NUMERICAL MATHEMATICS: Theory, Methods and Applications
Numer. Math. Theor. Meth. Appl., Vol. **1**, No. 3, pp. 321-339 (2008)

Fourth-Order Splitting Methods for Time-Dependant Differential Equations

Jürgen Geiser*

Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany.

Received 29 October, 2007; Accepted (in revised version) 17 April, 2008

Abstract. This study was suggested by previous work on the simulation of evolution equations with scale-dependent processes, e.g., wave-propagation or heat-transfer, that are modeled by wave equations or heat equations. Here, we study both parabolic and hyperbolic equations. We focus on ADI (alternating direction implicit) methods and LOD (locally one-dimensional) methods, which are standard splitting methods of lower order, e.g. second-order. Our aim is to develop higher-order ADI methods, which are performed by Richardson extrapolation, Crank-Nicolson methods and higher-order LOD methods, based on locally higher-order methods. We discuss the new theoretical results of the stability and consistency of the ADI methods. The main idea is to apply a higher-order time discretization and combine it with the ADI methods. We also discuss the discretization and splitting methods for first-order and second-order evolution equations. The stability analysis is given for the ADI method for first-order time derivatives and for the LOD (locally one-dimensional) methods for second-order time derivatives. The higher-order methods are unconditionally stable. Some numerical experiments verify our results.

AMS subject classifications: 35J60, 35J65, 65M99, 65N12, 65Z05, 74S10, 76R50, 80A20, 80M25

Key words: Partial differential equations, operator-splitting methods, evolution equations, ADI methods, LOD methods, stability analysis, higher-order methods.

1. Introduction

Using the classical operator-splitting methods, e.g., the Strang-Marchuk splitting method, we decouple the differential equation into more basic equations, so that each equation becomes simpler or contains only one operator, see [22, 23, 28]. These methods are often not sufficiently stable and also neglect the physical correlations between the operators, see [3]. The present work develops new efficient higher-order methods based on a stable variant of ADI or LOD methods, see [17, 20]. The decomposition ideas in exponential operators, see [3], which deal with the Sheng-Suzuki theorem, are based on the operator structure and conserve the physical characteristics. We contribute a further idea

*Corresponding author. *Email address:* `geiser@mathematik.hu-berlin.de` (J. Geiser)

that uses stable and conserved lower-order methods, improves them with extrapolation methods and derive new higher-order results. The standard second-order method consequently applies Richardson extrapolation to obtain fourth-order and higher-order results. The physical decoupling in new operators and the deriving of new, strong directions are presented in the applications, see [3,16], and can be included in our theory. We consider a more abstract decomposition method, that is motivated by spatial and temporal directions. The theoretical results are obtained by application of the Neumann linear stability analysis, see [18]. For the LOD method we apply weak formulations for the stability results, see [20]. At the least we obtain higher-order methods in time and space and derive new stability results. We compared our results to methods discussed in [29], which dealt with global extrapolation of the parallel splitting method and obtained at least second and third order. Such methods are simple to implement but are not proposed to fourth order methods. We can also take into account the higher-order discretization in time and space. Our extrapolation methods are applied to lower-order ADI and LOD methods and can balance the underlying higher-order space and time discretizations. Therefore we can expand our splitting method to fourth-order methods in time and space. Moreover, our methods can be used for heat as well as wave equations. The theoretical results are verified by numerical experiments and examine the stability and consistency of the proposed methods.

The paper is organized as follows. A mathematical model and the underlying discretization methods for the heat and wave equation are introduced in Section 2. The splitting method for the evolution equations is given in Section 3. The stability analysis of the higher-order splitting method is given in Section 3.4. We discuss the numerical results in Section 4. Finally, we consider future works in the area of splitting and decomposition methods.

2. Mathematical model and discretization

The study presented below was suggested by various real-life problems whose governing equations are of evolution type. The first group presents the discussion of heat-transfer problems, see, e.g., [13], which are modeled by parabolic equations. The second group presents computational simulation of earthquakes, see, e.g., [4] and the examination of seismic waves, see [1,2]. Their underlying equations are hyperbolic differential equations.

2.1. Heat equation

Further, we have the heat equation, see [13], for which the mathematical equations are given by

$$\partial_t u = D_1(x,y)\,\partial_{xx} u + D_2(x,y)\,\partial_{yy} u + D_3(x,y)\,\partial_{zz} u, \text{ in } \Omega \times [0,T], \tag{2.1}$$

$$u(x,y,0) = u_0(x,y), \text{ on } \Omega, \tag{2.2}$$

The unknown function $u = u(x,t)$ is considered to be in $\Omega \times (0,T) \subset \mathbb{R}^d \times \mathbb{R}$ where the spatial dimension is given by d. The function $\mathbf{D}(x,y) = (D_1(x,y), D_2(x,y), D_3(x,y))^t \in$

$\mathbb{R}^{3,+}$ describes the heat transfer in x, y, z. The function $u_0(x, y)$ is the initial condition for the heat equation.

The boundary conditions are given as

$$u(x, y, t) = o, \text{ on } \partial\Omega \times T: \text{ Dirichlet boundary condition,} \tag{2.3}$$

$$\frac{\partial u(x, y, t)}{\partial n} = 0, \text{ on } \partial\Omega \times T: \text{ Neumann boundary condition,} \tag{2.4}$$

$$\mathbf{D}\nabla u(x, y, t) = u_{out}, \text{ on } \partial\Omega \times T: \text{ outflow boundary condition.} \tag{2.5}$$

2.2. Wave equation

We concentrate on the wave equation, see [20], given by

$$\partial_{tt}\, u \;=\; D_1(x, y)\, \partial_{xx}\, u \;+\; D_2(x, y)\, \partial_{yy}\, u \;+\; D_3(x, y)\, \partial_{zz}\, u, \text{ in } \Omega \times [0, T], \tag{2.6}$$

$$u(x, y, 0) = u_0(x, y),\; u_t(x, y, 0) = u_1(x, y), \text{ on } \Omega, \tag{2.7}$$

The unknown function $u = u(x, t)$ is considered to be in $\Omega \times (0, T) \subset \mathbb{R}^d \times \mathbb{R}$ where the spatial dimension is given by d. The function

$$\mathbf{D}(x, y) = (D_1(x, y), D_2(x, y), D_3(x, y))^t \in \mathbb{R}^{3,+}$$

describes the wave propagation in x, y, z. The functions $u_0(x, y)$ and $u_1(x, y)$ are the initial conditions for the wave equation.

The boundary conditions are given in Eqs. (2.3)-(2.5). We can also deal with flux boundary conditions, see [8].

2.3. Discretization methods

We discuss the discretization methods based on higher-order finite difference schemes. We apply higher-order spatial discretization methods and discuss the time discretization for the parabolic and hyperbolic types.

2.3.1. Spatial discretization methods

For the spatial discretization methods we apply higher-order compact methods (HOC), see [18]. We concentrate on the two-dimensional case, but the three-dimensional case can be done in the same manner.

For our Eqs. (2.1) and (2.6) we can derive the following higher-order spatial discretization methods

$$L_x L_y \tilde{u} = (L_y A_x + L_x A_y) u + \mathcal{O}(h^p), \tag{2.8}$$

where $\tilde{u} = u_t$ (first-order time derivative) or $\tilde{u} = u_{tt}$ (second-order time derivative) and $p = 2, 4$.

We obtain the following operators with second-order in space:

$$L_x = 1, \quad A_x = D_1 \delta_{xx},$$
$$L_y = 1, \quad A_y = D_2 \delta_{yy},$$

with fourth-order in space, we obtain the following operators:

$$L_x = 1 + \frac{h_x^2}{12}\delta_{xx}, \quad A_x = D_1 \delta_{xx},$$
$$L_y = 1 + \frac{h_y^2}{12}\delta_{yy}, \quad A_y = D_2 \delta_{yy},$$

where $h = \max\{h_x, h_y\}$. $\delta_{xx} u_i = (u_{i+2} - 2u_i + u_{i-2})/4h^2$ and δ_{yy} are the central difference operators for the second derivative. For such schemes we can derive fourth-order ADI methods.

A further fourth-order scheme can be given with a nine-point compact scheme. Here we propose the following nine-point compact difference scheme (FOCS), as given in [15]:

$$\partial_{xx} u + \partial u_{yy} = \frac{1}{h^2} \begin{pmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{pmatrix} u + \mathcal{O}(h^4), \tag{2.9}$$

where we obtain an accuracy of $\mathcal{O}(h^4)$.

Remark 2.1. The generalization of the scheme (2.9) can also be done for diffusion-convection equations, see [15].

2.3.2. Time-discretization methods

For the time-discretization methods, we apply the standard discretization methods, such as the Crank-Nicolson scheme for the first order derivative, or the central difference scheme for the second order derivative. We obtain higher-order methods by Richardson extrapolation or weighted methods.

First-order derivative in time For the first-order derivative in time we use the second-order Crank-Nicolson method. We apply Richardson extrapolation to get fourth- and fifth-order methods.

The standard Crank-Nicolson method is given as

$$\begin{aligned} &\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}(f(u^n) + f(u^{n+1})), \\ &u(0) = u_0, \end{aligned} \tag{2.10}$$

where f is the right-hand side of the differential equation and depends on u. Here, u^n is the time-approximated solution at $t^n = n\Delta t$ and u^{n+1} is the time-approximated solution at $t^{n+1} = (n+1)\Delta t$. Further, we have $n \geq 0$ and Δt is the time increment.

The scheme is given as

$$S_{CN}(\Delta t, u^n) = u^{n+1,CN} = u^n + \frac{\Delta t}{2}(f(u^n) + f(u^{n+1})),$$
$$S_{CN}(\Delta t, u^n) = B\left(\frac{\Delta t}{2}, F(\frac{\Delta t}{2}, u^n)\right),$$

where

$$B\left(\frac{\Delta t}{2}, u^{n+1/2}\right) = u^{n+1,B} = u^{n+2} + \frac{\Delta t}{2} f(u^{n+1}),$$
$$F\left(\frac{\Delta t}{2}, u^n\right) = u^{n+1/2,F} = u^n + \frac{\Delta t}{2} f(u^n),$$

and B is the backward and F is the forward discretization scheme.

We can apply an extrapolation method, based on this second-order method, to obtain a higher-order method which we need for our modified ADI method. Here we apply the Richardson extrapolation to the second-order Crank-Nicolson method to obtain higher-order methods.

The idea of the extrapolation method is as follows:

$$D_4(\Delta t, u^n) = u^{n+1,4th} = \frac{4}{3} S_{CN}\left(\frac{\Delta t}{2}, S_{CN}(\frac{\Delta t}{2}, u^n)\right) - \frac{1}{3} S_{CN}(\Delta t, u^n). \tag{2.11}$$

To obtain fifth-order, we have to apply a Richardson extrapolation additionally, see [5,27]:

$$D_5(\Delta t, u^n) = u^{n+1,5th} = \frac{16}{15} D_4\left(\frac{\Delta t}{2}, D_4(\frac{\Delta t}{2}, u^n)\right) - \frac{1}{15} D_4(\Delta t, u^n). \tag{2.12}$$

These methods can be implemented by the basic time-discretization method. Another method, for which we can obtain a higher-order, is given in the following part.

Second-order derivative in time We apply the classical second-order time-discretization method and modify with the weighted method to at least fourth-order method, see [26].

The central difference method is given as

$$U_{tt,i} = \frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2}, \tag{2.13}$$
$$U(0) = u_0, U_t(0) = u_1, \tag{2.14}$$

where the index i refers to the space point x_i and $\Delta t = t^{n+1} - t^n$ is the time step.

The fourth-order time-discretization for the equation

$$U_{tt} = f(U), \tag{2.15}$$
$$U(0) = u_0, U_t(0) = u_1 \tag{2.16}$$

is given as the weighted method, see [26]

$$\frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2} = \eta f(U_i^{n+1}) + (1-2\eta)U_i^n + \eta U_i^{n-1} + \mathcal{O}(\Delta t^4), \tag{2.17}$$

$$U(0) = u_0, U_t(0) = u_1, \tag{2.18}$$

where the index i refers to the space point x_i and $\Delta t = t^{n+1} - t^n$ is the time step.

Remark 2.2. If we take into account higher smoothness of the solution, we can achieve higher-order by using the higher-order discretization terms, see [25]. For our method, we deal with at least fourth-order in time and in space.

3. Splitting methods

One of the classical splitting methods for differential equations of parabolic and hyperbolic type is the alternating direction implicit (ADI) method, see [9,21,23]. The idea is to decouple the different spatial dimensions in separate equations and to accelerate the solving process.

A further larger group of splitting methods for parabolic and hyperbolic equations is known as the local one-dimensional (LOD) method. This approach splits the multi-dimensional operators into local one-dimensional operators and sweeps implicitly over the directions. The methods are discussed in [14].

In the following we discuss the different methods.

3.1. First-order time derivative (ADI method)

To obtain a fourth-order ADI method, the underlying time-discretization method has to be at least fourth-order in time. So we provide our new method with the Richard extrapolation that uses the second-order ADI method, based on the Crank-Nicolson time discretization, and we reach at least a fourth-order method. Moreover, for the second-order ADI method we apply the second-order time-discretization using the Crank-Nicolson (CN) method.

The CN time-discretization is given as

$$(L_x L_y + \frac{\Delta t}{2} L^*) u^{n+1} = (L_x L_y - \frac{\Delta t}{2} L^*) u^n + \mathcal{O}(h^4) + \mathcal{O}(\Delta t^3), \tag{3.1}$$

where $L^* = L_y A_x + L_x A_y$ and the discretization is of order 2 in time and order 4 in space, see Eq. (2.8).

The ADI method, following [7], is given as follows

$$(L_x + \frac{\Delta t}{2} A_x) u^* = (L_x - \frac{\Delta t}{2} A_x)(L_y - \frac{\Delta t}{2} A_y) u^n, \tag{3.2}$$

$$(L_y + \frac{\Delta t}{2} A_y) u^{n+1} = u^*, \tag{3.3}$$

where we obtain a second-order ADI scheme.

By applying the Richardson extrapolation we obtain a fourth-order method for the CN scheme (see scheme (2.11)).

Proposition 3.1. *The ADI method based on the extrapolation and CN method is given in the next steps.*

Step 1 (first $\Delta t/2$ step), $\alpha = 1/2$:

$$(L_x + \alpha \frac{\Delta t}{2} A_x) u^{*,n+1/2} = (L_x - \alpha \frac{\Delta t}{2} A_x)(L_y - \alpha \frac{\Delta t}{2} A_y) u^n, \tag{3.4}$$

$$(L_y + \alpha \frac{\Delta t}{2} A_y) u^{n+1/2} = u^{*,n+1/2}, \tag{3.5}$$

Step 2 (Δt step), $\alpha = 1.0$:

$$(L_x + \alpha \frac{\Delta t}{2} A_x) u^{*,n+1} = (L_x - \alpha \frac{\Delta t}{2} A_x)(L_y - \alpha \frac{\Delta t}{2} A_y) u^n, \tag{3.6}$$

$$(L_y + \alpha \frac{\Delta t}{2} A_y) \tilde{u}^{n+1} = u^{*,n+1}, \tag{3.7}$$

Step 3 (second $\Delta t/2$ step), $\alpha = 1/2$:

$$(L_x + \alpha \frac{\Delta t}{2} A_x) u^{*,n+1} = (L_x - \alpha \frac{\Delta t}{2} A_x)(L_y - \alpha \frac{\Delta t}{2} A_y) u^{n+1/2}, \tag{3.8}$$

$$(L_y + \alpha \frac{\Delta t}{2} A_y) \tilde{\tilde{u}}^{n+1} = u^{*,n+1}, \tag{3.9}$$

Resulting step:

$$u^{n+1} = 4/3 \tilde{\tilde{u}}^{n+1} - 1/3 \tilde{u}^{n+1}, \tag{3.10}$$

where we obtain a fourth-order method, owing to the Richardson extrapolation with second-order methods.

3.2. First-order time derivative (LOD method)

In the following we introduce the LOD method, see [17], as an improved splitting method while using pre-stepping techniques. The method was discussed in [17] and is given by:

$$u^{n+1,0} - u^n = dt(A+B)u^n, \tag{3.11}$$

$$u^{n+1,1} - u^{n+1,0} = dt\eta A(u^{n+1,1} - u^n), \tag{3.12}$$

$$u^{n+1} - u^{n+1,1} = dt\eta B(u^{n+1} - u^n), \tag{3.13}$$

where $\eta \in (0.0, 0.5)$ and A, B are the spatial discretized operators.

If we eliminate the intermediate values in 3.11-3.13 we obtain

$$u^{n+1} - u^n \quad = \Delta t(A+B)(\eta u^{n+1} - (1-\eta)u^n)$$

and thus we obtain $\mathcal{O}(\Delta t^2)$.

So we obtain a second-order method for $\eta = 0.5$.

By applying the Richardson extrapolation we obtain a fourth-order method for the CN scheme (see scheme (2.11)).

Proposition 3.2. *The LOD method based on the extrapolation and CN method is given in the next steps.*

Step 1 (first $\Delta t/2$ step), $\alpha = 1/2$:

$$u^{n+1/2,0} - u^n = \alpha\, dt\, (A+B)u^n, \tag{3.14}$$

$$u^{n+1/2,1} - u^{n+1/2,0} = \alpha\, dt\, \eta\, A(u^{n+1/2,1} - u^n), \tag{3.15}$$

$$u^{n+1/2} - u^{n+1,1} = \alpha\, dt\, \eta\, B(u^{n+1/2} - u^n), \tag{3.16}$$

Step 2 (Δt step), $\alpha = 1.0$:

$$u^{n+1,0} - u^n = \alpha\, dt\, (A+B)u^n, \tag{3.17}$$

$$u^{n+1,1} - u^{n+1,0} = \alpha\, dt\, \eta\, A(u^{n+1,1} - u^n), \tag{3.18}$$

$$\tilde{u}^{n+1} - u^{n+1,1} = \alpha\, dt\, \eta\, B(\tilde{u}^{n+1} - u^n), \tag{3.19}$$

Step 3 (second $\Delta t/2$ step), $\alpha = 1/2$:

$$u^{n+1,0} - u^{n+1/2} = \alpha\, dt\, (A+B)u^{n+1/2}, \tag{3.20}$$

$$u^{n+1,1} - u^{n+1,0} = \alpha\, dt\, \eta\, A(u^{n+1,1} - u^{n+1/2}), \tag{3.21}$$

$$\tilde{\tilde{u}}^{n+1} - u^{n+1,1} = \alpha\, dt\, \eta\, B(\tilde{\tilde{u}}^{n+1} - u^{n+1/2}), \tag{3.22}$$

Resulting step:

$$u^{n+1} = 4/3\tilde{\tilde{u}}^{n+1} - 1/3\tilde{u}^{n+1}, \tag{3.23}$$

where we obtain a fourth-order method, owing to the Richardson extrapolation with second-order methods.

Remark 3.1. For $\eta \in (0, 0.5)$ we have unconditionally stable methods and for higher-order we use $\eta = \frac{1}{2}$. Then for sufficiently small time steps we get a conditionally stable splitting method.

3.3. Second-order time derivative

In the following we introduce the LOD method as an improved splitting method while using pre-stepping techniques. The method was discussed in [20] and is given by:

$$u^{n+1,0} - 2u^n + u^n - 1 = dt^2(A+B)u^n, \tag{3.24}$$
$$u^{n+1,1} - u^{n+1,0} = dt^2\eta A(u^{n+1} - 2u^n + u^{n-1}), \tag{3.25}$$
$$u^{n+1} - u^{n+1,1} = dt^2\eta B(u^{n+1} - 2u^n + u^{n-1}), \tag{3.26}$$

where $\eta \in (0.0, 0.5)$ and A, B are the spatial discretized operators.

If we eliminate the intermediate values in (3.24)-(3.26) we obtain

$$\begin{aligned} u^{n+1} - 2u^n + u^{n-1} = \Delta t^2(A+B)(\eta u^{n+1} - (1-2\eta)u^n + \eta u^{n-1} \\ + B_\eta(u^{n+1} - 2u^n + u^{n-1}), \end{aligned} \tag{3.27}$$

where $B_\eta = \eta^2\Delta t^2(AB)$ and thus $B_\eta(u^{n+1} - 2u^n + u^{n-1}) = \mathcal{O}(\Delta t^4)$.

So we obtain a higher-order method.

Remark 3.2. For $\eta \in (0.25, 0.5)$ we have unconditionally stable methods and for higher-order we use $\eta = \frac{1}{12}$. Then for sufficiently small time steps we get a conditionally stable splitting method.

3.4. Stability analysis for the splitting method

The stability analysis of the different splitting methods is discussed in the following.

3.4.1. First-order time derivatives

Our method is based on the ADI method of [6,18]. For the application of our new splitting method we apply the Crank-Nicolson method. This method is a stable second-order ADI method, see [7]. In our new splitting method, we can extend the stability to a fourth-order ADI method, that combines an extrapolation method with the second-order ADI method.

Theorem 3.1. *We assume unconditional stability of our second-order ADI method and for the Richardson extrapolation. Our higher-order ADI method is given as*

$$\begin{aligned} &D_{4,ADI}(\Delta t, u^n) = u^{n+1,4th} \\ &= \frac{4}{3}S_{CN,ADI}\left(\frac{\Delta t}{2}, S_{CN,ADI}\Big(\frac{\Delta t}{2}, u^n\Big)\right) - \frac{1}{3}S_{CN,ADI}(\Delta t, u^n), \end{aligned} \tag{3.28}$$

where $S_{CN,ADI}$ is the unconditional stable second-order ADI method. Then we obtain an unconditional stable method.

Proof. On the basis of the unconditional stability of a single ADI method, see [18], we have stability for each term of the Richardson extrapolation.

If we apply the Richardson extrapolation, which is stable, and we insert the stable ADI methods, we result at least in a stable method, see [26].

To study the stability, we apply the von Neumann linear stability analysis. Assume $u_{ij}^n = b^n \exp(I\theta_x i)\exp(I\theta_y j)$ be the value of u^n at node (i,j), where $I = \sqrt{-1}$, b^n is the amplitude at time level n, and $\theta_x = 2\pi\Delta x/\Lambda_1$ and $\theta_y = 2\pi\Delta y/\Lambda_2$ are phase angles with wavelength Λ_1 and Λ_2. Therefore for a stable method the amplification factor $\zeta(\theta_x,\theta_y) = b^{n+1}/b^n$ has to satisfy the stability condition $|\zeta(\theta_x,\theta_y)| \leq 1$, for all θ_x, θ_y in $[-\pi,\pi]$.

We apply our method, substitute the expressions $u_{i,j}^n$, $u_{i,j}^{n+1/2}$ and $u_{i,j}^{n+1}$ and obtain:

$$\zeta(\theta_x,\theta_y) = \frac{4}{3} g_{x,1}(\theta_x) g_{y,1}(\theta_y) g_{x,3}(\theta_x) g_{y,3}(\theta_y) - \frac{1}{3} g_{x,2}(\theta_x) g_{y,2}(\theta_y). \tag{3.29}$$

We have $g_{x,1} = g_{3,x}$ with $g_{x,1}$ given by

$$g_{x,1}(\theta_x) = \frac{\tilde{\gamma}_1 - \tilde{\gamma}_2}{\tilde{\gamma}_1 + \tilde{\gamma}_2}, \tag{3.30}$$

and $\tilde{\gamma}_1 = 1 - 1/3\sin^2(\theta_x/2)$, $\tilde{\gamma}_2 = \Delta t/\Delta^2 x \sin^2(\theta_x/2)$. Using the inequality $|\tilde{\gamma}_1 - \tilde{\gamma}_2| \leq |\tilde{\gamma}_1 + \tilde{\gamma}_2|$ we obtain $|g_{x,1} g_{3,x}| \leq 1$. The same can be done with $g_{x,2}$ and we obtain at last

$$|4/3 g_{x,1} g_{3,x} - 1/3 g_{2,x}| \leq 1, \quad \text{for all } \theta_x \in [-\pi,\pi].$$

Since we have a similar inequality for y, we conclude that our method is unconditionally stable. □

3.4.2. Second-order time derivative

In the following, we discuss the stability for the LOD method proposed for the hyperbolic equations. We assume discretization orders of $\mathcal{O}(h^p)$, $p = 2,4$, for the discretization in space where $h = h_x = h_y$ is the spatial grid width. Then we obtain the following consistency result for our method (3.24)-(3.26):

$$\tilde{B} = \theta^2 \Delta t^2 \tilde{A}_1 \tilde{A}_2.$$

Therefore we obtain a splitting error of $\tilde{B}(u^{n+1} - 2u^n + u^{n-1})$. Assuming sufficient smoothness we have $(u^{n+1} - 2u^n + u^{n-1}) = \mathcal{O}(\Delta t^2)$. Consequently, we obtain

$$\tilde{B}(u^{n+1} - 2u^n + u^{n-1}) = \mathcal{O}(\Delta t^4).$$

Thus we obtain a fourth-order method, if the spatial operators are also discretized as fourth-order terms.

In our new splitting method, we can generalize the stability to a fourth-order ADI method, with respect to the spatial mesh-size $h = \max\{h_x, h_y\}$, that underlies the same discretization order of $\mathcal{O}(h^p)$, $p = 2,4$, as in [20].

The stability of our new fourth-order splitting method is given in the following theorem.

Theorem 3.2. *The stability of our method is given by:*

$$\begin{aligned} &||(1-\Delta t^2\tilde{B})^{1/2}\partial_t^+ u^n||^2 + \mathscr{P}^+(u^n,\theta) \\ &\le ||(1-\Delta t^2\tilde{B})^{1/2}\partial_t^+ u^0||^2 + \mathscr{P}^+(u^0,\theta), \end{aligned} \tag{3.31}$$

where $\theta \in [0.25, 0.5]$ *and*

$$\mathscr{P}^{\pm}(u^j,\theta) := \theta(\tilde{A}u^j, u^j) + \theta(\tilde{A}u^{j\pm1}, u^{j\pm1}) + (1-2\theta)(\tilde{A}u^j, u^{j\pm1}).$$

Proof. We have to prove the theorem for a test function $\overline{\partial}_t u^n$, where $\overline{\partial}_t$ denotes the central difference. For $n \ge 1$, we have

$$((1-\Delta t^2\tilde{B})\overline{\partial}_{tt}u^n, \overline{\partial}_t u^n) + (\tilde{A}(\theta u^{j+1} - (1-2\theta)u^j + \theta u^{j-1}), \overline{\partial}_t u^n) = 0. \tag{3.32}$$

Multiplying with Δt and summing over j yields:

$$\sum_{j=1}^{n}((1-\Delta t^2\tilde{B})\overline{\partial}_{tt}u^j, \overline{\partial}_t u^j)\Delta t + (\tilde{A}(u^{j+1} - 2u^j + u^{j-1}), \overline{\partial}_t u^j)\Delta t = 0\,. \tag{3.33}$$

We can derive the identities,

$$\begin{aligned} &((1-\Delta t^2\tilde{B})\overline{\partial}_{tt}u^j, \overline{\partial}_t u^j)\Delta t \\ &= 1/2||(1-\Delta t^2\tilde{B})^{1/2}\partial_t^+ u^j||^2 - 1/2||(1-\Delta t^2\tilde{B})^{1/2}\partial_t^- u^h||^2, \end{aligned} \tag{3.34}$$

$$\begin{aligned} &(\tilde{A}(\theta u^{j+1} - (1-2\theta)u^j + \theta u^{j-1}), \overline{\partial}_t u^j)\Delta t \\ &= 1/2(\mathscr{P}^+(u^j,\theta) - \mathscr{P}^-(u^j,\theta))\,, \end{aligned} \tag{3.35}$$

which lead to (3.31). □

Remark 3.3. For $\theta = \frac{1}{12}$ we obtain a fourth-order method.

To compute the error of the local splitting we have to use the multiplier $\tilde{A}_1\tilde{A}_2$; thus for large constants we have an unconditional small time step.

Remark 3.4. 1. The unconditional stability of the LOD method is given for $\theta \in [0.25, 0.5]$.
2. The truncation error is $\mathscr{O}(\Delta t^2 + h^p)$, $p \ge 2$ for $\theta \in [0, 0.5]$.
3. If $\theta = 1/12$ we have a fourth-order method in time $\mathscr{O}(\Delta t^2 + h^p), p \ge 2$.
4. If $\theta = 0$ we have a second-order explicit scheme.
5. The CFL condition is important for all $\theta \in [0, 0.5]$ with $CFL = \Delta t^2/\Delta x_{max}^2 D_{max}$. Here x_{max} is the maximal spatial step and D_{max} is the maximal wave-propagation parameter in space.

4. Numerical examples for the spatial splitting methods

The test examples are discussed with respect to analytical solutions, boundary conditions and spatially-dependent propagation functions.

Table 1: Numerical results for the LOD method with $\theta = 1/12$.

Δt^2	η	err	ρ
1/10	1/12	3.1750×10^{-3}	
1/50	1/12	3.5413×10^{-6}	4.223
1/100	1/12	2.1242×10^{-7}	4.062
1/500	1/12	3.3137×10^{-10}	4.015
1/1000	1/12	1.9167×10^{-11}	4.115

Table 2: Numerical results for the LOD method.

Δt^2	η	err	ρ
1/10	0.0	0.1326	
1/50	0.0	0.0043	2.130
1/100	0.0	0.0011	1.967
1/500	0.0	4.1508×10^{-5}	2.036
1/1000	0.0	1.0356×10^{-5}	2.003

Table 3: Numerical results for the LOD method.

Δt^2	η	err	ρ
1/10	0.5	0.5793	
1/50	0.5	0.0215	2.047
1/100	0.5	0.0053	2.020
1/500	0.5	2.0753×10^{-4}	2.013
1/1000	0.5	5.1780×10^{-5}	2.003

4.1. Ordinary differential equation to test fourth-order method

We deal with an ordinary differential equation with constant coefficients where we can derive an analytical solution.

$$\partial_{tt} u = Au, \text{ with } \times (0,T), \tag{4.1}$$

$$u(0) = u_0, \tag{4.2}$$

where the time-interval is given as $[0,T] = [0,1]$. For the approximation error we choose the L_1-norm which is given by

$$err_{L_1} := |u(t^n) - u_{analy}(t^n)|. \tag{4.3}$$

The numerical convergence-rate is given as

$$\rho := \frac{\ln(err_{L_1(\Delta t_1)}/err_{L_1(\Delta t_2)})}{\ln(\Delta t_1/\Delta t_2)}. \tag{4.4}$$

The result for the different θ-steps is given in Tables 1-3. It is observed that we obtain fourth-order in the case of $\eta = \frac{1}{12}$ as we have seen in the theoretical analysis. The benefit of the method can be seen as a mixture of explicit and implicit methods, which is stable and of a higher-order.

4.2. Wave equation with analytical solution and Dirichlet boundary condition

We deal with a two-dimensional example with constant coefficients where we can derive an analytical solution:

$$\partial_{tt} u = D_1^2 \partial_{xx} u + D_2^2 \partial_{yy} u, \tag{4.5}$$

$$u(x,y,0) = \sin(\frac{1}{D_1}\pi x)\sin(\frac{1}{D_2}\pi y), \quad \partial_t u(x,y,0) = 0, \tag{4.6}$$

$$u(x,y,t) = \sin(\frac{1}{D_1}\pi x)\sin(\frac{1}{D_2}\pi y)\cos(\sqrt{2}\,\pi t), \text{ on } \partial\Omega \times (0,T), \tag{4.7}$$

where the initial conditions can be written as $u(x,y,t^n) = u(x,y,0)$ and $u(x,y,t^{n-1}) = u(x,y,t^{n+1}) = u(x,y,\Delta t)$. The analytical solution is given by

$$u_{analy}(x,y,t) = \sin(\pi x/D_1)\sin(\pi y/D_2)\cos(\sqrt{2}\,\pi t), \tag{4.8}$$

For the approximation error we choose the L_1-norm which is defined by

$$err_{L_1} := \sum_{i,j=1}^{m} \Delta x\, \Delta y\, |u(x_i,y_j,t^n) - u_{analy}(x_i,y_j,t^n)|, \tag{4.9}$$

where $u(x_i,y_j,t^n)$ is the numerical and $u_{analy}(x_i,y_j,t^n)$ the analytical solution.

Our test examples are organized as follows.

1. The non-stiff case: We choose $D_1 = D_2 = 1$ with a rectangle as our model domain $\Omega = [0,1] \times [0,1]$. We discretize with $\Delta x = 1/16$ and $\Delta y = 1/16$ and $\Delta t = 1/32$ and choose our parameter η between $0 \leq \eta \leq 1$. The exemplary function values u_{num} and u_{ana} are taken from the center of our domain.

2. The stiff case: We choose $D_1 = D_2 = 0.01$ with a rectangle as our model domain $\Omega = [0,1] \times [0,1]$. We discretize with $\Delta x = 1/32$ and $\Delta y = 1/32$ and $\Delta t = 1/64$ and choose our parameter η between $0 \leq \eta \leq 1$. The exemplary function values u_{num} and u_{ana} are taken from the point $(0.5, 0.5625)$.

The experiments are done with the uncoupled standard discretization method, i.e., the finite differences methods for time and space, and with the operator splitting methods, i.e., the classical operator-splitting method, for which we use the Strang-Marchuk splitting method, and the LOD method.

The non-stiff case can be analyzed in Tables 4-6. It is seen from the tables that we obtain a second-order method also for $\eta = 1/12$. Because of the spatial discretization and the bisection of the spatial step, the error reduction was dominated by the spatial dimension and is therefore of the second-order.

In Table 7, we fix the spatial step and deal with such fine grids that only the time error is dominant. In this case, it is observed that the rate is larger than 2.

In the experiments, we obtain the improved higher-order methods by considering the combination between explicit and implicit time-discretization with $\eta = 1/12$. It is important to discretize the spatial dimension with fine grids to be sure that the error is dominated by the time-discretization. Otherwise we should also use higher-order spatial discretization methods, e.g., high-order compact methods, see [18].

Table 4: Numerical results for the LOD method with $D_1 = D_2 = 1$, $\eta = 0$.

Δt	Δx	η	err	ρ
1/10	1/5	0.0	0.0135	
1/20	1/10	0.0	0.0035	1.948
1/50	1/25	0.0	5.6977×10^{-4}	1.981
1/70	1/35	0.0	2.9100×10^{-4}	1.997
1/100	1/50	0.0	1.4267×10^{-4}	1.998

Table 5: Numerical results for the LOD method with $D_1 = D_2 = 1$, $\eta = 1/12$.

Δt	Δx	η	err	ρ
1/10	1/5	1/12	0.0274	
1/20	1/10	1/12	0.0072	1.928
1/50	1/25	1/12	0.0011	2.050
1/70	1/35	1/12	5.8581×10^{-4}	1.872
1/100	1/50	1/12	2.8669×10^{-4}	2.003

Table 6: Numerical results for the LOD method with $D_1 = D_2 = 1$, $\eta = 0.5$.

Δt	Δx	η	err	ρ
1/10	1/5	0.5	0.0903	
1/20	1/10	0.5	0.0250	1.853
1/50	1/25	0.5	0.0040	2.000
1/70	1/35	0.5	0.0021	1.915
1/100	1/50	0.5	0.0010	2.080

Table 7: Numerical results for the LOD method with $D_1 = D_2 = 1$, $\eta = 1/12$.

Δt	Δx	η	err	ρ
1/20	1/50	1/12	2.013×10^{-2}	
1/50	1/50	1/12	3.085×10^{-3}	2.73
1/100	1/50	1/12	2.893×10^{-4}	3.42

4.3. Spatial-dependent wave equation

In this experiment we apply our method to the spatial-dependent problem. We have non-commuting matrices and the equation is given as

$$\partial_{tt} u = D_1(x,y)\partial_{xx} u + D_2(x,y)\partial_{yy} u, \tag{4.10}$$

$$u(x,y,t^n) = u_0, \quad \partial_t u(x,y,t^n) = u_1, \tag{4.11}$$

$$u(x,y,t) = u_2, \text{ on } \partial\Omega \times (0,T), \tag{4.12}$$

where $D_1(x,y) = 0.1x + 0.01y + 0.01$, $D_2(x,y) = 0.01x + 0.1y + 0.1$.

To compare the numerical results, we cannot use an analytical solution; that is why in

Table 8: Numerical results for the classical operator-splitting with spatial-dependent parameters and Dirichlet boundary (error to the reference solution).

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0032	-0.7251	-0.7154
0.1	0.0034	-0.7251	-0.7149
0.3	0.0037	-0.7251	-0.7139
0.5	0.0040	-0.7251	-0.7129
0.7	0.0044	-0.7251	-0.7120
0.9	0.0047	-0.7251	-0.7110
1.0	0.0049	-0.7251	-0.7105

Table 9: Numerical results for the LOD method with spatial-dependent parameters and Dirichlet boundary (error to the reference solution).

η	err_{L1}	u_{ana}	u_{num}
0.00	0.0032	-0.7251	-0.7154
0.1	0.7809×10^{-3}	-0.7251	-0.7226
0.122	0.6793×10^{-3}	-0.7251	-0.7242
0.3	0.0047	-0.7251	-0.7369
0.5	0.0100	-0.7251	-0.7512
0.7	0.0152	-0.7251	-0.7655
0.9	0.0205	-0.7251	-0.7798
1.0	0.0231	-0.7251	-0.7870

a first pre-step we are computing a reference solution. The reference solution is done with the finite difference scheme with fine time and space steps.

Concerning the choice of the time steps it is important to consider the CFL condition that is now based on the spatial coefficients. In our computations, we have assumed the following CFL condition:

$$\Delta t < 0.5\min(\Delta x, \Delta y)/\max_{x,y\in\Omega}(D_1(x,y), D_2(x,y)). \tag{4.13}$$

For the test example we define our model domain as a rectangle $\Omega = [0,1]\times[0,1]$. The reference solution is obtained by executing the finite differences method and setting $\Delta x = \Delta y = 1/256$ and the time step $\Delta t = 1/256 < 0.390625$.

The model domain is given by a rectangle with $\Delta x = 1/16$ and $\Delta y = 1/32$. The time steps are given by $\Delta t = 1/16$ and $0 \le \eta \le 1$.

The numerical results are given in Tables 8 and 9. In the experiments, we analyze the classical operator-splitting and the LOD method and show that the LOD method yields yet more accurate values. Based on the higher-order LOD method, we also obtain at least higher-order results.

Table 10: Numerical results for the LOD method without (left) and with (right) Richardson extrapolation with $\lambda_1 = 1$ and $\lambda_2 = 0.5$.

Δt	L_1-error without extrapolation	ρ	L_1-error with extrapolation	ρ
10	0.0052		2.2910×10^{-6}	
20	0.0012	2.11	1.1288×10^{-7}	4.35
40	2.7627×10^{-4}	2.12	6.2986×10^{-9}	4.15
80	6.7324×10^{-5}	2.01	3.7246×10^{-10}	4.08
160	1.6620×10^{-5}	2.01	2.2639×10^{-11}	4.07
320	4.1289×10^{-6}	2.01	1.6156×10^{-12}	3.82

4.4. First-order derivative

In this example we apply our higher-order splitting method for the problem

$$\partial_t u = -\lambda_1 u - \lambda_2 u, \tag{4.14}$$
$$u(0) = u_0 = 1, \tag{4.15}$$

where $[0,T] = [0,1]$. The analytical solution is given as $u_{analy}(t) = \exp(-(\lambda_1 + \lambda_2)t)$.

We compare the standard second-order LOD method with our improved fourth-order method. The results are presented by the L_1-error and the numerical convergence rate ρ. In the numerical results we obtain the benefit of the fourth-order method (see Table 10).

In the experiments we can observe the higher accuracy of the modified method. Because of the pure time-discretization we can verify the fourth-order method as a combination of LOD method with Richardson extrapolation.

4.5. Heat equation

In this experiment we apply our method to the heat equation given by

$$\partial_t u = D_x \partial_{xx} u + D_y \partial_{yy} u, \tag{4.16}$$
$$u(x,y,0) = u_{analy}(x,y,0), \text{ on } \Omega, \tag{4.17}$$
$$u(x,y,t) = 0, \text{ on } \partial\Omega \times (0,T), \tag{4.18}$$

where $\Omega = [0,1] \times [0,1]$, $T = 1$, $D_x = 1$, $D_y = 0.5$. The analytical solution is given as

$$u_{analy}(x,y,t) = \exp(-(D_x + D_y)\pi^2 t)\sin(\pi x)\sin(\pi y). \tag{4.19}$$

The numerical results for the classical 2nd-order method ADI method are given in Table 11; while the fourth-order improvement is presented in Table 12. In both cases, $D_1 = 1$ and $D_2 = 0.5$. The refinements are controlled by $\Delta x = \Delta y = \kappa$ and $\Delta t = \kappa^2$.

In the experiments we can obtain higher accuracy with respect to control of the spatial discretization. The splitting method and the time-discretization are given as fourth-order methods $\mathcal{O}(\Delta t^4)$, but the spatial discretization is given as Δx^2 so it is a second-order

Table 11: Numerical results for the LOD method without Richardson extrapolation.

κ	L_1-error	ρ
2	9.2997×10^{-8}	
4	7.2568×10^{-8}	0.35
8	2.6880×10^{-8}	1.435
16	7.0733×10^{-9}	1.922
32	1.7868×10^{-9}	2.056
64	4.4780×10^{-10}	2.02

Table 12: Numerical results for the LOD method with Richardson extrapolation.

κ	L_1-error	ρ
2	1.9297×10^{-6}	
4	1.5095×10^{-7}	3.675
8	3.0613×10^{-8}	4.604
16	7.2872×10^{-9}	4.143
32	1.7999×10^{-9}	4.016
64	4.4862×10^{-10}	4.048

method. We refine the spatial dimension based on the Courant criterion to ensure we see only the time error. With this control we obtain at least the fourth-order method for the splitting scheme.

5. Conclusions and discussions

We have presented different time splitting methods for differential equations of first and second-order in time. The reconstruction of higher-order splitting methods, based on ADI and LOD methods, can be done by using a higher-order method in time and space. We discuss the stability for the two methods and derive unconditional stability. The methods can be easily extended to multi-dimensional problems and mixed discretizations. It is simpler, incidentally, to derive higher-order splitting methods for the wave equation, because of its natural connection to higher-order time-discretization methods. On the other hand, we can deal with higher-order Richardson extrapolation to get higher-order splitting methods for a heat equation. The numerical examples verify the efficiency of the proposed higher-order splitting methods for the differential equations. In the future we will present further higher-order ADI methods based on geometrical integrators.

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