) < 1/\|T'_{m^*}\|_{L^\infty(\mathbb{R})}.$$

Since $W(\cdot)$ is monotonic and $W(0) = 0$, such m^* exists. For any $y \in \mathbb{R}$, we claim that

$$h(y, T^*(y); x) \ge T_{m^*}(x) \qquad \forall x \in \mathbb{R}. \tag{4.9}$$

When $x = y$, the inequality holds and is strict since either $T^*(y) = \infty$ or $\Phi^+(\psi(y), T^*(y) - T(y)) = 1$ whereas $\Phi^+(\psi(y), T_{m^*}(y) - T(y)) = \max\{\psi(y), m^*\}$. If (4.9) does not hold for all x, there exists $\hat{x} \in \mathbb{R} \setminus \{y\}$ such that $h(y, T^*(y); \hat{x}) = T_{m^*}(\hat{x})$ and $h(y, T^*(y); x) > T_{m^*}(\hat{x})$ for all x between y and $\hat{x}$. Consequently, $|h'(y, T^*(y); \hat{x})| \le |T'_{m^*}(\hat{x})| \le \|T'_{m^*}\|_{L^\infty(\mathbb{R})}$. In addition, $\psi(\hat{x}) < m^*$ since $T_{m^*}(\hat{x}) = h(y, T^*(y); \hat{x}) > T(\hat{x})$. Thus,

$$\|T'_{m^*}\|_{L^\infty(\mathbb{R})} \ge |h'(y, T^*(y); \hat{x})| = 1/W(\Phi^+(\psi(\hat{x}), T_{m^*}(\hat{x}) - T(\hat{x}))) = 1/W(m^*),$$

which contradicts to the definition of m^*. Hence (4.9) holds for all y.

From (4.9), we obtain $\Phi^+(\psi(x), h(y, T^*(y); x) - T(x)) \ge \Phi^+(\psi(x), T_{m^*}(x) - T(x)) = \max\{\psi(x), m^*\} \ge m^*$, so that $|h'(y, T^*(y); x)| = 1/W(\Phi^+(\psi, h - T)) \le 1/W(m^*)$ for all $x, y \in \mathbb{R}$. Thus, $\|H'\|_{L^\infty(\mathbb{R})} \le \sup_y \|h'(y, T^*(y); \cdot)\| \le 1/W(m^*)$. This proves the assertion (d) and also completes the proof of assertion (2) of the lemma when $(a, b) = \mathbb{R}$.

Next we consider the case when at least one of a or b is finite. We define $\tilde{T}(x) = T(x)$ and $\tilde{\psi}(x) = \psi(x)$ for $x \in (a, b)$, $\tilde{T}(x) = T(a)$ and $\tilde{\psi}(x) = \psi(a)$ for $x \in (-\infty, a]$, and $\tilde{T}(x) = T(b)$ and $\tilde{\psi}(x) = \psi(b)$ for $x \in [b, \infty)$. With such defined $\tilde{T}, \tilde{\psi}$ and $\tilde{a} := -\infty$

and $\tilde{b} := \infty$, we denote the corresponding functions in (4.4) and (4.2) by $\tilde{T}^*(y)$ and $\tilde{h}(y, \tilde{T}^*(y); x)$. Simple comparison gives, for all $x \in [a, b] \cap \mathbb{R}$,

$$\begin{aligned}
&\tilde{h}(y, \tilde{T}^*(y); x) = h(y, T^*(y); x) \quad \text{if } \ y \in (a, b), \\
&\tilde{h}(y, \tilde{T}^*(y); x) > h(a, T(a); x) \quad \text{if } \ y \leq a \text{ and } a > -\infty, \\
&\tilde{h}(y, \tilde{T}^*(y); x) > h(b, T(b); x) \quad \text{if } \ y \geq b \text{ and } b < \infty.
\end{aligned}$$

We extend $h(a, T(a); \cdot) \equiv \infty$ if $a = \infty$ and $h(b, T(b); \cdot) \equiv \infty$ if $b = \infty$. Then, for all $x \in [a, b] \cap \mathbb{R}$,

$$H(T, \psi, a, b; x) = \min\{h(a, T(a); x),\ h(b, T(b); x), H(\tilde{T}, \tilde{\psi}, -\infty, \infty; x)\}.$$

The rest of the proof then follows similarly to that for the case $(a, b) = \mathbb{R}$. Here we need the condition (4.7) to establish the existence and (uniform) Lipschitz continuity of $h(a, T(a); \cdot)$ and $h(b, T(b); \cdot)$ in case a and/or b is finite.

This completes the proof of the lemma.

4.2 Expansion of the "−" phase region

For any point $(x_0, t_0) \in Q^-$, there are two driving forces that may change the phase at x_0. The first is an external force coming from the neighboring points on the "+" phase, but it will not be large enough to change the phase at x_0 if v at x_0 is positive. The other is an internal force due to the nucleation, yet it will not change the phase at x_0 if $v > -1$. Thus, as long as $v > 0$ at x_0, the "−" phase at x_0 will not change. Consequently, $v(x_0, t) = \Phi^-(v(x_0, t_0), t - t_0)$ is valid at least up to the time v becomes zero. Based on this idea, we prove the following lemma concerning the expansion of the "−" phase region.

Lemma 4.2 *Let (v, Q^+, Q^-) be a solution to $(\mathbf{P})$ and $(x_0, t_0) \in \overline{Q^-}$ be a point such that $W(v(x_0, t_0)) > 0$. Let $[A, B]$ be a finite interval such that $x_0 \in (A, B)$ and the equation, for $h(\cdot)$,*

$$\operatorname{sgn}(x - x_0) \frac{d\, x}{d\, h(x)} = W(v(x, h(x))) > 0 \quad \forall\, x \in [A, B] \setminus \{x_0\}, \quad h(x_0) = t_0 \qquad (4.10)$$

has a solution on $[A, B]$. Then for all $x \in [A, B]$ and $t \in (h(x), h(x) + \int_{v(x,h(x))}^{0} \frac{ds}{G^-(s)})$,

$$(x, t) \in Q^- \quad \text{and} \quad v(x, t) = \Phi^-(v(x, h(x)), t - h(x))\,.$$

Proof Recall that $G^- < 0$ on $[0, \infty)$. The integral $\int_{v(x,h(x))}^{0} \frac{ds}{G^-(s)}$ in the lemma is well–defined and positive since $v(x, h(x)) > 0$.

Let $(x_j, t_j)_{j=1}^{\infty}$ be a sequence in Q^- such that $(x_j, t_j) \to (x_0, t_0)$ as $j \to \infty$. Let $h^j(x)$ be the solution to

$$\operatorname{sgn}(x - x_j) \frac{dx}{dh^j(x)} = W(v(x, h^j)) - \tfrac{1}{j} > 0 \quad \forall\, x \in [A, B] \setminus \{x_j\}, \qquad h^j(x_j) = t_j.$$

Since $W(v(x,t))$ is Lipschitz in t, for all large enough j, h^j exists and is unique. In addition, as $j \to \infty$, $h^j \to h$ uniformly on $[A,B]$. Since $W(v(x,h(x))) > 0$ for all $x \in [A,B]$, there is $J \gg 1$ such that $h^j(x) \le J$ and $v(x,h^j(x)) \ge 2/J$ for all $j \ge J$ and all $x \in [A,B]$. For each $j \ge J$, let

$$T_j^0(x) = \sup\{t \le j \mid v(x,\tau) \ge \tfrac{1}{j} \text{ for all } \tau \in [h^j(x),t]\}$$

and $\mathcal{D}_j = \{(x,t) \mid x \in [A,B], t \in [h^j(x), T_j^0(x)]\}$. We claim that $\mathcal{D}_j \subset Q^-$ for all $j \ge J$.

Suppose for the contrary that $\mathcal{D}_j \setminus Q^- \ne \emptyset$ for some $j \ge J$. Then since $\mathcal{D}_j \setminus Q^-$ is compact, there is $(x^*,t^*) \in \mathcal{D}_j \setminus Q^-$ such that $t^* = \min\{t \mid (x,t) \in \mathcal{D}_j \setminus Q^- \text{ for some } x\}$, i.e., $\mathcal{D}_j \cap \{t < t^*\} \subset Q^-$.

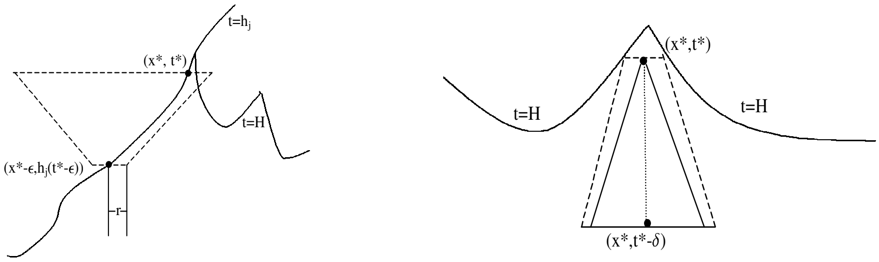
First, we show that $t^* = h^j(x^*)$. Suppose $t^* \ne h^j(x^*)$. Then $t^* > h^j(x^*)$ and, since $v(x^*,t^*) > 0$, there is a small positive δ such that $v > 0$ on $\bar{B}(x^*,(1+M)\delta) \times [t^*-\delta, t^*]$, and $(x^*, t^*-\delta) \in \mathcal{D}_j \cap \{t < t^*\} \subset Q^-$. As Q^- is open, there exists a small positive $r \in (0,\delta)$ such that $B(x^*,r) \times \{t^*-\delta\} \subset Q^-$. Applying the **propagation** criterion in the definition 1 of the weak solution with $(x_0,t_0,r_0) = (x^*, t^*-\delta, r)$ then gives $B(x^*, r+c^-\delta) \times \{t^*\} \subset Q^-$ where $c^- = \min_{\bar{B}(x^*,r+M\delta)\times[t^*-\delta,t^*]}\{W(v)\} > 0$. It then follows that $(x^*,t^*) \in Q^-$, contradicting to the assumption that $(x^*,t^*) \in \mathcal{D}_j \setminus Q^-$. This contradiction shows that $t^* = h^j(x^*)$.

Since $(x_j,t_j) \in Q^-$, we must have $x^* \ne x_j$. Hence, Without loss of generality, we assume that $x^* \in (x_j, B]$. As $v(x^*,t^*) = v(x^*,h^j(x^*)) \ge 2/J$, there exists $\eta > 0$ such that $v \ge 1/J$ on $\bar{B}(x^*,(1+M)\eta) \times [t^*-\eta, t^*]$. Using $\frac{dh^j}{dx} > 0$ on $(x_j, B]$, we see that for every sufficiently small $\epsilon > 0$, $(x^*-\epsilon, h^j(x^*-\epsilon)) \in D_j \cap \{t < t^*\} \subset Q^-$. See Figure 7 (a). It then follows that there exists $r = r(\epsilon) > 0$ such that $B(x^*-\epsilon, r) \times \{h^j(x^*-\epsilon)\} \subset Q^-$. Denote $\delta(\epsilon) = h^j(x^*) - h^j(x^*-\epsilon)$. Applying the **propagation** criterion in the definition of solution with $(x_0,r_0,t_0,\delta) = (x^*-\epsilon, r(\epsilon), h^j(x^*-\epsilon), \delta(\epsilon))$ then gives us that $B(x^* - \epsilon, r + c^-(\epsilon)\delta(\epsilon)) \times \{t^*\} \subset Q^-$, where $c^-(\epsilon) = \min_{\bar{B}(x^*-\epsilon, r+M\delta(\epsilon))\times[h^j(x^*-\epsilon),h^j(x^*)]}\{W(v)\}$. Note that as $\epsilon \searrow 0$, $c^-(\epsilon) \to W(v(x^*,t^*))$ and $\frac{\epsilon}{\delta(\epsilon)} \to \frac{dx}{dh^j(x^*)} = W(v(x^*,t^*)) - 1/j$. Hence, for sufficiently small positive ϵ, $c^-(\epsilon)\delta(\epsilon) > \epsilon$, which implies $x^* \in B(x^* - \epsilon, r + c^-(\epsilon)\delta(\epsilon))$; that is, $(x^*,t^*) \in Q^-$, contradicting to the assumption that $(x^*,t^*) \in \mathcal{D}_j \setminus Q^-$. This contradiction shows that $\mathcal{D}_j \subset Q^-$ for all $j \ge J$.

Sending $j \to \infty$ we then conclude that $\mathcal{D} := \{(x,t) \mid x \in [A,B], t \in (h(x), T^0(x))\} \subset Q^-$ where $T^0(x) = \lim_{j\to\infty} T_j^0(x) = \sup\{t \mid v(x,\tau) > 0 \ \forall \tau \in (h(x),t)\}$. Finally, in $\mathcal{D}$, we have $v_t = G^-(v)$ so that $v = \Phi^-(v(x,h(x)), t-h(x))$. This then implies that $T^0(x) = T(x) + \int_{v(x,h(x))}^0 \frac{ds}{G^-(s)}$, thereby completing the proof of the lemma.

5. A Local Existence and Uniqueness Result

In this section, we show that the curve $t = H(T,\psi,a,b;x)$ defined in (4.3) is actually a component of the interface, and the solution can be uniquely solved below and near $t = H$.



(a)Figure for the proof of Lemma 4.2 (b)Figure for the proof of Theorem 3(II)a

Figure 7:

Theorem 3 *Let* $(T,\psi,\Omega_+,\Omega_-)\in\mathbf{S}$ *and* $(a,b)\subset\Omega_+$ *be an interval such that* $a\notin\Omega_+$, $b\notin\Omega_+$, *and* (4.7) *holds. Let* $H(x)=H(T,\psi,a,b;\cdot)$ *be defined as in Lemma* 4.1. *Set*

$$\begin{aligned}
&\mathcal{D}=\{(x,t)\mid x\in(a,b),T(x)\le t<H(x)\}\ ,\\
&\hat{T}=H,\quad \hat{\psi}=\Phi^+(\psi,H-T),\quad \hat{\Omega}_-=(\Omega_-\cup[a,b])\cap\mathbb{R},\quad \hat{\Omega}_+=\Omega_+\setminus(a,b),\\
&E_\eta=\left\{(x,t)\mid x\in(a-\eta,b+\eta),H(x)<t<H(x)+\textstyle\int_{\hat{\psi}(x)}^{0}\frac{ds}{G^-(s)}\right\}.
\end{aligned}$$

Then the followings hold:

(I) $(\hat{T},\hat{\psi},\hat{\Omega}_+,\hat{\Omega}_-)\in\mathbf{S}$, $\hat{T}\gneqq T$, *and* $\hat{\Gamma}_0=\Gamma_0\setminus\{a,b\}$ *where* $\hat{\Gamma}_0:=\partial\hat{\Omega}_\pm$ *and* $\Gamma_0:=\partial\Omega_\pm$;

(II) *If* (v,Q^+,Q^-) *is a solution to* $(\mathbf{P})$ *on* $\{t\ge T(x)\}$ *with Cauchy data* $(T,\psi,\Omega_+,\Omega_-)$, *then*

(a) $\mathcal{D}\subset Q^+$ *and* $v(x,t)=\Phi^+(\psi(x),t-T(x))$ *on* $\bar{\mathcal{D}}$,

(b) $E_\eta\subset Q^-$ *and* $v(x,t)=\Phi^-(\hat{\psi}(x),t-H(x))$ *on* $\bar{E}_\eta$ *for some* $\eta>0$, *and*

(c) *the following defined* $(\hat{v},\hat{Q}^+,\hat{Q}^-)$ *solves* $(\mathbf{P})$ *on* $\{t\ge\hat{T}(x)\}$ *with Cauchy data* $(\hat{T},\hat{\psi},\hat{\Omega}_+,\hat{\Omega}_-)$:

$$\hat{v}=v,\quad \hat{Q}^-=Q^-\cup\{(x,H(x))\mid x\in[a,b]\},\quad \hat{Q}^+=Q^+\setminus\mathcal{D}\ ;$$

(III) *If* $(\hat{v},\hat{Q}^+,\hat{Q}^-)$ *is a solution to* $(\mathbf{P})$ *on* $\{t\ge\hat{T}(x)\}$ *with Cauchy data* $(\hat{T},\hat{\psi},\hat{\Omega}_+,\hat{\Omega}_-)$, *then the following defined* (v,Q^+,Q^-) *is a solution to* $(\mathbf{P})$ *on* $\{t\ge T(x)\}$ *with Cauchy data* $(T,\psi,\Omega_+,\Omega_-)$:

$$\begin{aligned}
&v(x,t)=\begin{cases}\hat{v}(x,t) & \text{if}\ \ t\ge\hat{T}(x),\\ \Phi^+(\psi(x),t-T(x)) & \text{if}\ \ T(x)\le t<\hat{T}(x),\end{cases}\\
&Q^-=\hat{Q}^-\setminus\{(x,\hat{T}(x))\mid x\in[a,b]\},\qquad Q^+=\hat{Q}^+\cup\mathcal{D};
\end{aligned}$$

(IV) **(P)** *has a unique solution on* $\{t \geq T(x)\}$ *with Cauchy data* $(T, \psi, \Omega_+, \Omega_-)$ *if and only if* **(P)** *has a unique solution on* $\{t \geq \hat{T}(x)\}$ *with Cauchy data* $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-)$.

Proof Recall that $(T, \psi, \Omega_+, \Omega_-) \in \mathbf{S}$ implies (4.5). Hence, by Lemma 4.1, $H(x) := H(T, \psi, a, b; x)$ is well–defined. Recall also that $G^- < 0$ on $[0, \infty)$, so that $\int_\alpha^0 \frac{ds}{G^-(s)} > 0$ for all $\alpha > 0$. As $\hat{\psi} = \Phi^+(\psi, H - T) > 0$ on (a, b), $\psi(a) > 0$ (if a is finite), and $\psi(b) > 0$ (if b is finite), there exists $\eta > 0$ such that $(a - \eta, a) \cup (b, b + \eta) \subset \Omega^-$ and $\hat{\psi} > 0$ for all $x \in (a - \eta, b + \eta)$, so that E_η is well defined. In the sequel, we fix such an $\eta > 0$.

(I). Since $\hat{\psi} > 0$ on $[a, b] \cap \mathbb{R}$, $-W(\hat{\psi})\,|H'| \leq 0 < 1$ on $[a, b] \cap \mathbb{R}$. It then follows that $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-) \in \mathbf{S}$. The assertion **(I)** of the theorem thus follows.

(II)(a). Let $\epsilon > 0$ be any small number. Set $a_\epsilon = \max\{a + \epsilon, -1/\epsilon\}$ and $b_\epsilon = \min\{b - \epsilon, 1/\epsilon\}$. Let $\psi_\epsilon : [a_\epsilon, b_\epsilon] \to (-\infty, 1)$ be a Lipschitz continuous function such that $\psi_\epsilon > \psi$ on $[a_\epsilon, b_\epsilon]$, $W(\psi_\epsilon(a_\epsilon)) > 0$, and $W(\psi_\epsilon(b_\epsilon)) > 0$, and $\psi_\epsilon < \psi + \epsilon$ on $[a_\varepsilon + \varepsilon, b_\epsilon - \epsilon]$. Such function ψ_ϵ exists since ψ is Lipschitz and $\psi < 1$ on $\Omega_+ \supset (a, b)$.

Let $H_\epsilon(x) = H(T, \psi_\epsilon, a_\epsilon, b_\epsilon; x)$ for all $x \in [a_\epsilon, b_\epsilon]$, where $H(\cdots)$ is as in Lemma 4.1. Set

$$\mathcal{D}_\epsilon = \{(x, t) \mid x \in [a_\epsilon, b_\epsilon], T(x) \leq t \leq H_\epsilon(x)\}, \qquad v_\epsilon = \Phi^+(\psi_\epsilon, t - T).$$

We claim that $\mathcal{D}_\epsilon \subset Q^+$. Suppose the claim is not true. Then since $\mathcal{D}_\epsilon \setminus Q^+$ is compact, there exists $(x^*, t^*) \in \mathcal{D}_\epsilon \setminus Q^+$ such that $t^* = \min\{t \mid \ (x, t) \in \mathcal{D}_\epsilon \setminus Q^+ \text{ for some } x\ \}$, i.e., $\mathcal{D}_\epsilon \cap \{t < t^*\} \subset Q^+$.

First, $t^* \in (T(x^*), H_\epsilon(x^*)]$ and $x^* \in (a_\epsilon, b_\epsilon)$ since $\{(x, T(x)) \mid x \in [a_\epsilon, b_\epsilon]\} \subset Q^+$ and $H_\epsilon = T$ at a_ϵ and b_ϵ.

Next, $B(x^*, c_\epsilon(\delta)\delta) \times \{t^* - \delta\} \subset \mathcal{D}_\epsilon$ for all small $\delta > 0$, where $c_\epsilon(\delta) = \min_{\bar{B}(x^*, M\delta) \times [t^* - \delta, t^*]}\{W(v_\epsilon)\}$ when $t^* = H_\epsilon(x^*)$ (by Lemma 4.1 (2)(c)), and $c_\epsilon(\delta) = M$ when $t^* < H_\epsilon(x^*)$ (as (x^*, t^*) is in the interior of $\mathcal{D}_\epsilon$). See Figure 7b.

Using $\mathcal{D}_\epsilon \cap \{t < t^*\} \subset Q^+$, we obtain $v = \Phi^+(\psi, t - T)$ on $\mathcal{D}_\epsilon \cap \{t \leq t^*\}$. Consequently, $v(x^*, t^*) = \Phi^+(\psi(x^*), t^* - T(x^*)) < \Phi^+(\psi_\epsilon(x^*), t^* - T(x^*)) = v_\epsilon(x^*, t^*) \leq 1$ since $\psi < \psi_\epsilon$. The continuity of v then implies $v < 1$ on $\bar{B}(x^*, 2M\delta) \times [t^* - \delta, t^*]$ for all sufficiently small positive δ. Therefore, as $B(x^*, c_\epsilon(\delta)\delta) \times \{t^* - \delta\} \subset \mathcal{D}_\epsilon \cap \{t < t^*\} \subset Q^+$, we can apply the **propagation** criterion in the definition 1 for the solution (v, Q^+, Q^-) with $(x_0, r_0, t_0) = (x^*, c_\epsilon(\delta)\delta, t^* - \delta)$ to conclude that $B(x^*, [c_\epsilon(\delta) - c(\delta)]\delta) \times \{t^*\} \subset Q^+$ where $c(\delta) = -c^+ := -\min_{\bar{B}(x^*, (c_\epsilon(\delta) + M)\delta)) \times [t^* - \delta, t^*]}\{-W(v)\}$. Note that

$$\lim_{\delta \searrow 0} c_\epsilon(\delta) \geq W(v_\epsilon(x^*, t^*)) > W(v(x^*, t^*)) = \lim_{\delta \searrow 0} c(\delta).$$

Thus, for all sufficiently small positive δ, $c_\epsilon(\delta) > c(\delta)$, which implies that $(x^*, t^*) \in Q^+$, contradicting to the definition of (x^*, t^*). This contradiction shows that $\mathcal{D}_\epsilon \subset Q^+$ for every small positive ϵ.

Sending $\epsilon \searrow 0$, we then obtain the assertion **(II)**(a). (The equation $\cup_{0 < \epsilon < \epsilon_0} \mathcal{D}_\epsilon = \mathcal{D}$, for all small positive ϵ_0, can be proven by using Lemma 4.1 (2)(a) and its corresponding proof.)

(II)(b). Let $x \in (a-\eta, b+\eta)$ be any point. If $\{a$ is finite and $x \in (a-\eta, a]\}$, or if $\{b$ is finite and $x \in [b, b+\eta)\}$, or if $\{x \in (a,b)$ and $v(x, H(x)) = 1\}$, then $(x, H(x)) \in \overline{Q^-}$ (cf. Remark 2.1) and $W(v(x, H(x))) > 0$, and therefore, applying Lemma 4.2 with $(x_0, t_0) = (x, H(x))$ and $[A, B] = [x-\epsilon, x+\epsilon]$ for a small positive ϵ gives

$$(x,t) \in Q^- \quad \forall\, t \in \left(H(x), H(x) + \int_{\hat{\psi}(x)}^{0} \frac{ds}{G^-(s)} ds\right). \tag{5.1}$$

It remains to consider the case when $x \in (a,b)$ and $v(x, H(x)) < 1$. By Lemma 4.1 (2)(a), there exists $y = y^x \in [a,b] \cap \mathbb{R}$ such that $H(\cdot) = h(y, T^*(y); \cdot)$ in the closed interval bounded by x and y. Since $\{\Phi^+(y, T^*(y)) = 1$ if $y \in (a,b)$ $\}$ and $\{T^*(y) = T(y)$ if $y \in \{a,b\} \cap \mathbb{R}\}$, we see that $(y, T^*(y)) \in \overline{Q^-}$ and $W(v(y, T^*(y))) > 0$. Hence, we again can apply Lemma 4.2 with $(x_0, t_0) = (y, T^*(y))$ to obtain (5.1) with $H(\cdot)$ replaced by $h(\cdot)$, the solution to (4.10). To conclude the proof of **(II)**(b), it remains to show that $h(\cdot) = H(\cdot)$ on the closed interval bounded by x and y. Since v is Lipschitz in t, the solution $h(\cdot)$ to (4.10) is unique. On the other hand, on the interval bounded by x and y, $H(\cdot) = h(y, T^*(y); \cdot)$ solves $H' = 1/W(\Phi^+(\psi, H-T)) = 1/W(v)$ by **(II)**(a). Hence, $H(\cdot)$ is indeed the unique solution to (4.10) on the closed interval bounded by x and y. Thus, $h(x) = H(x)$ and (5.1) holds. This proved **(II)**(b).

(II)(c). The assertion **(II)**(c) follows directly from **(II)**(a) (b) and the definition of $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-)$.

(III) Since $(\hat{v}, \hat{Q}^+, \hat{Q}^-)$ is a solution on $\{t \geq H(x)\}$ and since $v < 1$ in $\mathcal{D}$, one can verify directly all the requirements in definitions 2 and 1 for (v, Q^+, Q^-) being a solution to **(P)** on $\{t \geq T(x)\}$ with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ except the following:

> *If* $B(x_0, r_0) \times \{t_0\} \subset \mathcal{D}$ *and* $v < 1$ *in* $\bar{B}(x_0, r_0 + M\delta) \times [t_0, t_0+\delta]$, *then* $B(x_0, r_0 + c^+\delta) \times \{t_0 + \delta\} \subset \mathcal{D}$ *where* $c^+ = \min_{\bar{B}(x_0, r_0+M\delta)\times[t_0,t_0+\delta]}\{-W(v)\}$.

We now verify this last condition by a contradiction argument. Replacing $B(x_0, r_0)$ by $B(x_0, r_0 - \epsilon)$ and taking $\epsilon \searrow 0$ if necessary, we can assume without loss of generality that $\bar{B}(x_0, r_0) \times \{t_0\} \subset \mathcal{D}$.

Suppose the condition does not hold. Then since $\bar{B}(x_0, r_0) \times \{t_0\} \subset \mathcal{D}$, the curve $t = H(x)$, for $x \in (a,b)$, will intersect the closed trapezoid $U := \cup_{\tau \in [0,\delta]} \bar{B}(x_0, r_0 + c^+\tau) \times \{t_0 + \tau\}$. Therefore

$$c^+ = \min_{\bar{B}(x_0, r_0+M\delta)\times[t_0,t_0+\delta]}\{-W(v)\} = -\max_{\bar{B}(x_0, r_0+M\delta)\times[t_0,t_0+\delta]}\{W(v)\} < 0$$

since $W(v) = W(\Phi^+(\psi, H-T)) > 0$ on the curve $t = H(x)$ $(x \in (a,b))$. In addition, there exists $x^* \in B(x_0, r_0)$ such that $(x^*, H(x^*)) \in U$.

Let $y = y^{x^*} \in [a,b]$ be the point given in the Lemma 4.1 (2)(a). Without loss of generality, we assume that $y \leq x^*$. Let $\hat{y} = \max\{y, x_0 - r_0\} \in [y, x^*]$. We now compare, for $x \in [\hat{y}, x^*]$, the curve $t = H(x) = h(y, T^*(y); x)$ with the curve $t = \ell(x) :=$

$\min\{t_0+\delta, t_0+(x-x_0+r_0)/|c^+|\}$ which represents the left lateral and top boundary of the trapezoid U.

We claim that $H(\hat{y}) > \ell(\hat{y})$. Indeed, if $y \geq x_0 - r_0$, then $\hat{y} = y$ and $H(\hat{y}) = T^*(y) > \ell(\hat{y})$ since $v(y, T^*(y)) = 1$ and $v < 1$ in $B(x_0, r_0 + M\delta) \times [t_0, t_0 + \delta]$; if $y < x_0 - r_0$, then $\bar{B}(x_0, r_0) \times \{t_0\} \subset \mathcal{D}$ implies $H(\hat{y}) = H(x_0 - r_0) > t_0 = \ell(\hat{y})$.

Now for $x \in [\hat{y}, x^*]$, $H(x) = h(y, T^*(y); x)$, so that $H'(x) = 1/W(v(x, H)) \geq -1/c^+ \geq \ell'(x)$. An integration of this inequality over $[\hat{y}, x^*]$ then gives $H(x^*) > \ell[x^*]$, contradicting to the assumption that $(x^*, H(x^*)) \in U$. This contradiction shows that $U \subset \mathcal{D}$ and verifies the condition needed. Hence (v, Q^+, Q^-) is a solution as required.

(IV). Finally, the last assertion **(IV)** follows from **(II)**(c) and **(III)**. This completes the proof.

6. Proof of Theorem 2

The idea of the proof of Theorem 2 is to use repeatedly Theorem 3 (and it's companion for the case $(a, b) \subset \Omega_-$) to reduce the problem into a simple case where $\Gamma_0 = \partial\Omega_\pm = \emptyset$. Then use again Theorem 3 for the case $(a, b) = \mathbb{R}$ to construct, layer by layer in the space-time domain, a unique solution.

Proof of Theorem 2 Let $(T, \psi, \Omega_+, \Omega_-) \in \mathbf{S}$ be given. We prove the existence of a unique solution to $(\mathbf{P})$ on $\{t \geq T(x)\}$ with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ in two steps.

Step 1 We assume that $\Gamma_0 \neq \emptyset$; otherwise, we go directly to Step 2.

First we find a maximal connected component (a, b), of either Ω_+ or Ω_-, for which we can apply Theorem 3 (or its companion for " $-$ ") to transfer the Cauchy problem to a simpler one.

We assign every point in the set $\Sigma := \{-\infty\} \cup \Gamma_0 \cup \{\infty\}$ a letter either "R" or "L", depending on the initial direction (Right or Left) of the motion of interface at that point. As a default, we assign "R" to $\{-\infty\}$ and "L" to $\{\infty\}$. Since $W(\psi) \neq 0$ on Γ_0, the assignment is well–defined. For example, for $a \in \Gamma_0$, the letter assigned is "R" if $W(\psi(a)) > 0$ and $(a-\epsilon, a) \subset \Omega^-$ or if $W(\psi(a)) < 0$ and $(a-\epsilon, a) \subset \Omega^+$ for some small positive ϵ; otherwise, the letter assigned is "L". Now appending all the letters assigned to Σ in the same order as the corresponding points in Σ appeared on the real line, we obtain a word consisting of two letters, "R" and "L". By the default, this word begins with "R" and ends with "L". Hence, there is a first place where the letter "R" is followed by "L". Let's denote the corresponding points by a and b respectively. Then either (i) $(a, b) \subset \Omega_+$, $W(\psi(a)) > 0$ (if a is finite) and $W(\psi(b)) > 0$ (if b is finite), or (ii) $(a, b) \subset \Omega_-$, $W(\psi(a)) < 0$ (if a is finite) and $W(\psi(b)) < 0$ (if b is finite). Without loss of generality, we assume that (i) happens.

Now with the given $(T, \psi, \Omega_+, \Omega_-) \in \mathbf{S}$ and such (uniquely) chosen interval (a, b), we can apply Theorem 3 to obtain a new Cauchy data $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-) \in \mathbf{S}$ such that $(\mathbf{P})$ with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ has a unique solution if and only if $(\mathbf{P})$ with Cauchy data $(\hat{T}, \hat{\psi}, \hat{\Omega}_+, \hat{\Omega}_-)$ has a unique solution. One notices that $\hat{\Gamma}_0 := \partial\hat{\Omega}_\pm = \Gamma_0 \setminus \{a, b\}$ has at least one point less than Γ_0 does.

Applying this process finitely many times, we then find $(\tilde{T}, \tilde{\psi}, \tilde{\Omega}_+, \tilde{\Omega}_-) \in \mathbf{S}$ such that either $\tilde{\Omega}_- = \mathbb{R}$ or $\tilde{\Omega}_+ = \mathbb{R}$, and that problem $(\mathbf{P})$ on $\{t \geq T(x)\}$ with Cauchy data $(T, \psi, \Omega_+, \Omega_-)$ is equivalent to $(\mathbf{P})$ on $\{t \geq \tilde{T}(x)\}$ with Cauchy data $(\tilde{T}, \tilde{\psi}, \tilde{\Omega}_+, \tilde{\Omega}_-)$.

Step 2 Assume either $\Omega_- = \mathbb{R}$ or $\Omega_+ = \mathbb{R}$. Without loss of generality, we assume that $\Omega_+ = \mathbb{R}$. We consider separately the following three cases:

(i) $G^+(1) < 0$; (ii) $G^+(1) > 0$ and $G^-(-1) > 0$; (iii) $G^+(1) > 0$ and $G^-(-1) < 0$.

Case (i): $G^+(1) < 0$. This case is either *bistable* (when $G^-(-1) > 0$) or *excitable* (when $G^-(-1) < 0$).

Since $\psi < 1$ on $\Omega_+ = \mathbb{R}$, the definition of T^* in (4.4) gives $T^*(\cdot) \equiv \infty$, so that $H(T, \psi, -\infty, \infty; \cdot) \equiv \infty$. By Theorem 3 **(II)**(a) with $(a, b) = \mathbb{R}$, the unique solution is given by

$$Q^- = \emptyset,\ Q^+ = \{(x,t) \mid x \in \mathbb{R}, t \geq T(x)\},\ v(x,t) = \Phi^+(\psi(x), t - T(x)) \ \text{ in } \ Q^+. \tag{6.1}$$

Case (ii): $G^+(1) > 0$ and $G^-(-1) > 0$. This corresponds to an *excitable* case.

By Lemma 4.1 with $(a, b) = \mathbb{R}$, either $H(\cdot) = H(T, \psi, -\infty, \infty; \cdot) \equiv \infty$ or $H(x) < \infty$ for all $x \in \mathbb{R}$.

If $H \equiv \infty$, there is a unique solution and it is given by (6.1).

If $H(x) < \infty$ for all $x \in \mathbb{R}$, we first apply Theorem 3 to $(T, \psi, \mathbb{R}, \emptyset)$ and then apply a companion of Theorem 3 for the "$-$" phase change for $(H, \Phi^+(\psi, H - T), \emptyset, \mathbb{R})$ to conclude that there is a unique solution, given by

$$\begin{aligned} &Q^- = \{t > H(x)\}, \quad Q^+ = \{T(x) \leq t < H(x)\}, \\ &v(x,t) = \begin{cases} \Phi^+(\psi(x), t - T(x)), & (x,t) \in \overline{Q^+}, \\ \Phi^-(v(x,\ H(x)), t -\ H(x)), & (x,t) \in Q^-. \end{cases} \end{aligned} \tag{6.2}$$

Case (iii): $G^+(1) > 0$ and $G^-(-1) < 0$. We consider three different situations:
(iii)(a) $\max_{[-1,0]}\{G^-\} \geq 0$;
(iii)(b) $G^- < 0$ on $[-1, \infty)$ and $\min_{[0,1]}\{G^+\} \leq 0$;
(iii)(c) $G^- < 0$ on $[-1, \infty)$ and $G^+ > 0$ on $(-\infty, 1]$.
As we shall see, cases (iii)(a) and (iii)(b) are *excitable* and (iii)(c) is *oscillatory.*

Case (iii)(a). If $T_1 = H(T, \psi, -\infty, \infty; x)$ is finite, then by Lemma 4.1 (2) (b), $\psi_1 := \Phi^+(\psi, H - T) > 0$ on $\mathbb{R}$. It then follows $T_1^*(y) \equiv \infty$ where

$$T_1^*(y) := \sup\{t \geq T_1(y) \mid \Phi^-(\psi_1(y), \tau - T_1(y)) > -1\ \forall \tau \in [T_1(y), t)\} \quad \forall y \in \mathbb{R}. \tag{6.3}$$

Hence, same as the case (ii), the solution is unique, given by (6.1) (when $H \equiv \infty$) or (6.2) (when $H < \infty$).

Case (iii)(b). If $T_1 := H(T, \psi, -\infty, \infty; \cdot) \equiv \infty$. Then the unique solution is given by (6.1).

Suppose $T_1(x) < \infty$ for all $x \in \mathbb{R}$. Then T_1^* defined by (6.3) is bounded, since $G^- < 0$ on $[-1, \infty)$. Applying the companion Theorem 3 for the "$-$" case and using a similar reasoning as above we then conclude that there is a finite $T_2(\cdot) > T_1(\cdot)$ such that the solution is given uniquely by $Q^+ = \{T(x) \le t < T_1\} \cup \{t > T_2\}$, $Q^- = \{T_1(x) < t < T_2(x)\}$, and $v = \Phi^+(\psi, t - T)$ in $\{t \le T_1\}$, $v = \Phi^-(\psi_1, t - T_1)$ in Q^-, and $v = \Phi^+(\psi_2, t - T_2)$ in $\{t \ge T_2\}$ where $\psi_2 = \Phi^-(\psi_1, T_2 - T_1)$.

Case (iii)(c). Same as before, we first apply Theorem 3 to obtain $(T_1, \psi_1, \Omega_+^1, \Omega_-^1) := (H(T, \psi, -\infty, \infty; \cdot), \Phi^+(\psi, H - T), \emptyset, \mathbb{R}) \in \mathbf{S}$. Note that $T_1 = H \le T^* < \infty$ since $G^+ > 0$ on $(-\infty, 1]$. Applying a companion of Theorem 3 for the Cauchy data $(T_1, \psi_1, \Omega_+^1, \Omega_-^1)$ we then obtain $(T_2, \psi_2, \Omega_+^2, \Omega_-^2)$ where $\Omega_+^2 = \mathbb{R}$ and $\Omega_-^2 = \emptyset$, and $T_2 < \infty$ since $G^- < 0$ on $[-1, \infty)$. Repeating this process we obtain a sequence $\{(T_j, \psi_j, \Omega_+^j, \Omega_-^j)\}_{j=1}^\infty$ in $\mathbf{S}$, where $T_j < T_{j+1} < \infty$ for all j, $\Omega_+^j = \emptyset$ if j is odd, $\Omega_+^j = \mathbb{R}$ if j is even. Hence in $\cup_{j=1}^\infty \{T(x) \le t \le T^j(x)\}$ the solution is uniquely determined.

To complete the proof, it remains to show that $\lim_{j\to\infty} T_j(x) = \infty$ for any $x \in \mathbb{R}$.

Suppose for the contrary that there exists $x_0 \in \mathbb{R}$ such that $\lim_{i\to\infty} T_i(x_0) = t_0 < \infty$. We consider the closed set $\mathbf{K} = \{(x,t) \mid t \le t_0,\ |x - x_0| \le 2M(t_0 - t)\}$, a cone with vertex at (x_0, t_0) and a (large) open angle $2\arctan(2M)$. Apparently, each curve $t = T_j(x)$, $j \ge 1$, regarded as a function, attains a minimum value in $\mathbf{K}$ at some point, say, $(x_j, T_j(x_j)) \in \mathbf{K}$. We claim that for all $j \ge 1$, $v(x_j, T_j(x_j)) = (-1)^{j+1}$, i.e., $(x_j, T_j(x_j))$ is actually a point of nucleation.

Suppose the claim that $v(x_j, T_j(x_j)) = (-1)^{j+1}$ for all $j \ge 1$ is not true. Then $v(x_k, T_k(x_k)) \ne (-1)^{k+1}$ for some $k \ge 1$. Without loss of generality, we assume that k is odd. Let $y_k = y^{x_k}$ be the point given by Lemma 4.1 2(a) with $(a, b) = \mathbb{R}$. Then $T_k(\cdot) = h(y_k, T_k^*(y_k); \cdot)$ on the closed interval bounded by y_k and x_k. Since $T_k(y_k) = T_k^*(y_k)$, we have $v(y_k, T_k(y_k)) = 1$ so that $y_k \ne x_k$. Without loss of generality, we assume that $y_k < x_k$. Now since x_k is the minimum of $t = T_k(\cdot)$ in $\mathbf{K}$ and since $T_k(\cdot) = h(y_k, T_k^*(y_k); \cdot)$ is an increasing function on $[y_k, x_k]$, we see that x_k must lie on the left lateral boundary of $\mathbf{K}$. Consequently, since $h'(y_k, T_k^*(y_k); x_k) = 1/W(v) \ge 1/M$, which is larger than the slope $\frac{1}{2M}$ of the lateral boundary of the cone $\mathbf{K}$, we see that for all x sufficiently close to and on the left–hand side of x_k, $(x, T_k(x)) = (x, h(y_k, T_k^*(y_k); x))$ is in $\mathbf{K}$. However since $h(y_k, T_k^*(y_k); x) < h(y_k, T_k^*(y_k); x_k)$, this contradicts to the assumption that x_k is the minimum of the curve $t = T_k(\cdot)$ in the cone $\mathbf{K}$. Hence, we must have $v(x_j, T_j(x_j)) = (-1)^{j+1}$ for all $j \ge 1$.

If j is odd, then from Lemma 4.1 (2)(b) we have $W(v) > 0$ on $t = T_j$, i.e., $v(x, T_j(x)) > 0$ for all $x \in \mathbb{R}$. It then follows, since x_j is the minimum of $t = T_j(\cdot)$ in

K, that

$$T_{j+1}(x_{j+1}) - T_j(x_j) \geq T_{j+1}(x_{j+1}) - T_j(x_{j+1})$$
$$= \int_{v(x_{j+1},T_j(x_{j+1}))}^{v(x_{j+1},T_{j+1}(x_{j+1}))} \frac{ds}{G^-(s)} \geq \int_0^{-1} \frac{ds}{G^-(s)}.$$

Similarly, if j is even, $T_{j+1}(x_{j+1}) - T_j(x_j) \geq \int_0^1 \frac{ds}{G^+(s)}$. Therefore,

$$t_0 > T_{j+1}(x_0) \geq T_{j+1}(x_{j+1}) > T_1(x_1) + j * \min\{\int_0^{-1} \frac{ds}{G^-(s)}, \int_0^1 \frac{ds}{G^+(s)}\}.$$

Sending $j \to \infty$ we then conclude that $t_0 = \infty$, contradicting to our assumption $t_0 < \infty$. Therefore, $\lim_{j\to\infty} T_j(x) = \infty$ for all $x \in \mathbb{R}$. This completes the proof of Theorem 2.

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