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ON A HIGH ORDER SPIN WAVE SYSTEM WITH A NONLINEAR FREE TERM

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Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

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Abstract In this paper, we deduct a new spin wave model in lattices, which is a nonlinear high order degenerate parabolic system with a nonlinear free term. In a further theoretical study, by using a parameter ϵ approximation, the existence of a weak solution has been obtained.

Key Words Spin wave; high order system; degenerate parabolic equations; existence of solution.

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1. Modelling and Problem

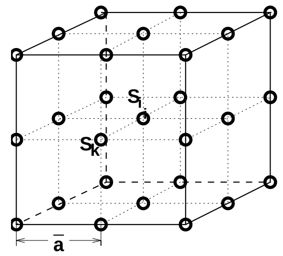
In Solid State Physics an important concept is that of collective excitations ([1-3]). Collective excitations are the low-lying excited states of systems where a strong coupling between particles is present. Their nature and origin can be varied and depends on the system and interaction considered. For example, the most notable of these are the lattice vibrations of a crystalline structure, which, when properly quantized are called phonons. Other collective excitations of particular importance, and that we will consider here, are spin wave excitations. These low-lying excitations occur in ferromagnets and correspond to the oscillations of the electron-spin-density fluctuations. To be more specific, in a ferromagnet below the Curie temperature, due to the exchange interaction, the magnetic moments associated with each lattice site are lined up so that they all statistically point in the same direction. This corresponds to the ground state of the system. Spin waves are the excited eigenstates of the system Hamiltonian which, in a classical sense, correspond to the propagation of spin deviations from the original direction.

Spin wave excitations in ferromagnetic lattices can be characterized by a spin-exchange Hamiltonian which is invariant under lattice translation. That is, after calculating the quantum equation of motion with the spin Hamiltonian, spin vectors $\boldsymbol{S}$

should satisfy the following relationship, (see [3-5]),

$$\frac{\hbar}{2}\frac{\partial \boldsymbol{S}_i}{\partial t} = \sum_{k\neq i}\frac{A}{2}[\boldsymbol{S}_i, \boldsymbol{S}_k] + [\boldsymbol{S}_i, \boldsymbol{h}],$$

where i points to the ith atom. The summing index k points to a neighbor atom of the ith site; A is the exchange integral; $\hbar$ is Planck constant; the vector $\boldsymbol{h}$ is a given function which may depend on $\boldsymbol{S}_i$. The square brackets $[\cdot,\cdot]$ denote the commutator of the two vectors which describes the effect between the atoms.



simple cubic lattice

A kind of materials, such as $\alpha-$Fe (see [3-5]),with a ferromagnetic property is a simple cubic lattice with lattice constant $\bar{a}$. Suppose that a smooth function $\boldsymbol{S}$ values $\boldsymbol{S}_i$ at the ith atom, i.e. $\boldsymbol{S}$ is continuous and smooth enough and $\boldsymbol{S}(x) = \boldsymbol{S}_i(x)$, then $\boldsymbol{S}_k(x) = \boldsymbol{S}(x \pm \bar{a})$, where $\boldsymbol{S}_k$ correspond to those atoms adjacent the ith atom, $x = (x_1, x_2, x_3)$. Those $\boldsymbol{S}_k$s can be expanded and expressed by $\boldsymbol{S}$ as follows:

$$\boldsymbol{S}(x_1 \pm \bar{a}, x_2, x_3) = \boldsymbol{S}(x) \pm \frac{\partial \boldsymbol{S}}{\partial x_1}(x) + \frac{1}{2!}\frac{\partial^2 \boldsymbol{S}}{\partial x_1^2}(x) \pm \frac{1}{3!}\frac{\partial^3 \boldsymbol{S}}{\partial x_1^3}(x) + \cdots,$$

the same way for $\boldsymbol{S}(x_1, x_2 \pm \bar{a}, x_3)$, $\boldsymbol{S}(x_1, x_2, x_3 \pm \bar{a})$, $\boldsymbol{S}(x_1 \pm \bar{a}, x_2 \pm \bar{a}, x_3)$, $\cdots$ and so on.

Sum all $\boldsymbol{S}_k$ around x. By the symmetry of the lattices, all the items with odd differential degree are eliminated. Therefore, we have

$$\frac{A}{2}\sum_k \boldsymbol{S}_k(x) = \sum_{m=0}^{\infty}\widetilde{\triangle}^m \boldsymbol{S}(x),$$

where

$$\widetilde{\triangle}^m = \sum_{|\alpha|=m} a_\alpha D_x^{2\alpha}, \qquad (m = 1, 2, \cdots) \tag{1.1}$$

are elliptic operators. α is a $N-$tuple index, i.e. $\alpha = (\alpha_1, \cdots, \alpha_N)$, with $\{\alpha_i\}_{i=1,\cdots,N}$ are non-negative integrals, $|\alpha| = \sum_{j=1}^N \alpha_j = m$, $m = 1, 2, \cdots, M$. For any α defined above, a_α is a positive constant depending on $\bar{a}$. Then

$$\boldsymbol{S}_t = \boldsymbol{S}\times \sum_{m=0}^{\infty}\widetilde{\triangle}^m \boldsymbol{S} + \boldsymbol{S}\times\boldsymbol{h}, \tag{1.2}$$

where $\times$ denotes vectorial product.

It can be seen that the Heisenberg system

$$\boldsymbol{S}_t = \boldsymbol{S}\times\boldsymbol{S}_{xx} + \boldsymbol{S}\times\boldsymbol{h} \tag{1.3}$$

is a primary approximation of this model (e.g. see [4]). For this second order system, [6] and [7] obtained the existence of the weak solution.

In this paper, we consider this spin wave system in a more general case, where the sum index in the first item in the right side of (1.2) is up to M. That is, the terms whose differential orders are higher than $2M$ in (1.2) are neglected. We also allow the zero order term $\boldsymbol{h}$ depending on $\boldsymbol{S}$.

Denoting the unknown by $\boldsymbol{Z}$, the problem we considered now is described as follows

$$\boldsymbol{Z}_t = \boldsymbol{Z}\times \sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{Z} + \boldsymbol{f}(\boldsymbol{Z}, x, t), \qquad \text{in } Q_T = \Omega\times[0, T), \tag{1.4}$$

with the boundary-initial conditions

$$\frac{d^l \boldsymbol{Z}}{d\gamma^l}\Big|_{\partial\Omega} = 0, \quad l = 0, 1, \cdots, M-1, \qquad \text{on } S_T = \partial\Omega\times[0, T), \tag{1.5}$$

$$\boldsymbol{Z}(x, 0) = \boldsymbol{Z}_0(x), \qquad \text{on } \Omega, \tag{1.6}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain and $\partial\Omega \in C^{M-1,1}$; $M, N \geq 1$ are integers; The unknown $\boldsymbol{Z} = (Z_1, Z_2, Z_3)$ and the free term $\boldsymbol{f} = (f_1, f_2, f_3)$ are 3-dimensional (3D) vector functions; γ is the unit outward vector to $\partial\Omega$.

Throughout this paper, when we mention free term, we mean zero order term $\boldsymbol{f}$. From (1.2), in this paper, we consider $\boldsymbol{f}$ has the the following structure:

$$\boldsymbol{f} = \boldsymbol{Z}\times\boldsymbol{h}(\boldsymbol{Z}) + \boldsymbol{\xi}(\boldsymbol{Z}, x, t), \tag{1.7}$$

where

$$\boldsymbol{h}(\boldsymbol{Z}) = (h_1(Z_1), h_2(Z_2), h_3(Z_3)); \quad h_i(Z) : \mathbf{R} \to \mathbf{R},\ i = 1, 2, 3$$

$$\boldsymbol{\xi}(\boldsymbol{Z}, x, t) = (\xi_1(\boldsymbol{Z}, x, t), \xi_2(\boldsymbol{Z}, x, t), \xi_3(\boldsymbol{Z}, x, t)) : \mathbf{R}^3\times\mathbf{R}^N\times\mathbf{R} \to \mathbf{R}^3.$$

This is a nonlinear problem for a high order, strong degenerate parabolic system. The coefficient matrix of the highest order elliptic operator of the system (1.4) is

$$A_{2M}(\boldsymbol{Z}) = \begin{pmatrix} 0 & -Z_3 & Z_2 \\ Z_3 & 0 & -Z_1 \\ -Z_2 & Z_1 & 0 \end{pmatrix},$$

which is a null definite matrix (see [6]).

The author started discussing this high order model with a linear free term under a condition $N < 2M$ in [8]. An ϵ approximation was used for help to the degeneration.

The existence of a weak solution for that problem has been obtained. The problem has been further developed in [9], where the periodic problem for this high order model was considered, with a linear free term.

In this paper, we have considered nonlinearity of the free term. The linearity of the free term now becomes a special case of our discussion. The main difficulty, which is essential, is looking for necessary estimates, when the high order derivatives of the nonlinear term will bring about a lot of great troublesome terms. These terms are very hard to be dealt with in bound. We use the structure coming from the original model to avoid this difficulty. However, when $M=1$, this difficulty is not serious, such structure of nonlinearity can be loosed. Moreover, in this paper, we have gotten off the condition $N<2M$.

The hypotheses on the given functions for this problem are as follows:

(H1) The initial function $\boldsymbol{Z}_0(x):\ \mathbf{R}^N\to\mathbf{R}^3$ belongs to $H_0^M(\Omega)$ and $\|\boldsymbol{Z}_0\|_{H^M(\Omega)}\le C$.

(H2) The free term $\boldsymbol{f}(\boldsymbol{Z},x,t)$ is defined in (1.7), in which

$$|h_i(Z_i)|\le C(|Z_i|+1),\ \xi_i(\boldsymbol{Z},x,t)=\sum_{j=1}^{3}\xi_{ji}(x,t)Z_j+\xi_{0i}(x,t),\ i=1,2,3,$$

where $\xi_{ij}\in L_\infty(0,T;C^M(\Omega))$, $h_i\in C(\Omega)$ and C is a positive constant.

If the order of the system is $M=1$, i.e. the system is a general second order parabolic problem, the hypothesis of the free term can be loosed as

(H2a) In the free term $\boldsymbol{f}(\boldsymbol{Z},x,t)$ defined in (1.7), $\boldsymbol{h}$ is defined as in (H2), and $\boldsymbol{\xi}\in C^1(\mathbf{R}^3\times\Omega\times\mathbf{R})$ with $|\boldsymbol{\xi}(\boldsymbol{Z},x,t)|\le C(|\boldsymbol{Z}|+1)$, where C is a positive constant.

We seek a weak solution for this problem. The sense of the weak solution is given in the following definition.

Definition 1.1 *$\boldsymbol{Z}\in L_2(0,T;H_0^M(\Omega))$ is a weak solution of the problem* (1.4)-(1.6) *if for any 3D vector $\boldsymbol{\phi}(x,t)\in C^1(0,T;C_0^M(\Omega))$, $\boldsymbol{\phi}(x,T)\equiv 0$, there holds*

$$\int_{Q_T}\Big[\boldsymbol{\phi}_t\cdot\boldsymbol{Z}+\sum_{0<|\alpha|\le M}(-1)^{|\alpha|}a_\alpha D_x^\alpha\boldsymbol{Z}\cdot\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}\boldsymbol{Z})+\boldsymbol{\phi}\cdot\boldsymbol{f}(\boldsymbol{Z},x,t)\Big]dxdt+\int_\Omega\boldsymbol{\phi}(x,0)\cdot\boldsymbol{Z}_0(x)dx=0. \tag{1.8}$$

The following are the main theorems obtained in this paper.

Theorem 1.2 *Under the hypotheses* (H1) *and* (H2)*, the problem* (1.4)-(1.6) *has at least one weak solution belonging to space*

$$\mathcal{Z}=L_\infty(0,T;H_0^M(\Omega))\cap W_\infty^1(0,T;H^{-(M+J)}(\Omega)), \tag{1.9}$$

in the sense of (1.8), *with*

$$J = [N/2] + 1 \quad ([A] \text{ - the integer part of } A). \tag{1.10}$$

Theorem 1.3 *Under the hypotheses* (H1) *and* (H2a) *and* $M = 1$, *the problem* (1.4)-(1.6) *has at least one weak solution belonging to the space*

$$\mathcal{Z} = L_\infty\Big(0, T; H_0(\Omega)\Big) \cap W^1_\infty\Big(0, T; H^{-J-1}(\Omega)\Big),$$

in the sense of (1.8), *where* J *is defined as in* (1.10).

The outline of the paper is as follows. Section 2 lists some preparatory lemmas. To deal with the degeneration of the main operator, as in [8], (see also [6, 7], [9]) the problem with the parameter ϵ is considered in Section 3,

$$\boldsymbol{Z}_{\epsilon t} = \epsilon(-1)^{M+1}\Big(\sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{Z}_\epsilon + a_0 \boldsymbol{Z}_\epsilon\Big) + \boldsymbol{Z}_\epsilon \times \sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{Z}_\epsilon + \boldsymbol{Z}_\epsilon \times \boldsymbol{h}(\boldsymbol{Z}_\epsilon) + \boldsymbol{\xi}(\boldsymbol{Z}_\epsilon, x, t), \tag{1.11}$$

with the boundary-initial condition (1.5)-(1.6), where $(-1)^M a_0$ in (1.11) is a large positive constant independent of ϵ, which will be chosen later. To overcome the difficulty brought about the lack of the condition $N < 2M$ (i.e. $\boldsymbol{Z}_\epsilon$ is only estimated in H^{2M}, not in L_∞), we first consider $[\boldsymbol{Z}_\epsilon]^k \times \sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{Z}_\epsilon$ instead of $\boldsymbol{Z}_\epsilon \times \sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{Z}_\epsilon$ then let $k \to \infty$. Next, in Section 4, we give some estimates uniformly with respect to ϵ in $\mathcal{Z}$ defined in (1.9). The ϵ limit process is discussed in Section 5. It is proved that the limit of the ϵ-solution is just the weak solution of the problem (1.4)-(1.6). Thus, the proofs of Theorem 1.2, 1.3 are completed. Some further remarks are collected in Section 6.

2. Preliminary Lemmas

We mention some preliminary results relevant to the later proofs here. The following three lemmas can be found in references, which we write here without proofs.

Lemma 2.1 *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded domain having the cone property. If* u *belongs to* $H^{2M}(\Omega) \cap H^M_0(\Omega)$, *the following interpolation inequality holds*

$$\|u\|_{H^{2M}(\Omega)} \le C\left(\|\widetilde{\triangle}^M u\|_{L_2(\Omega)} + \|u\|_{L_2(\Omega)}\right), \tag{2.1}$$

where C *is a positive constant depending only on* M *and* N. $\widetilde{\triangle}^M$ *is defined in* (1.1).

This result can be found in [10], p243.

Lemma 2.2 (Nirenberg-Gagliardo inequality) *Suppose* $u \in W^M_q(\Omega) \cap L_r(\Omega)$. *If* $1 \le p, q, r \le \infty$, $\theta \in [0, 1 - \frac{j}{M}]$ *and* $j - \frac{N}{p} \le (1-\theta)(M - \frac{N}{q}) - \theta\frac{N}{r}$, *then*

$$\|u\|_{W^j_p(\Omega)} \le C\|u\|^{1-\theta}_{W^M_q(\Omega)} \|u\|^\theta_{L_r(\Omega)},$$

where $\Omega \subset \mathbf{R}^N$ *has a cone property,* C *is some positive constant.*

In particular, when $r = +\infty$, $q = 2$, $p = \frac{2M}{j}$, *we have* $\theta = 1 - \frac{j}{M}$, $j = 0, 1, \cdots, M$.

This result can be found in [11], p69.

Lemma 2.3 *Suppose that* X, B, Y *are Banach Spaces satisfying* $X \subset B \subset Y$ *with compact imbedding* $X \to B$, *and for* $0 < \theta < 1$,

$$\|u\|_B \le C\|u\|_X^{1-\theta}\|u\|_Y^{\theta}. \tag{2.2}$$

Let $1 < q_1, q_2 < \infty$. *Then each set bounded in both* $L_{q_1}(0,T;X)$ *and* $W^1_{q_2}(0,T;Y)$ *is bounded and relatively compact in* $L_q(0,T;B)$, *for all* $q < \frac{q_1 q_2}{q_2(1-\theta)+q_1(1-q_2)\theta}$, *provided* $q_2(1-\theta) + q_1(1-q_2)\theta > 0$; *If* $q_2 = 1$, $q < \frac{q_1}{1-\theta}$; *If* $q_1 = q_2 = \infty$, $q = \infty$. *If* $q_2(1-\theta) + q_1(1-q_2)\theta < 0$, $L_{q_1}(0,T;X) \cap W^1_{q_2}(0,T;Y)$ *is bounded and relatively compact in* $C(0,T,B)$. *In particular,*

1) *if* $X = H^2(\Omega)$, $B = L_2(\Omega)$, $Y = L_1(\Omega)$, $q_1 = 2$, $q_2 = 1$, *then space* $L_2(0,T;H^{2M}(\Omega)) \cap W^1_1(0,T;L_1(\Omega))$ *is compactly imbedded into* $L_2(Q_T)$;

2) *if* $X = H^M(\Omega)$, $B = H^{M-1}(\Omega)$, $Y = H^{-(M+J)}(\Omega)$, $q_1 = q_2 = \infty$, *then the space* $L_\infty(0,T;H^M(\Omega)) \cap W^1_\infty(0,T;H^{-(M+J)}(\Omega))$ *is compactly imbedded in* $L_\infty(0,T;H^{M-1}(\Omega))$, *for* J *defined in* (1.10).

Proof This lemma can be found in [12], p89. We only need to verify the special cases. In fact, 1) from Lemma 2.2, we can choose $\theta = \frac{4}{N+4}$, so that $L_2(0,T;H^{2M}(\Omega)) \cap W^1_1(0,T;L_1(\Omega))$ is imbedded into $L_q(0,T;L_2(\Omega))$ with $2 \le q < \frac{2}{1-\frac{4}{N+4}}$. 2) is direct.

3. The Existence of the Solution of the ϵ-Problem

In this section, the existence of the solution of the ϵ-problem (1.11) with boundary-initial condition (1.5)-(1.6) has been discussed. The solution is considered in space

$$\mathcal{G} = L_\infty(0,T;H_0^M(\Omega)) \cap L_2(0,T;H^{2M}(\Omega)) \cap W^1_1(0,T;L_1(\Omega)). \tag{3.1}$$

Proposition 3.1 *Let* $\mathcal{G}$ *as in* (3.1), *then* $\mathcal{G}$ *is compactly imbedded into* $L_2(Q_T)$.

This is the corollary of 1) in Lemma 2.3.

Proposition 3.2 *There exists a unique solution of the boundary-initial problem for the following linear high order parabolic system in* Q_T,

$$\boldsymbol{u}_t = (-1)^{M+1}\epsilon\widetilde{\triangle}^M\boldsymbol{u} + [\boldsymbol{g}(x,t)]^k \times \Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{u} + c_0\boldsymbol{u}\Big) + \sum_{|\alpha|<2M} A_\alpha(x,t)D^\alpha\boldsymbol{u} + \boldsymbol{B}(x,t), \tag{3.2}$$

with the boundary and initial condition

$$\frac{d^l\boldsymbol{u}}{d\gamma^l}\Big|_{\partial\Omega} = 0, \qquad l = 0,1,\cdots,M-1, \qquad \text{on } S_T, \tag{3.3}$$

$$\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \qquad \text{on } \Omega, \tag{3.4}$$

where $(-1)^M c_0$ *is a large positive constant dependent of* ϵ (*specialized in the proof*); $\boldsymbol{g}(x,t), \boldsymbol{B}(x,t) \in L_2(Q_T)$ *are 3D vectors,* $A_\alpha(x,t) \in L_\infty(Q_T)$, $|\alpha| < 2M$, *are* 3×3 *matrices;*

$$[\boldsymbol{g}]^k = [(g_1,g_2,g_3)]^k = ([g_1]^k,[g_2]^k,[g_3]^k); \quad [g]^k = \begin{cases} g, & \textit{if } |g| \le k, \\ k\ \mathrm{sign}(g), & \textit{otherwise;} \end{cases}$$

and $\boldsymbol{u}_0(x) \in H_0^M(\Omega)$. *Moreover, the solution belongs to* $\mathcal{G}$ *and has the estimate*

$$\sup_{0\le t\le T} \|\boldsymbol{u}(\cdot,t)\|_{H^M(\Omega)} + \|\boldsymbol{u}_t(\cdot,t)\|_{L_1(Q_T)} + \|\boldsymbol{u}\|_{L_2(0,T;H^{2M}(\Omega))} \tag{3.5}$$
$$\le C\Big(\|\boldsymbol{u}_0(x)\|_{H^M(\Omega)}, \|\boldsymbol{B}(x,t)\|_{L_2(Q_T)}, \|\boldsymbol{g}(x,t)\|_{L_2(Q_T)}, \sum_{|\alpha|<2M} \|A_\alpha(x,t)\|_{L_\infty(Q_T)}\Big),$$

where C *also depends on* ϵ, a_α, T, N, M *and* Ω, *it is independent of* k.

Proof The proof is similar to the one of Lemma 1 in [8], which requires $A_{2M} \in L_\infty(Q_T)$. However, we need to obtain a new estimate which only depends on the $L_2(Q_T)$ norm of $\boldsymbol{g}$, which is in a part of the main coefficient matrix A_{2M}.

First, easy to see, the main coefficient matrix $A_{2M}(x,t) = \begin{pmatrix} \epsilon & -[g_3]^k & [g_2]^k \\ [g_3]^k & \epsilon & -[g_1]^k \\ -[g_2]^k & [g_1]^k & \epsilon \end{pmatrix}$ is positive definite.

Taking (3.2) get the scalar product by vector $(-1)^M\Big(\sum_{m=1}^M \widetilde{\triangle}^m \boldsymbol{u} + c_0\boldsymbol{u}\Big)$, noticing that $\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{v}) = 0$, then summing and integrating the result over Ω by parts to obtain

$$\frac{(-1)^M}{2}\frac{d}{dt}\Big((-1)^{|\alpha|}\sum_{|\alpha|=M,|\alpha|<M} a_\alpha\|D_x^\alpha \boldsymbol{u}\|_{L_2}^2 + c_0\|\boldsymbol{u}\|_{L_2}^2\Big) + \epsilon\int_\Omega \widetilde{\triangle}^M\boldsymbol{u}\cdot\Big(\sum_{m=1}^M \widetilde{\triangle}^m\boldsymbol{u} + c_0\boldsymbol{u}\Big)dx$$
$$\le \sum_{|\alpha|<2M}\int_\Omega\Big|A_\alpha(x,t)D^\alpha\boldsymbol{u}\cdot\Big(\sum_{m=1}^M \widetilde{\triangle}^m\boldsymbol{u} + c_0\boldsymbol{u}\Big)\Big|dx + \int_\Omega\Big|\boldsymbol{B}(x,t)\cdot\Big(\sum_{m=1}^M \widetilde{\triangle}^m\boldsymbol{u} + c_0\boldsymbol{u}\Big)\Big|dx,$$

then integrating above inequality with respect to t, and using Lemma 2.1, 2.2 to deal with derivative terms $D^\alpha\boldsymbol{u}$ for $0 < |\alpha| < 2M$, we have

$$\frac{1}{4}\sum_{|\alpha|=M}\|D_x^\alpha\boldsymbol{u}(\cdot,t)\|_{L_2(\Omega)}^2 + \Big((-1)^M c_0 - \bar{C}\Big)\|\boldsymbol{u}\|_{L_2(\Omega)}^2 + \frac{\epsilon}{2}\int_0^t \|\widetilde{\triangle}^M\boldsymbol{u}\|_{L_2(\Omega)}^2 d\tau$$
$$\le C\Big(\|\boldsymbol{B}\|_{L_2}^2 + \|\boldsymbol{u}_0\|_{H^M}^2\Big) + C\Big(1 + \sum_{|\alpha|<2M}\|A_\alpha\|_\infty^2\Big)\int_0^t \|\boldsymbol{u}\|_{L_2(\Omega)}^2 d\tau = C_m\Big(1 + \int_0^t \|\boldsymbol{u}\|_{L_2(\Omega)}^2 d\tau\Big).$$

where $\bar{C}$ depends only on the given data a_α $(0 < |\alpha| \le M)$, it is independent of ϵ, k and $\boldsymbol{g}$. Choose $(-1)^M c_0$ such that $(-1)^M c_0 - \bar{C} > 0$, then c_0 is independent of ϵ, k and $\boldsymbol{g}$. In the above estimate, C_m depends only on ϵ and on the given data. By Gronwall inequality, we have estimates for $\|\boldsymbol{u}\|_{L_\infty(0,T;H^M(\Omega))}^2$ and $\|\widetilde{\triangle}^M\boldsymbol{u}\|_{L_2(Q_T)}^2$.

Now let us estimate $\boldsymbol{u}_t$, from the equation, we have for $0 < t < T$,

$$\begin{aligned}\int_{Q_t} |\boldsymbol{u}_t| dx d\tau \le\ & C\Big(\epsilon \int_{Q_t} |\widetilde{\triangle}^M \boldsymbol{u}| dx d\tau + \sum_{m=1}^{M} \int_{Q_t} |[\boldsymbol{g}]^k \times \widetilde{\triangle}^m \boldsymbol{u}| dx d\tau \\ & + c_0 \int_{Q_t} |[\boldsymbol{g}]^k \times \boldsymbol{u}| dx d\tau + \sum_{|\alpha|<2M} \int_{Q_t} |A_\alpha D^\alpha \boldsymbol{u}| dx d\tau + \int_{Q_t} |\boldsymbol{B}| dx d\tau\Big) \\ \le\ & C\Big(1 + \int_{Q_t} |\boldsymbol{g}|^2 dx d\tau + \int_{Q_t} |\boldsymbol{u}|^2 dx d\tau + \int_{Q_t} |\widetilde{\triangle}^M \boldsymbol{u}|^2 dx d\tau\Big) \le C C_m.\end{aligned} \tag{3.6}$$

Now we have

$$\|\boldsymbol{u}\|^2_{L_\infty(0,T;H^M(\Omega))} + \|\boldsymbol{u}_t\|_{L_1(Q_T)} + \|\widetilde{\triangle}^M \boldsymbol{u}\|^2_{L_2(Q_T)} \le C\Big(\|D_x^M \boldsymbol{u}_0\|^2_{L_2}, \|\boldsymbol{B}\|_{L_2}, \|\boldsymbol{g}\|_{L_2}, \sum_{|\alpha|<2M} \|A_\alpha\|_{L_\infty}\Big),$$

where C also depends on ϵ, a_α, N, M, Ω and T and other given data , it is independent of k. i.e. by Lemma 2.1, (3.5) has been obtained.

By the well known theory of linear parabolic system (see [13, 14]),noticing the bound of $[\boldsymbol{g}]^k$, the solution for problem (3.2)-(3.4) exists and is in $\mathcal{G}$. Its norm is independent of k. The other part of the proof is standard as the one of Lemma 1 in [8].

Therefore, we have the existence and uniqueness results of the system (3.2) - (3.4).

Remark 3.3 For $\boldsymbol{g} \in L_2(Q_T)$, by a limit process, it is not difficult to show in Q_T,

$$\boldsymbol{u}_t = (-1)^{M+1} \epsilon \widetilde{\triangle}^M \boldsymbol{u} + \boldsymbol{g}(x,t) \times \Big(\sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{u} + c_0 \boldsymbol{u}\Big) + \sum_{|\alpha|<2M} A_\alpha(x,t) D^\alpha \boldsymbol{u} + \boldsymbol{B}(x,t), \tag{3.7}$$

with boundary-initial condition (3.3)-(3.4) has a solution in $\mathcal{G}$.

Remark 3.4 If $N \ge 2$, $\boldsymbol{g} \in L_{2N/N-2}(Q_T)$, the solution $\boldsymbol{u}$ of the problem (3.2) (or (3.7)), (3.3), (3.4) belongs to the space $W^1_{p'}(0,T; L_{p'}(\Omega))$ for $p' = N/(N-1)$. If $\boldsymbol{g} \in L_\infty(Q_T)$, the solution $\boldsymbol{u}$ belongs to $\mathcal{G} \cap H^1(0,T; L_2(\Omega))$.

Now consider a map $\mathbf{T}_\sigma(\boldsymbol{u})$ in space $\mathcal{B} = L_2(Q_T)$. For any $\boldsymbol{u} \in \mathcal{B}$ and $\sigma \in [0,1]$, the image $\mathbf{T}_\sigma(\boldsymbol{u})$ of this map is the solution of the following problem

$$\begin{aligned}\boldsymbol{Z}_{\epsilon k t} = (-1)^{M+1} \epsilon \Big(\sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{Z}_{\epsilon k} + a_0 \boldsymbol{Z}_{\epsilon k}\Big) + \sigma [\boldsymbol{u}]^k \times \Big(\sum_{m=1}^{M} \widetilde{\triangle}^m \boldsymbol{Z}_{\epsilon k} + a_0 \boldsymbol{Z}_{\epsilon k}\Big) \\ + \sigma [\boldsymbol{u}]^k \times \boldsymbol{h}(\boldsymbol{u}) + \sigma \boldsymbol{\xi}(\boldsymbol{u}, x, t), \quad \text{in } Q_T,\end{aligned} \tag{3.8}$$

$$\frac{d^l \boldsymbol{Z}_{\epsilon k}}{d\gamma^l}\Big|_{\partial\Omega} = 0, \qquad l = 1, \cdots, M-1, \quad \text{on } S_T, \tag{3.9}$$

$$\boldsymbol{Z}_{\epsilon k}(x,0) = \sigma \boldsymbol{Z}_0(x), \quad \text{on } \Omega. \tag{3.10}$$

where $\boldsymbol{Z}_0$ satisfies (H1), $\boldsymbol{h}(\cdot)$ and $\boldsymbol{\xi}(\cdot, x, t)$ satisfy (H2) or (H2a).

For the sake of continuity argument of $\mathbf{T}_\sigma$, we first consider the problem with $[\cdot]^k$, then let k goes to infinity to obtain the solution for problem (1.11),(1.5)-(1.6).

From Proposition 3.2, we know that the solution of the problem (3.8)-(3.10) exists in $\mathcal{G}$ provided $(-1)^M a_0 > (-1)^M c_0$ where c_0 is defined in Proposition 3.2, depending only on the given data a_α, for $0 < |\alpha| \le M$. $\mathcal{G}$ is compactly imbedded into $\mathcal{B}$ by Corollary 3.1. That is, the map $\mathbf{T}_\sigma(\boldsymbol{u})$ is well defined and compact. For any $k > 0$, the map is continuous by noticing $|[u_1]^k - [u_2]^k| \le |u_1 - u_2|$, we have,

$$\|\boldsymbol{Z}_{\epsilon k1} - \boldsymbol{Z}_{\epsilon k2}\|_{\mathcal{B}} \le C\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathcal{B}}.$$

Also, $\mathbf{T}_\sigma(\boldsymbol{u}) = 0$ when $\sigma = 0$. Then, if there is a uniform estimate for every fixed point $\boldsymbol{Z}_{\epsilon k}$ in $\mathcal{B}$, the map $\mathbf{T}_\sigma$ has at least a fixed point in $\mathcal{B}$ by Leray-Schauder's fixed point theorem. i.e. for $\sigma = 1$, the fixed point $\boldsymbol{Z}_{\epsilon k}$ satisfies

$$\begin{aligned}\boldsymbol{Z}_{\epsilon kt} = (-1)^{M+1}\epsilon\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big) + [\boldsymbol{Z}_{\epsilon k}]^k\times\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)\\ +[\boldsymbol{Z}_{\epsilon k}]^k\times\boldsymbol{h}(\boldsymbol{Z}_{\epsilon k}) + \boldsymbol{\xi}(\boldsymbol{Z}_{\epsilon k}, x, t), \quad \text{in } Q_T,\end{aligned} \tag{3.11}$$

with boundary-initial condition (3.9)-(3.10).

Now, let us find out the uniform estimate for the map.

Take (3.11) to scalar product by $(-1)^M(\sum_{m=1}^M \widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k} + a_0\boldsymbol{Z}_{\epsilon k})$, where $(-1)^M a_0$ is a large positive constant independent of ϵ and k, which will be chosen later. Then integrate the result over Q_t by parts. Let us estimate terms one by one:

$$\begin{aligned} s\int_{Q_t}\boldsymbol{Z}_{\epsilon kt}\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)dxd\tau = \frac{s}{2}\int_0^t d\tau\frac{d}{d\tau}\int_\Omega\Big(\sum_{|\alpha|\le M}(-1)^{\alpha}a_\alpha|D^\alpha\boldsymbol{Z}_{\epsilon k}|^2 + a_0|\boldsymbol{Z}_{\epsilon k}|^2\Big)dx\\ \ge \frac{s}{2}\int_\Omega a_0|\boldsymbol{Z}_{\epsilon k}|^2dx + \frac{s}{2}\Big((-1)^M\sum_{|\alpha|=M} + (-1)^{|\alpha|}\sum_{0<|\alpha|<M}\Big)a_\alpha\|D_x^\alpha\boldsymbol{Z}_{\epsilon k}\|^2_{L_2(\Omega)} - C(1+\|\boldsymbol{Z}_0\|_{H^M}),\end{aligned}\tag{3.12}$$

where $s = (-1)^M$, C is independent of k and σ.

The third term in the left hand side of (3.12) containing several sub terms whose signs are changing, can be bounded by

$$s(-1)^{|\alpha|}\sum_{|\alpha|<M}a_\alpha\|D_x^\alpha\boldsymbol{Z}_{\epsilon k}(\cdot,t)\|^2_{L_2(\Omega)} \ge \sum_{|\alpha|=M}\frac{a_M}{2}\|D_x^\alpha\boldsymbol{Z}_{\epsilon k}(\cdot,t)\|^2_{L_2(\Omega)} - \hat{C}\|\boldsymbol{Z}_{\epsilon k}(\cdot,t)\|^2_{L_2(\Omega)} \tag{3.13}$$

with the help of Lemma (2.2), where $\hat{C} = C(M, N, a_M, a_m)$ is independent of ϵ and k. Here $a_M = \min_{|\alpha|=M}\{a_\alpha\}$, $a_m = \max_{0<|\alpha|<M}\{a_\alpha\}$. So, we can choose a_0 such that $(-1)^M a_0 > 2C(M, N, a_M, a_m)$.

$$\epsilon\int_{Q_t}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k} + a_0\boldsymbol{Z}_{\epsilon k}\Big)\cdot\Big(\sum_{m=1}^{m}\widetilde{\triangle}^M\boldsymbol{Z}_{\epsilon k} + a_0\boldsymbol{Z}_{\epsilon k}\Big)dx \ge 0 \tag{3.14}$$

We need pay special attention on the estimate of the free item. Denoting

$$\boldsymbol{H}(\boldsymbol{\beta})=\Big(\int_0^{\beta_1}h_1(\eta)d\eta,\int_0^{\beta_2}h_2(\eta)d\eta,\int_0^{\beta_3}h_3(\eta)d\eta\Big),\tag{3.15}$$

for $\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)$ and using (3.11) and (H2) or (H2a) we have the following calculation:

$$\begin{aligned}&\Big|\int_{Q_t}\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)\cdot[\boldsymbol{Z}_{\epsilon k}]^k\times\boldsymbol{h}(\boldsymbol{Z}_{\epsilon k})dxd\tau\Big| \qquad (3.16)\\ &=\Big|\int_{Q_t}\boldsymbol{h}(\boldsymbol{Z}_{\epsilon k})\cdot\Big([\boldsymbol{Z}_{\epsilon k}]^k\times\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)\Big)dxd\tau\Big|\\ &=\Big|\int_{Q_t}\boldsymbol{h}(\boldsymbol{Z}_{\epsilon k})\cdot\Big(\boldsymbol{Z}_{\epsilon kt}+(-1)^M\epsilon\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)-\boldsymbol{\xi}(\boldsymbol{Z}_{\epsilon k},x,t)\Big)dxd\tau\Big|\\ &\le\Big|\int_\Omega\boldsymbol{H}(\boldsymbol{Z}_{\epsilon k})dx-\int_\Omega\boldsymbol{H}(\boldsymbol{Z}_0)dx\Big|+\frac{\epsilon}{4}\int_{Q_t}\Big|\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big|^2dxd\tau+C\Big(\int_{Q_t}|\boldsymbol{Z}_{\epsilon k}|^2dxd\tau+1\Big)\\ &\le C_0\Big(\int_\Omega|\boldsymbol{Z}_{\epsilon k}|^2dx+\int_\Omega|\boldsymbol{Z}_0|^2dx\Big)+\frac{\epsilon}{4}\int_{Q_t}\Big|\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big|^2dxd\tau+C\Big(\int_{Q_t}|\boldsymbol{Z}_{\epsilon k}|^2dxd\tau+1\Big),\end{aligned}$$

where C_0 only depends on the given data, it is independent of k, σ and ϵ. If we choose $(-1)^Ma_0>4C_0$, the first and second terms of the right side of (3.16) will be eliminated. C depends on the given data, it is independent of k and σ. With all conditions above, we should choose a_0 such that

$$(-1)^Ma_0>4\max\{\hat{C},C_0,c_0\},\tag{3.17}$$

where $\hat{C}$ is given in (3.13), C_0 is defined in (3.16) and c_0 is defined in Proposition 3.2, therefore a_0 can be chosen and is independent of ϵ, k and σ. At last,

$$\begin{aligned}&\Big|\int_{Q_t}\boldsymbol{\xi}(\boldsymbol{Z}_{\epsilon k},x,t)\cdot\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)dxd\tau\Big| \qquad (3.18)\\ &\quad\le\frac{\epsilon}{4}\int_{Q_t}\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)\cdot\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)dxd\tau+C_\epsilon\Big(1+\int_0^t\|\boldsymbol{Z}_{\epsilon k}\|^2_{L_2(\Omega)}d\tau\Big),\end{aligned}$$

where C_ϵ depends on ϵ and is independent of k and σ.

All together from (3.12)-(3.18) we can obtain

$$\begin{aligned}&\frac{(-1)^M}{4}\int_\Omega a_0|\boldsymbol{Z}_{\epsilon k}|^2dx+\frac14\sum_{|\alpha|=M}a_\alpha\|D_x^\alpha\boldsymbol{Z}_{\epsilon k}\|^2_{L_2(\Omega)}+\frac{\epsilon}{2}\int_{Q_t}\Big|\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big|^2dxd\tau\\ &\quad\le C_\epsilon\Big(1+\|\boldsymbol{Z}_0\|^2_{H^M(\Omega)}+\int_0^t\|\boldsymbol{Z}_{\epsilon k}\|^2_{L_2(\Omega)}d\tau\Big), \qquad (3.19)\end{aligned}$$

where C_ϵ depends on the norm of $\boldsymbol{Z}_0$ given by (H1), the norm of $\boldsymbol{f}$ given by (H2) or (H2a) and other given data. It depends on ϵ, but is independent of k and σ.

The left hand side of (3.19) is positive. By Gronwall inequality it yields

$$\max_{0\le t\le T}\Big(\|\boldsymbol{Z}_{\epsilon k}\|^2_{L_2(\Omega)}+\|D_x^M\boldsymbol{Z}_{\epsilon k}\|^2_{L_2(\Omega)}\Big)+\frac{\epsilon}{2}\|\boldsymbol{Z}_{\epsilon k}\|^2_{H^{2M}(Q_T)}\le C_\epsilon. \tag{3.20}$$

By the imbedding theorem, we have $\|\boldsymbol{Z}_{\epsilon k}\|_{\mathcal{B}}\le C_\epsilon$ for all fixed point $\boldsymbol{Z}_{\epsilon k}$, where C_ϵ depends on ϵ but is independent of k and σ, i.e. $\boldsymbol{Z}_{\epsilon k}$ has an uniform estimate in space $\mathcal{B}$. That is, the map $\mathbf{T}_\sigma$ has at least one fixed point in space $\mathcal{B}$ when $\sigma=1$. That is, the solution of the problem (3.11), (1.5)-(1.6) exists.

Next, let $k\to\infty$, we intend to prove that the limit $\boldsymbol{Z}_\epsilon$ of $\boldsymbol{Z}_{\epsilon k}$ is just the solution of the problem (1.11), (1.5), (1.6).

In fact, for $\boldsymbol{Z}_{\epsilon k}\in\mathcal{B}$, with uniform (3.20), we also can obtain $\boldsymbol{Z}_{\epsilon kt}\in L_1(Q_T)$ and $\boldsymbol{Z}_{\epsilon k}\in L_\infty(0,T;H^M(\Omega))\cap L_2(0,T;H^{2M}(\Omega))$, whose norms are independent of k. Thus $\boldsymbol{Z}_{\epsilon k}\in\mathcal{G}$ uniformly with respect to k. By Corollary 3.1, $\mathcal{G}$ is compactly imbedded into $\mathcal{B}$, so that, noticing (3.20) to be uniform with respect to k, we have, when $k\to\infty$,

$$\begin{aligned}
&\boldsymbol{Z}_{\epsilon k}\longrightarrow\boldsymbol{Z}_\epsilon, \quad \text{in } \mathcal{B} \text{ strongly}; \qquad [\boldsymbol{Z}_{\epsilon k}]^k\longrightarrow\boldsymbol{Z}_\epsilon, \quad \text{in } \mathcal{B} \text{ strongly};\\
&\boldsymbol{Z}_{\epsilon k}\longrightarrow\boldsymbol{Z}_\epsilon, \quad \text{in } \mathcal{G} \text{ weakly}; \qquad \boldsymbol{Z}_{\epsilon k}\longrightarrow\boldsymbol{Z}_\epsilon, \quad \text{a.e. in } Q_T.
\end{aligned}$$

That is, $\boldsymbol{Z}_\epsilon\in\mathcal{G}$. Thus, for any $\boldsymbol{\phi}\in C^\infty(Q_T)$, as $k\to\infty$, $[\boldsymbol{Z}_{\epsilon k}]^k,\boldsymbol{Z}_{\epsilon k}\to\boldsymbol{Z}_\epsilon$ in $\mathcal{B}$, and

$$\begin{aligned}
&\int_{Q_T}\Big[-\boldsymbol{\phi}\cdot(\boldsymbol{Z}_{\epsilon kt}-\boldsymbol{Z}_{\epsilon t})+\boldsymbol{\phi}\cdot(-1)^{M+1}\epsilon\Big(\sum_{m=1}^M\widetilde{\triangle}^m(\boldsymbol{Z}_{\epsilon k}-\boldsymbol{Z}_\epsilon)+a_0(\boldsymbol{Z}_{\epsilon k}-\boldsymbol{Z}_\epsilon)\Big)\\
&\quad+\boldsymbol{\phi}\cdot\Big([\boldsymbol{Z}_{\epsilon k}]^k-\boldsymbol{Z}_\epsilon\Big)\times\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_{\epsilon k}+a_0\boldsymbol{Z}_{\epsilon k}\Big)+\boldsymbol{\phi}\cdot\boldsymbol{Z}_\epsilon\times\Big(\sum_{m=1}^M\widetilde{\triangle}^m(\boldsymbol{Z}_{\epsilon k}-\boldsymbol{Z}_\epsilon)+a_0(\boldsymbol{Z}_{\epsilon k}-\boldsymbol{Z}_\epsilon)\Big)\\
&\quad+\boldsymbol{\phi}\cdot\Big([\boldsymbol{Z}_{\epsilon k}]^k-\boldsymbol{Z}_\epsilon\Big)\times\boldsymbol{h}(\boldsymbol{Z}_{\epsilon k})+\boldsymbol{\phi}\cdot\boldsymbol{Z}_\epsilon\times\Big(\boldsymbol{h}(\boldsymbol{Z}_{\epsilon k})-\boldsymbol{h}(\boldsymbol{Z}_\epsilon)\Big)+\boldsymbol{\phi}\cdot\Big(\boldsymbol{\xi}(\boldsymbol{Z}_{\epsilon k})-\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon)\Big)\Big]dxdt\to 0,\\
&\int_{Q_T}\Big[\boldsymbol{\phi}\cdot\Big([\boldsymbol{Z}_{\epsilon k}]^k\times a_0\boldsymbol{Z}_{\epsilon k}\Big)\Big]dxdt\to\int_{Q_T}\Big[\boldsymbol{\phi}\cdot\Big(\boldsymbol{Z}_\epsilon\times a_0\boldsymbol{Z}_\epsilon\Big)\Big]dxdt=0.
\end{aligned}$$

That is, $\boldsymbol{Z}_\epsilon$ is the solution of the problem (1.11), (1.5)-(1.6). We have

Theorem 3.5 *Problem* (1.11), (1.5)-(1.6) *admits at least one solution in space* $\mathcal{G}$.

4. The ϵ-Independent Estimates

In this section, we want to obtain the following estimates, which are the key estimates in this paper:

$$\sup_{0\le t\le T}\|\boldsymbol{Z}_\epsilon(\cdot,t)\|_{H^M(\Omega)}\le C, \tag{4.1}$$

$$\sup_{0\le t\le T}\|\boldsymbol{Z}_{\epsilon t}(\cdot,t)\|_{H^{-(M+J)}(\Omega)}\le C, \tag{4.2}$$

where C is independent of ϵ, J is defined in (1.10).

First, we suppose (H2) is satisfied. Now let us estimate (4.1).

Taking (1.11) get the scalar product by the vector $\boldsymbol{Z}_\epsilon$, then integrating the result in domain Q_t, we have

$$\int_{Q_t}\boldsymbol{Z}_\epsilon\cdot\boldsymbol{Z}_{\epsilon t}dxd\tau=(-1)^{M+1}\epsilon\int_{Q_t}\boldsymbol{Z}_\epsilon\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau \tag{4.3}$$
$$+\int_{Q_t}\boldsymbol{Z}_\epsilon\cdot\boldsymbol{Z}_\epsilon\times\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon dxd\tau+\int_{Q_t}\boldsymbol{Z}_\epsilon\cdot\boldsymbol{Z}_\epsilon\times\boldsymbol{h}(\boldsymbol{Z}_\epsilon)dxd\tau+\int_{Q_t}\boldsymbol{Z}_\epsilon\cdot\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)dxd\tau,$$

where

$$(-1)^{M+1}\epsilon\int_{\Omega}\boldsymbol{Z}_\epsilon\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau \tag{4.4}$$
$$\leq-\epsilon C\Big[\sum_{|\alpha|=M}\frac{a_\alpha}{2}\|D_x^\alpha\boldsymbol{Z}_\epsilon\|_2^2-C(M,N,a_M,a_m)\|\boldsymbol{Z}_\epsilon\|_2^2+(-1)^Ma_0\|\boldsymbol{Z}_\epsilon\|_2^2\Big]\leq 0$$

as long as (3.17) holds. So that, noting the second and third term in the right side of the inequality (4.3) equal to 0 and take count on (H2), $\int_\Omega|\boldsymbol{Z}_\epsilon|^2dx\leq C\Big(1+\int_0^t\int_\Omega|\boldsymbol{Z}_\epsilon|^2dxd\tau\Big)$. Thus by Gronwell Inequality we have ,

$$\max_{0\leq t\leq T}\int_\Omega|\boldsymbol{Z}_\epsilon|^2dx\leq C, \tag{4.5}$$

where C is independent of ϵ.

Taking (1.11) get the scalar product by the vector $(-1)^M(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon)$, then integrating the result in the domain Q_t, we have

$$\int_{Q_t}(-1)^M\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\boldsymbol{Z}_{\epsilon t}dxd\tau \tag{4.6}$$
$$=-\epsilon(-1)^{2M}\int_{Q_t}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau$$
$$+(-1)^M\Big[\int_{Q_t}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\boldsymbol{Z}_\epsilon\times\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon dxd\tau$$
$$+\int_{Q_t}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\boldsymbol{Z}_\epsilon\times\boldsymbol{h}(\boldsymbol{Z}_\epsilon)dxd\tau+\int_{Q_t}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)dxd\tau\Big],$$

where the left side of the above inequality can be estimated as

$$\int_{Q_t}(-1)^M\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\boldsymbol{Z}_{\epsilon t}dxd\tau$$
$$=\frac{1}{2}\int_\Omega\Big(\sum_{|\alpha|=1}^{M}(-1)^{M+|\alpha|}a_\alpha(|D_x^\alpha\boldsymbol{Z}_\epsilon|^2-|D_x^\alpha\boldsymbol{Z}_0|^2)+(-1)^Ma_0(|\boldsymbol{Z}_\epsilon|^2-|\boldsymbol{Z}_0|^2)\Big)dx$$

Treat it as (3.13) and (4.4), then the first item of the right side of above inequality can be found a low bound as

$$\begin{aligned}&\int_\Omega\Big(\sum_{|\alpha|=1}^{M}(-1)^{M+|\alpha|}a_\alpha|D_x^\alpha \boldsymbol{Z}_\epsilon|^2+(-1)^M a_0|\boldsymbol{Z}_\epsilon|^2\Big)dx\\
&\quad\geq\Big[\sum_{|\alpha|=M}\frac{a_\alpha}{2}\|D_x^\alpha \boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)}-C(M,N,a_M,a_m)\|\boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)}+(-1)^M a_0\|\boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)}\Big]\\
&\quad\geq\Big[\sum_{|\alpha|=M}\frac{a_\alpha}{2}\|D_x^\alpha \boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)}+(-1)^M\frac{a_0}{2}\|\boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)}\Big]>0 \end{aligned}\tag{4.7}$$

$$-\epsilon\int_{Q_t}(-1)^{2M}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau\leq 0,\tag{4.8}$$

$$\int_{Q_t}(-1)^M\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\boldsymbol{Z}_\epsilon\times\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon dxd\tau=0.\tag{4.9}$$

By (H2) and (1.11), for the free term, denoting $\boldsymbol{H}(\cdot)$ as defined in (3.15), we have

$$\begin{aligned}&\int_{Q_t}\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon\cdot\boldsymbol{Z}_\epsilon\times\boldsymbol{h}(\boldsymbol{Z}_\epsilon)dxd\tau=\int_{Q_t}\boldsymbol{h}(\boldsymbol{Z}_\epsilon)\cdot\Big(\boldsymbol{Z}_\epsilon\times\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon\Big)dxd\tau \qquad (4.10)\\
&=\int_{Q_t}\boldsymbol{h}(\boldsymbol{Z}_\epsilon)\cdot\Big(\boldsymbol{Z}_{\epsilon t}+\epsilon(-1)^M\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)-\boldsymbol{Z}_\epsilon\times\boldsymbol{h}(\boldsymbol{Z}_\epsilon)-\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)\Big)dxd\tau\\
&=\int_\Omega(\boldsymbol{H}(\boldsymbol{Z}_\epsilon)-\boldsymbol{H}(\boldsymbol{Z}_0))dx+\epsilon(-1)^M\int_{Q_t}\boldsymbol{h}(\boldsymbol{Z}_\epsilon)\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau-\int_{Q_t}\boldsymbol{h}\cdot\boldsymbol{\xi}dxd\tau,\end{aligned}$$

where, by (H2) the last term of above (4.10) is bounded by $C(1+\int_{Q_T}|\boldsymbol{Z}_\epsilon|^2dxd\tau)$, then bounded by a constant which independent of ϵ from (4.5). Take count on (4.5), the first term in the right side of (4.10) can be estimated as

$$\Big|\int_\Omega\boldsymbol{H}(\boldsymbol{Z}_\epsilon)dx-\int_\Omega\boldsymbol{H}(\boldsymbol{Z}_0)dx\Big|\leq C\Big(1+\|\boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)}+\|\boldsymbol{Z}_0\|^2_{L_2(\Omega)}\Big)\leq C'\tag{4.11}$$

where C' is independent of ϵ and by (H2) the second term of the right side of (4.10) has the estimate

$$\begin{aligned}&\Big|\,\epsilon(-1)^M\int_{Q_t}\boldsymbol{h}(\boldsymbol{Z}_\epsilon)\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau\Big| \qquad (4.12)\\
&\quad\leq\frac{\epsilon}{2}\int_{Q_t}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau+\frac{\epsilon}{2}\int_{Q_t}|\boldsymbol{h}(\boldsymbol{Z}_\epsilon)|^2dxd\tau\\
&\quad\leq\frac{\epsilon}{2}\int_{Q_t}\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau+C\Big(1+\int_{Q_t}|\boldsymbol{Z}_\epsilon|^2dxd\tau\Big)\end{aligned}$$

The first term in the right side of the above inequality (4.12) can be eliminated by (4.8) and the second term is bounded by a constant which independent of ϵ from (4.5).

Now let us treat the last term of the right side of (4.10). By (H2), we have

$$\begin{aligned}&\Big|\int_{Q_t}\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dxd\tau\Big| \qquad (4.13)\\ &=\Big|\int_{Q_t}\sum_{m=1}^{M}D^m\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)\cdot\sum_{m=1}^{M}D^m\boldsymbol{Z}_\epsilon dxd\tau+a_0\int_{Q_t}\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)\cdot\boldsymbol{Z}_\epsilon dxd\tau\Big|\\ &\le C\int_{Q_t}\Big(\sum_{|\alpha|=M}|D_x^\alpha\boldsymbol{Z}_\epsilon|^2+|\boldsymbol{Z}_\epsilon|^2+1\Big)dxd\tau\end{aligned}$$

where C depends on the $L_\infty(0,T;H^M(\Omega))$ norms of ξ_{ij}. It is independent of ϵ.

All together of above estimates, we have

$$\sum_{|\alpha|=M}\|D_x^\alpha\boldsymbol{Z}_\epsilon\|_{L_2}^2(t)+\|\boldsymbol{Z}_\epsilon\|_{L_2}^2(t)\le C\Big(1+\int_0^t[\sum_{|\alpha|=M}\|D_x^\alpha\boldsymbol{Z}_\epsilon\|_{L_2}^2(\tau)+\|\boldsymbol{Z}_\epsilon\|_{L_2}^2(\tau)]d\tau\Big),$$

which leads to (4.1) by Gronwall Inequality, where C depends only on the given data. It is independent of ϵ.

Secondly, let us consider (4.2).

Now, choose $\boldsymbol{\psi}(x)\in H_0^{M+J}(\Omega)$, where J is defined in (1.10). Take (1.11) to scalar product by $\boldsymbol{\psi}(x)$, then integrate it over Ω, there comes

$$\begin{aligned}\int_\Omega\boldsymbol{\psi}(x)\cdot\boldsymbol{Z}_{\epsilon t}(x,t)dx&=\int_\Omega(-1)^{M+1}\epsilon\boldsymbol{\psi}\cdot\Big(\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)dx+\int_\Omega\boldsymbol{\psi}\cdot\boldsymbol{Z}_\epsilon\times\sum_{m=1}^{M}\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon dx\\ &\quad+\int_\Omega\boldsymbol{\psi}\cdot\boldsymbol{Z}_\epsilon\times\boldsymbol{h}(\boldsymbol{Z}_\epsilon)dx+\int_\Omega\boldsymbol{\psi}\cdot\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)dx\\ &\le\epsilon C\|\boldsymbol{\psi}\|_{H_0^M(\Omega)}\|\boldsymbol{Z}_\epsilon\|_{H_0^M(\Omega)}+C\sum_{0<|\beta|\le M}\|D_x^\beta\boldsymbol{\psi}\|_{L_\infty(\Omega)}\sum_{|\alpha|\le M}\|D_x^\alpha\boldsymbol{Z}_\epsilon\|_{L_2(\Omega)}^2\\ &\quad+C\Big(\|\boldsymbol{Z}_\epsilon\|_{L_2(\Omega)}+1\Big)\|\boldsymbol{\psi}\|_{L_\infty(\Omega)}\le C\|\boldsymbol{\psi}\|_{H_0^{M+J}(\Omega)}.\end{aligned}$$

where C depends only on the given data. It is independent of ϵ.

Since $H_0^{M+J}\to C_0^M$, for all M, $\boldsymbol{Z}_{\epsilon t}\in H^{-(M+J)}(\Omega)$, a.e.in $[0,T]$. (4.2) follows, and

$$\boldsymbol{Z}_\epsilon\in\mathcal{Z}=L_\infty\Big(0,T;H_0^M(\Omega)\Big)\cap W_\infty^1\Big(0,T;H^{-(M+J)}(\Omega)\Big), \qquad (4.14)$$

and $\boldsymbol{Z}_\epsilon$ has a uniform bound in space $\mathcal{Z}$. By 2) of Lemma 2.3, space $\mathcal{Z}$ is imbedded into space $L_\infty(0,T;H^{M-1}(\Omega))$ compactly.

Now we consider a special case $M=1$, when the hypothesis on the free term $\boldsymbol{f}$ can be more general as (H2a). Actually, viewing all estimates we have done above, we can find out that the most difficult part is to get an ϵ independent bound for the term $\Big|\int_\Omega\Big(\sum_{m=1}^M\widetilde{\triangle}^m\boldsymbol{Z}_\epsilon+a_0\boldsymbol{Z}_\epsilon\Big)\cdot\boldsymbol{f}(\boldsymbol{Z}_\epsilon)dx\Big|$.

On (H2a), the estimate of $\boldsymbol{Z}_\epsilon\times\boldsymbol{h}(\boldsymbol{Z}_\epsilon)$, the first part of $\boldsymbol{f}$, should be the same as the one of (H2). For the second part $\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)$, where $\boldsymbol{\xi}$ is no more linear with respect to $\boldsymbol{Z}_\epsilon$, should be treated in a different way.

If $\boldsymbol{\xi}$ is nonlinear and the order M is high, it will bring about many trouble terms. However if $M=1$, we can directly calculate it as

$$\begin{aligned}\int_{Q_t} \triangle \boldsymbol{Z}_\epsilon \cdot \boldsymbol{\xi}(\boldsymbol{Z}_\epsilon) dx d\tau &= \sum_{|\alpha|=1} \int_{Q_t} D_x^\alpha \boldsymbol{Z}_\epsilon \cdot D_x^\alpha \boldsymbol{\xi}(\boldsymbol{Z}_\epsilon, x, t) dx d\tau \\ &\le \sum_{|\alpha|=1} \int_{Q_t} \Big(\Big|\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{Z}_\epsilon}\Big| |D_x^\alpha \boldsymbol{Z}_\epsilon|^2 + \sum_{|\alpha|=1} \Big|\frac{\partial \boldsymbol{\xi}}{\partial x_\alpha}\Big| |D_x^\alpha \boldsymbol{Z}_\epsilon| \Big) dx d\tau \\ &\le C\Big(\int_0^t \Big(\sum_{|\alpha|=1} \|D_x^\alpha \boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)} + \|\boldsymbol{Z}_\epsilon\|^2_{L_2(\Omega)} \Big) d\tau + 1 \Big),\end{aligned}$$

where C is independent of ϵ.

So that, in the case of $M=1$ and (H2a), we obtain (4.1) from (4.10) as well. The other estimates should be same or simpler to be obtained.

5. The Limit Process

Now we consider the limit process when ϵ tends to 0. From the result of Section 3, $\boldsymbol{Z}_\epsilon \in \mathcal{G}$ exists and for any 3D vector $\boldsymbol{\phi}(x,t) \in C^1(0,T; C_0^M(\Omega))$, $\boldsymbol{\phi}(x,T) \equiv 0$, we have

$$\begin{aligned}\int_{Q_T} \Big[\boldsymbol{\phi}_t \cdot \boldsymbol{Z}_\epsilon + \epsilon \sum_{|\alpha| \le M} (-1)^{M+|\alpha|+1} a_\alpha D_x^\alpha \boldsymbol{\phi} \cdot D_x^\alpha \boldsymbol{Z}_\epsilon \\ + \sum_{0<|\alpha|\le M} (-1)^{|\alpha|} a_\alpha D_x^\alpha \boldsymbol{Z}_\epsilon \cdot \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \Big(D_x^\beta \boldsymbol{\phi} \times D_x^{\alpha-\beta} \boldsymbol{Z}_\epsilon \Big) \\ + \boldsymbol{\phi} \cdot \boldsymbol{Z}_\epsilon \times \boldsymbol{h}(\boldsymbol{Z}_\epsilon) + \boldsymbol{\phi} \cdot \boldsymbol{\xi}(\boldsymbol{Z}_\epsilon, x, t) \Big] dx dt + \int_\Omega \boldsymbol{\phi}(x,0) \cdot \boldsymbol{Z}_0(x) dx = 0.\end{aligned} \tag{5.1}$$

Since $\boldsymbol{Z}_\epsilon \in \mathcal{Z}$ uniformly with respect to ϵ and $\mathcal{Z}$ is compact in $L_\infty(0,T;H^{M-1}(\Omega))$, when $\epsilon \to 0$, there exists $\boldsymbol{Z} \in \mathcal{Z}$ such that

$\boldsymbol{Z}_\epsilon \longrightarrow \boldsymbol{Z}$ in $L_\infty(0,T;H^{M-1}(\Omega))$ strongly; $\boldsymbol{Z}_\epsilon \longrightarrow \boldsymbol{Z}$ in $L_\infty(0,T;H^M(\Omega))$ weakly;
$\boldsymbol{Z}_\epsilon \longrightarrow \boldsymbol{Z}$ a.e. in Q_T.

So, when $\epsilon \to 0$,

$$\int_{Q_T} \boldsymbol{\phi}_t \cdot \boldsymbol{Z}_\epsilon dx dt \longrightarrow \int_{Q_T} \boldsymbol{\phi}_t \cdot \boldsymbol{Z} dx dt;$$

$$\epsilon \Big| \int_{Q_T} \sum_{|\alpha|\le M} (-1)^{M+|\alpha|+1} a_\alpha D_x^\alpha \boldsymbol{\phi} \cdot D_x^\alpha \boldsymbol{Z}_\epsilon dx dt \Big| \le \epsilon C \max_{0\le t\le T} \|\boldsymbol{Z}_\epsilon\|_{H^M(\Omega)} \longrightarrow 0;$$

$$\begin{aligned}&\int_{Q_T} \sum_{0<|\alpha|\le M} (-1)^{|\alpha|} a_\alpha D_x^\alpha \boldsymbol{Z}_\epsilon \cdot \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \Big(D_x^\beta \boldsymbol{\phi} \times D_x^{\alpha-\beta} \boldsymbol{Z}_\epsilon \Big) dx dt \\ &\quad = \int_{Q_T} \sum_{0<|\alpha|\le M} (-1)^{|\alpha|} a_\alpha D_x^\alpha \boldsymbol{Z}_\epsilon \cdot \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} \Big(D_x^\beta \boldsymbol{\phi} \times D_x^{\alpha-\beta} (\boldsymbol{Z}_\epsilon - \boldsymbol{Z}) \Big) dx dt\end{aligned}$$

$$
\begin{aligned}
&+\int_{Q_T}\sum_{0<|\alpha|\le M}(-1)^{|\alpha|}a_\alpha D_x^\alpha(\boldsymbol{Z}_\epsilon-\boldsymbol{Z})\cdot\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\Big(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}\boldsymbol{Z}\Big)dxdt\\
&+\int_{Q_T}\sum_{0<|\alpha|\le M}(-1)^{|\alpha|}a_\alpha D_x^\alpha\boldsymbol{Z}\cdot\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\Big(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}\boldsymbol{Z}\Big)dxdt\\
&\longrightarrow\int_{Q_T}\sum_{0<|\alpha|\le M}(-1)^{|\alpha|}a_\alpha D_x^\alpha\boldsymbol{Z}\cdot\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\Big(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}\boldsymbol{Z}\Big)dxdt;
\end{aligned}
$$

$$
\begin{aligned}
&\int_{Q_T}\boldsymbol{\phi}\cdot\boldsymbol{Z}_\epsilon\times\boldsymbol{h}(\boldsymbol{Z}_\epsilon)dxdt\longrightarrow\int_{Q_T}\boldsymbol{\phi}\cdot\boldsymbol{Z}\times\boldsymbol{h}(\boldsymbol{Z})dxdt;\\
&\int_{Q_T}\boldsymbol{\phi}\cdot\boldsymbol{\xi}(\boldsymbol{Z}_\epsilon,x,t)dxdt\longrightarrow\int_{Q_T}\boldsymbol{\phi}\cdot\boldsymbol{\xi}(\boldsymbol{Z},x,t)dxdt.
\end{aligned}
$$

The third limit above should be treated carefully, it holds because when $\beta=0$,

$$
D_x^\alpha\boldsymbol{Z}_\epsilon\cdot(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}\boldsymbol{Z}_\epsilon)=D_x^\alpha\boldsymbol{Z}\cdot(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}\boldsymbol{Z})=0,
$$

so that the first term of the second equation of it can be bounded by

$$
\begin{aligned}
&\Big|\int_{Q_T}\sum_{0<|\alpha|\le M}(-1)^{|\alpha|}a_\alpha D_x^\alpha\boldsymbol{Z}_\epsilon\cdot\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\Big(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}(\boldsymbol{Z}_\epsilon-\boldsymbol{Z})\Big)dxdt\Big|\\
&\le C\int_0^T\|\boldsymbol{Z}_\epsilon\|_{H^M(\Omega)}\|\boldsymbol{Z}_\epsilon-\boldsymbol{Z}\|_{H^{M-1}(\Omega)}dt\longrightarrow 0,
\end{aligned}
$$

and the second term turns to

$$
\int_{Q_T}\sum_{0<|\alpha|\le M}(-1)^{|\alpha|}a_\alpha D_x^\alpha(\boldsymbol{Z}_\epsilon-\boldsymbol{Z})\cdot\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\Big(D_x^\beta\boldsymbol{\phi}\times D_x^{\alpha-\beta}\boldsymbol{Z}\Big)dxdt\longrightarrow 0.
$$

Therefore, the limit $\boldsymbol{Z}$ is just the weak solution of Problem (1.4)-(1.6) defined in Definition 1.1. The main Theorem 1.2 and 1.3 have been proved.

6. Some Remarks

Remark 6.1 Linear free term, as in [8], is a special case of (H2)when $\boldsymbol{h}=0$.

Remark 6.2 When $M=1$, the nonlinearity of the free term $\boldsymbol{f}$ can be free of the structure (1.7), when simply let $\boldsymbol{h}=0$. However, in (H2a), the condition for the nonlinearity of $\boldsymbol{\xi}(\boldsymbol{Z})$ cannot cover the case of $\boldsymbol{Z}\times\boldsymbol{h}(\boldsymbol{Z})$. For this reason, when we discuss the case $M=1$, we still let $\boldsymbol{f}$ be as in (1.7) with a nonlinear $\boldsymbol{\xi}(\boldsymbol{Z})$.

Remark 6.3 Instead of (1.5), if the problem has the boundary condition

$$
\frac{d^l\boldsymbol{Z}}{d\gamma^l}\Big|_{\partial\Omega}=0,\quad l=M,\cdots,2M-1,\qquad\text{on }\partial\Omega\times[0,T]\tag{6.1}
$$

then we also have the same results as Theorem 1.2 and Theorem 1.3.

The proof of the theorem does not require many changes with boundary condition (6.1) instead of (1.5). The integral by parts can still be available in this case.

Remark 6.4 If boundary condition (1.5) is replaced by the non-homogeneous one:

$$\frac{d^l \boldsymbol{Z}}{d\gamma^l}\Big|_{\partial\Omega} = \boldsymbol{g}_l, \quad l = 0, \cdots, M-1, \qquad \text{on } \partial\Omega\times[0,T), \tag{6.2}$$

where $\boldsymbol{g}_l$s are in some suitable spaces, we also have the same results as Theorem 1.2 and Theorem 1.3 by transforming the problem into a homogeneous one.

Remark 6.5 If $N < 2M$ is satisfied, the imbedding theorem of H^M to L_∞ is available. The solution of the problem is then in the space $L_\infty(Q_T)$. Moreover, if there exists j, $0 \le j < M$, such that $N < 2(M-j)$, and $N \ge 2(M-j-1)$, the solution also belongs to the space $L_\infty(0,T;C^{j,M-j-\frac{N}{2}}(\Omega))$.

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