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The Jackson Inequality for the Best L^2-Approximation of Functions on $[0,1]$ with the Weight x

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Abstract. Let $L^2([0,1],x)$ be the space of the real valued, measurable, square summable functions on $[0,1]$ with weight x, and let $\mathscr{L}_n$ be the subspace of $L^2([0,1],x)$ defined by a linear combination of $J_0(\mu_k x)$, where J_0 is the Bessel function of order 0 and $\{\mu_k\}$ is the strictly increasing sequence of all positive zeros of J_0. For $f \in L^2([0,1],x)$, let $E(f,\mathscr{L}_n)$ be the error of the best $L^2([0,1],x)$, i.e., approximation of f by elements of $\mathscr{L}_n$. The shift operator of f at point $x \in [0,1]$ with step $t \in [0,1]$ is defined by

$$T(t)f(x) = \frac{1}{\pi}\int_0^{\pi} f\left(\sqrt{x^2+t^2-2xt\cos\theta}\right)d\theta.$$

The differences $(I-T(t))^{r/2}f = \sum_{j=0}^{\infty}(-1)^j\binom{r/2}{j}T^j(t)f$ of order $r \in (0,\infty)$ and the $L^2([0,1],x)-$ modulus of continuity $\omega_r(f,\tau) = \sup\{\|(I-T(t))^{r/2}f\| : 0 \le t \le \tau\}$ of order r are defined in the standard way, where $T^0(t) = I$ is the identity operator. In this paper, we establish the sharp Jackson inequality between $E(f,\mathscr{L}_n)$ and $\omega_r(f,\tau)$ for some cases of r and τ. More precisely, we will find the smallest constant $\mathscr{K}_n(\tau,r)$ which depends only on n, r, and τ, such that the inequality $E(f,\mathscr{L}_n) \le \mathscr{K}_n(\tau,r)\omega_r(f,\tau)$ is valid.

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1. Introduction

1.1. Some histories

The Jackson inequalities with the first and higher modulus of continuity in various function spaces of one and several variables have a long history. The *Jackson inequality* usually means the following relation between the value $d(f,L,X)$ of the best approximation of a

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function f in a normed function space X by elements of a subspace L and the structure characterization of the function f in terms of some seminorm (or quasi-seminorm) $|\cdot|_X$:

$$d(f,L,X) \le K(L,X)|f|_X \quad \text{for all} \quad f \in X. \tag{1.1}$$

The greatest lower bound of the $K(L,X)$ is called the *sharp constant* or *Jackson constant* in Jackson inequality (1.1).

We recall only some fundamental results in Jackson inequalities concerning direct theorems. Firstly, we introduce some necessary notation. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R}$ be the set of all real numbers and $\mathbb{R}_+$ be the set of all positive real numbers. Denote by $C(\mathbb{T})(\mathbb{T}=[-\pi,\pi])$ the space of continuous, 2π -periodic functions $f:\mathbb{R}\to\mathbb{R}$, with the uniform norm $\|f\|_{C(\mathbb{T})}=\max\{|f(x)|: x\in\mathbb{R}\}$, by $L^2(\mathbb{T})$ the space of real-valued, 2π-periodic, measurable functions which are square summable on $\mathbb{T}$ with the following $L^2(\mathbb{T})$—norm,

$$\|f\|_{L^2(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^2\,dx\right)^{1/2}, \tag{1.2}$$

by $L^2(\mathbb{R})$ the space of real-valued, measurable, square summable functions in the real line $\mathbb{R}$ with the $L^2(\mathbb{R})$-norm,

$$\|f\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2},$$

by $\mathscr{T}_n$ the set of all trigonometric polynomials of degree not higher than n, and by W_σ^2, $\sigma\ge 0$, the collection of all entire functions of exponential type σ which as functions of a real $x\in\mathbb{R}$ lie in $L^2(\mathbb{R})$.

Denote by X a normed space of some functions defined on $\mathbb{R}$ with the norm $\|\cdot\|_X$. For any $r\in\mathbb{N}$, the structure characterization of the function $f\in X$ is the modulus of continuity of order r of f:

$$\omega_r(f,\delta)_X = \sup\{\|\Delta_t^r f\|_X : t\in\mathbb{R},\ |t|\le\delta\}, \quad \delta\ge 0, \tag{1.3}$$

where

$$\Delta_t^r f(x) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(x+jt), \tag{1.4}$$

while $\binom{r}{0}=1$, $\binom{r}{j}=r(r-1)\cdots(r-j+1)/j!$, $j=1,2,\cdots,r$.

In the most cases, the mathematicians consider Jackson inequality (1.1) for the cases $X=C(\mathbb{T})$, $X=L^2(\mathbb{T})$ or $X=L^2(\mathbb{R})$, and correspondingly $L=\mathscr{T}_n(n\in\mathbb{N})$ or $L=W_\sigma^2(\sigma\in\mathbb{R}_+)$. In 1911, Jackson [15] proved the inequality (1.1) for the case $X=C(\mathbb{T})$, $L=\mathscr{T}_n$. He obtained that for any function $f\in C(\mathbb{T})$, the quantity $d(f,L,X)=E_n(f)_{C(\mathbb{T})}$ of the best uniform approximation of $f\in C(\mathbb{T})$ by trigonometric polynomials of order (at most) n tends to zero (as $n\to\infty$) not slower than $\omega_1(f,1/n)_{C(\mathbb{T})}$, which is defined as (1.3) and (1.4) with taking $X=C(\mathbb{T})$ and $r=1$. More precisely, the inequality

$$E_n(f)_{C(\mathbb{T})} \le M_1\omega_1(f,1/n)_{C(\mathbb{T})}, \quad f\in C(\mathbb{T}), \quad n\ge 1,$$

holds with some finite positive constant M_1 which does not depend on f and n.

Denote by $E_{n-1}(f)_{L^2(\mathbb{T})}$ the best $L^2(\mathbb{T})$-approximation of $f \in L^2(\mathbb{T})$ by the trigonometric polynomials of order $n-1$. When $d(f,L,X) = E_{n-1}(f)_{L^2(\mathbb{T})}$, $|f|_X = \omega_r(f,\delta)_{L^2(\mathbb{T})}$, we denote the greatest lower bound of the $K(L,X)$ in (1.1) by $K_{n,r}(\delta)$. In this case, the formula (1.1) is also called to be Jackson inequality. In 1967, Chernykh [10,11] established the Jackson inequality (1.1) for the case $X = L^2(\mathbb{T})$, $L = \mathscr{T}_{n-1}$. He proved that for any integer $n \geq 1$ and real number $\delta \geq \pi/n$, the exact inequality

$$E_{n-1}(f)_{L^2(\mathbb{T})} < \frac{1}{\sqrt{2}}\,\omega_1(f,\delta)_{L^2(\mathbb{T})}, \quad f \in L^2(\mathbb{T}), f \not\equiv \text{const}, \tag{1.5}$$

holds. It is also shown in [3] that inequality (1.5) with the constant $\frac{1}{\sqrt{2}}$ is not valid for $0 < \delta < \pi/n$. Thus,

$$K_{n,1}(\delta) = \frac{1}{\sqrt{2}}, \quad \delta \geq \frac{\pi}{n}; \quad K_{n,1}(\delta) > \frac{1}{\sqrt{2}}, \quad 0 < \delta < \frac{\pi}{n}.$$

Chernykh obtained a similar result for modulus of continuity of order $r > 1$ as well. Namely, he proved in [10] that the inequality

$$E_{n-1}(f)_{L^2(\mathbb{T})} < \frac{1}{\sqrt{\binom{2r}{r}}}\,\omega_r(f,\delta)_{L^2(\mathbb{T})}, \quad f \in L^2(\mathbb{T}), f \not\equiv \text{const}, \tag{1.6}$$

holds for any $n \geq 1$, $r \geq 2$ and $\delta \geq 2\pi/n$.

Moreover, from the works of Arestov and Babenko [2], Kozko and Rozhdestvenskii [16], and Vasil'ev [22], we see that the following estimate

$$K_{n,r}(\delta) \geq \frac{1}{\sqrt{\binom{2r}{r}}}, \quad n \geq 1, \quad r \geq 2, \quad \delta > 0, \tag{1.7}$$

is valid.

Thus, by (1.6), (1.7), when $n \geq 1$, $r \geq 2$, $\delta \geq 2\pi/n$, it is easy to see that the inequality (1.7) is exact, i.e.,

$$K_{n,r}(\delta) = \frac{1}{\sqrt{\binom{2r}{r}}}.$$

Further, Vasil'ev improved the results (see paper [2] by Arestov and Babenko for details).

Using ideas from Chernykh's papers [10,11], Ibragimov and Nasibov [13] and, independently, Popov [20] obtained the Jackson inequalities for the case of $\sigma > 0$, $X = L^2(\mathbb{R})$, $L = W_\sigma^2$, $d(f,L,X) = E_\sigma(f)_{L^2(\mathbb{R})}$, and $|f|_X = \omega_r(f,\delta)_{L^2(\mathbb{R})}$, where $E_\sigma(f)_{L^2(\mathbb{R})}$ represents the best approximation of a function $f \in L^2(\mathbb{R})$ by the space W_σ^2, in the metric of the space $L^2(\mathbb{R})$, that is

$$E_\sigma(f)_{L^2(\mathbb{R})} = \inf\{\|f - g\|_{L^2(\mathbb{R})} : g \in W_\sigma^2\},$$

and $\omega_r(f,\delta)_{L^2(\mathbb{R})}$ is defined by the formula (1.3) and (1.4) for $X = L^2(\mathbb{R})$.

Ibragimov and Nasibov [13] proved that for any function $f \in L^2(\mathbb{R})$, $f \not\equiv 0$, the inequalities

$$E_\sigma(f)_{L^2(\mathbb{R})} < \frac{1}{\sqrt{2}}\,\omega_1\left(f, \frac{\pi}{\sigma}\right)_{L^2(\mathbb{R})}; \quad E_\sigma(f)_{L^2(\mathbb{R})} < \frac{1}{2}\,\omega_2\left(f, \frac{\pi}{\sigma}\right)_{L^2(\mathbb{R})}$$

hold. Popov [20] proved that for any non-zero function $f \in L^2(\mathbb{R})$ the inequalities

$$\begin{aligned} E_\sigma(f)_{L^2(\mathbb{R})} &< \frac{1}{\sqrt{2}}\,\omega_1\left(f, \delta\right)_{L^2(\mathbb{R})}, \quad \text{for } \delta \geq \frac{\pi}{\sigma}, \\ E_\sigma(f)_{L^2(\mathbb{R})} &< \frac{1}{\sqrt{\binom{2r}{r}}}\,\omega_r\left(f, \delta\right)_{L^2(\mathbb{R})}, \quad \text{for } \delta \geq \frac{2\pi}{\sigma}, r \geq 2 \end{aligned}$$

are valid and sharp. One may also refer to Arestov and Babenko's paper [2] and Arestov's paper [1] for above mentioned results.

For the multidimensional case, we only briefly list some classical important sharp results pertaining to the Jackson inequality. The exact Jackson constant in the space $L^2(\mathbb{T}^d)$, $(d \in \mathbb{N}, d > 1)$ was obtained by Yudin [24] (the order of the modulus of continuity is $r = 1$) in 1981. Similarly, in the space $L^2(\mathbb{R}^d)$, the exact Jackson inequalities were proved by Popov [21] for the case $d = 2, 3, \cdots$, and $r = 1$. In the space $L^2(S^{d-1})$ (S^{d-1} is the unit sphere of $\mathbb{R}^d$), $d \geq 3$, the exact constants in the Jackson inequalities were established by Arestov and Popov [4] ($d = 3, 4$, and $r \in \mathbb{N}$), and by Babenko [5] ($d \geq 5$, $r > 0$). Berdysheva [9] obtained the exact Jackson inequality in $L^2(\mathbb{R}^d)$ on the centrally symmetric convex closed set for $r = 1$.

We now recall some development for the L^p−normed function spaces with some weight function. In 1997, Moskovskiĭ [19] proved the Jackson theorem in the spaces $L^p(\mathbb{R}_+, x^\alpha dx)$ (the weight is x^α) for $1 \leq p < 2$, $\alpha \geq 0$. In 1998, Babenko [7] proved the Jackson inequalities in the space $L^2(\mathbb{R}_+, x^{2\nu+1}dx)$ (the weight is $x^{2\nu+1}$) with Fourier-Bessel integral for $\nu \geq -\frac{1}{2}$ and $r > 0$. For any real number $\alpha > \beta \geq -\frac{1}{2}$, Babenko [6] constructed the sharp Jackson inequality (1.1) for the case $X = L^2_{\alpha,\beta}$, $L = \mathfrak{C}_{n-1}$, where $L^2_{\alpha,\beta}$ is the space of real measurable even 2π-periodic functions $F(x) = f(\cos x)$, $x \in \mathbb{R}$, with scalar product

$$(F, G)_{\alpha,\beta} = \int_0^\pi F(x)G(x)\left(\sin\frac{x}{2}\right)^{2\alpha+1}\left(\cos\frac{x}{2}\right)^{2\beta+1} dx$$

and norm $\|F\|_{\alpha,\beta} = (F, F)^{1/2}_{\alpha,\beta}$; $\mathfrak{C}_{n-1}$ is the space of cosine-polynomials of order $n-1$, that is

$$\mathfrak{C}_{n-1} = \left\{ H : H(x) = \sum_{k=0}^{n-1} a_k \cos kx,\ a_k \in \mathbb{R} \right\}.$$

The best approximation of $F \in L^2_{\alpha,\beta}$ in the space $\mathfrak{C}_{n-1}$ is defined to be

$$E_{n-1}(F)_{L^2_{\alpha,\beta}} = \min\{\|F - H\|_{\alpha,\beta} : H \in \mathfrak{C}_{n-1}\}.$$

The shift operator with step $t \in \mathbb{R}$ on an arbitrary function $F(x) = f(\cos x) \in L^2_{\alpha,\beta}$ is defined as follows:

$$T_t F(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \int_0^1 \int_0^\pi f(\cos\Psi)(1-\rho^2)^{\alpha-\beta-1}\rho^{2\beta+1}(\sin\xi)^{2\beta} d\xi d\rho,$$

where $t, x \in \mathbb{R}$, $\Gamma(x)$ is the gamma function, and $\cos\Psi$ is calculated by the following formula,

$$\cos\Psi = \cos t \cos x + \rho \sin t \sin x \cos\xi + \frac{1}{2}(\rho^2-1)(1-\cos t)(1-\cos x).$$

For any real number $r > 0$, the author constructed the corresponding difference operator of (real) order $r > 0$ with step $t \in \mathbb{R}$,

$$\Delta_t^r = (I - T_t)^{r/2} = \sum_{k=0}^{\infty} (-1)^k \binom{r/2}{k} T_t^k,$$

where $T_t^0 = I$, I is the identity operator, $T_t^k = T_t T_t^{k-1}$ and $\binom{a}{0} = 1$, $\binom{a}{k} = a(a-1)\cdots(a-k+1)/k!$, $k = 1, 2, \cdots$, are binomial coefficients for the positive real number a. The modulus of continuity of order $r > 0$ of $F(x) \in L^2_{\alpha,\beta}$ is defined to be the following function of $\tau > 0$:

$$\omega_r(F,\tau)_{\alpha,\beta} = \sup\left\{\|\Delta_t^r F\|_{\alpha,\beta} : |t| \le \tau\right\}.$$

Babenko [6] proved that if $\alpha > \beta \ge -\frac{1}{2}$ and $n \ge \max\{2, 1 + \frac{\alpha-\beta}{2}\}$, then for any function $F \in L^2_{\alpha,\beta}$,

$$E_{n-1}(F)_{L^2_{\alpha,\beta}} \le \omega_r(F, 2x_n^{\alpha,\beta})_{\alpha,\beta}, \quad r \ge 1,$$
$$E_{n-1}(F)_{L^2_{\alpha,\beta}} \le 2^{(1-r)/2}\omega_r(F, 2x_n^{\alpha,\beta})_{\alpha,\beta}, \quad 0 < r < 1,$$

where $x_n^{\alpha,\beta}$ is the first positive zero of the Jacobi cosine-polynomial $\phi_n^{\alpha,\beta}(x) = R_n^{\alpha,\beta}(\cos x)$. Meantime, for $\tau \ge 2x_n^{\alpha,\beta}$, Babenko gave the Jackson constant $K_n^{\alpha,\beta}(\tau, r)$ with Jacobi weight as follows,

$$K_n^{\alpha,\beta}(\tau,r) = 1, \quad r \ge 1,$$
$$1 \le K_n^{\alpha,\beta}(\tau,r) \le 2^{(1-r)/2}, \quad 0 < r < 1.$$

In the present paper, inspired by Babenko's work above-mentioned (see [6]), we are interested in establishing the sharp Jackson inequality of the best approximation of f from $L^2([0,1],x)$ by the finite sum of functions $J_0(\mu_k x), k = 1, 2, \cdots, n$, for the Bessel function J_0 of order 0, and the modulus of continuity of order r (see in the next section), where $L^2([0,1],x)$ be the space of the real valued, measurable, square summable functions on $[0,1]$ with weight x, and $\{\mu_k\}_{k\in\mathbb{N}}$ is the increasing sequence of all positive zeros of J_0.

1.2. Our main results

In order to state our main results, we need the following notations. Denote by $L^2([0,1],x)$ the space of the real valued, measurable, square summable functions on $[0,1]$ with scalar product

$$(f,g)=\int_0^1 f(x)g(x)xdx, \tag{1.8}$$

and norm $\|f\|=(f,f)^{1/2}$. Let J_0 be the Bessel function of order 0 and μ_k be the k-th positive root of $J_0(x)$, that is $J_0(\mu_k)=0$, $k\in\mathbb{N}$. Using the property of J_0, we know that the system $\{J_0(\mu_k x)\}_{k\in\mathbb{N}}$ is orthogonal on $[0,1]$ with weight x, namely,

$$\int_0^1 J_0(\mu_k x)J_0(\mu_l x)xdx=\begin{cases} 0, & k\neq l,\\ \dfrac{1}{2}[J_1(\mu_k)]^2, & k=l, \end{cases} \tag{1.9}$$

where $J_1(x)$ is the Bessel function of order 1. Moreover, the system $\{J_0(\mu_k x)\}_{k\in\mathbb{N}}$ forms a complete system on $[0,\ 1]$(see [18, Chap.7]).

Since $J_0'=-J_1$, we also have

$$\int_0^1 [J_0(\mu_k x)]^2 xdx=\frac{1}{2}[J_0'(\mu_k)]^2, \quad k\in\mathbb{N}.$$

For convenience, we denote $J_0(\mu_k x)$ by $\varphi_k(x),\ x\in[0,\infty)$.

The space $\mathscr{L}_n$ is defined as follows,

$$\mathscr{L}_n=\left\{g:\ g(x)=\sum_{k=1}^n b_k\varphi_k(x), \quad b_k\in\mathbb{R}, \quad x\in[0,1]\right\}. \tag{1.10}$$

Then the best approximation of $f\in L^2([0,1],x)$ by the space $\mathscr{L}_n$ is defined to be

$$E(f,\mathscr{L}_n)=\min\{\|f-g\|:\ g\in\mathscr{L}_n\}. \tag{1.11}$$

For any function $f\in L^2([0,1],x)$, it can be expended as Fourier-Bessel series as follows:

$$f(x)=\sum_{k=1}^\infty a_k\varphi_k(x), \quad a_k=\frac{(f,\varphi_k)}{(\varphi_k,\varphi_k)}, \tag{1.12}$$

where the convergence is understood in the metric of $L^2([0,1],x)$. In the next section, we will prove the series $\sum_{k=1}^\infty a_k\varphi_k(x)$ also converges in the metric of $L^2([1,3],x)$, i.e., there is a function f_1 defined on $[1,3]$ such that

$$\int_1^3 \left|\sum_{k=1}^n a_k\varphi_k(x)-f_1(x)\right|^2 x\,dx\to 0, \ \text{ as } n\to\infty.$$

Thus, we may extend the definition of the function f from on $[0,1]$ to on $[0,3]$ as follows:

$$\widetilde{f}(x)=\begin{cases} f(x), & x\in[0,1],\\ f_1(x), & x\in(1,3].\end{cases} \tag{1.13}$$

The shift operator with step $t\in[0,1]$ is defined to be the operator $T(t)$ that acts on $f\in L^2([0,1],x)$ by the rule

$$T(t)f(x)=\frac{1}{\pi}\int_0^{\pi} f\left(\sqrt{x^2+t^2-2xt\cos\theta}\right)d\theta. \tag{1.14}$$

Note that when $x,t\in[0,1],\theta\in[0,\pi]$, the inequality $\sqrt{x^2+t^2-2xt\cos\theta}>1$ is possible. Then in (1.14) we actually use the definition of the function $\widetilde{f}$ in the place of f, but with no confusion, we still use f for brevity.

The corresponding difference operator $(I-T(t))^{r/2}$ of the real order $r>0$ with step $t\in[0,1]$ is defined as follows:

$$(I-T(t))^{r/2}=\sum_{k=0}^{\infty}(-1)^k\binom{r/2}{k}T^k(t), \tag{1.15}$$

where I is the identity operator and $\binom{a}{0}=1$, $\binom{a}{k}=\frac{a(a-1)\cdots(a-k+1)}{k!}$, $k=1,2,\cdots$, are binomial coefficients for the positive real number a. The modulus of continuity of order $r>0$ of $f\in L^2([0,1],x)$ is defined by the following function of $\tau>0$,

$$\omega_r(f,\tau)=\sup\left\{\|(I-T(t))^{r/2}f\|:\,0\le t\le\tau\right\}. \tag{1.16}$$

In this paper, we consider the classical problem on the sharp Jackson constant $\mathscr{K}=\mathscr{K}_n(\tau,r)$ in the Jackson inequality in $L^2([0,1],x)$:

$$E(f,\mathscr{L}_n)\le\mathscr{K}\,\omega_r(f,\tau),\quad f\in L^2([0,1],x). \tag{1.17}$$

That is, the problem of calculating the constant

$$\mathscr{K}=\mathscr{K}_n(\tau,r)=\sup\left\{\frac{E(f,\mathscr{L}_n)}{\omega_r(f,\tau)}:\,f\in L^2([0,1],x),\quad f(t)\neq0,a.e.\ t\in[0,1]\right\}. \tag{1.18}$$

In the next section, we will show that $\omega_r(f,\tau)=0$ if and only if $\|f\|=0$. It is why we restrict $f(t)\neq0,a.e.\ t\in[0,1]$ in (1.18).

By partly using the analogous method with the one used in Babenko [6], we established the sharp Jackson inequality (1.17). Our results can be stated as follows:

Theorem 1.1. *Let $n\in\mathbb{N}$, r be a real number ≥1 and $\mathscr{K}_n(\tau,r)$ be defined as* (1.18)*. Then*

$$\mathscr{K}_n(\tau,r)=1,$$

for $\tau\ge2\mu_1/\mu_{n+1}$.

Theorem 1.2. *Let $\mathscr{L}_n$, $E(f,\mathscr{L}_n)$ and $\omega_r(f,\tau)$ be defined as* (1.10), (1.11) *and* (1.16)*, respectively. Then for any $n\in\mathbb{N}$ and any $f\in L^2([0,1],x)$, we have*

$$E(f,\mathscr{L}_n)\le2^{(1-r)/2}\omega_r(f,\tau),\quad if\quad 0<r<1,$$

for $\tau\ge2\mu_1/\mu_{n+1}$.

2. Preliminary lemmas

We begin by introducing some properties of the shift operator $T(t)$ (see(1.14)). Let us consider the case of $f(x) = \varphi_k(x)$. Then we have

$$T(t)\varphi_k(x) = \frac{1}{\pi}\int_0^\pi J_0(\mu_k\sqrt{x^2+t^2-2xt\cos\theta})d\theta. \tag{2.1}$$

Since

$$J_0(R) = J_0(r_1)J_0(r_2) + 2\sum_{m=1}^\infty J_m(r_1)J_m(r_2)\cos m\theta,$$

where $R = \sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}$ (see [18, p.414]), and $\int_0^\pi \cos m\theta d\theta = 0$ for any $m \in \mathbb{N}$, we have

$$\frac{1}{\pi}\int_0^\pi J_0(R)d\theta = J_0(r_1)J_0(r_2),$$

that is

$$\frac{1}{\pi}\int_0^\pi J_0\left(\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}\right)d\theta = J_0(r_1)J_0(r_2). \tag{2.2}$$

Therefore, we can rewrite (2.1) as follows,

$$T(t)\varphi_k(x) = J_0(\mu_k x)J_0\left(\mu_k t\right) = \varphi_k(x)\varphi_k(t). \tag{2.3}$$

Then by (1.13), (1.14), (2.1) and (2.3), we have

$$T(t)f(x) = \sum_{k=1}^\infty a_k\varphi_k(x)\varphi_k(t), \quad a_k = \frac{(f,\varphi_k)}{(\varphi_k,\varphi_k)}, \tag{2.4}$$

which implies that the operator $T(t)$ on $L^2([0,1],x)$ is linear.

From the fact that $|J_0(x)| \le 1$ for all $x \in \mathbb{R}$ and (2.4), it is easy to see that the norm of $T(t)$ in $L^2([0,1],x)$ is not more than one. Therefore, using the scheme of Grünwald-Letnikov [12, 17], it is reasonable to construct the corresponding difference operator $(I - T(t))^{r/2}$ in (1.15). From (2.4), using the Parseval's formula in the space $L^2([0,1],x)$, we see that

$$\|(I-T(t))^{r/2}f\|^2 = \frac{1}{2}\sum_{k=1}^\infty a_k^2[J_1(\mu_k)]^2(1-\varphi_k(t))^r,$$

and hence $\|(I-T(t))^{r/2}f\|^2 = 0$ if and only if $\|f\| = 0$.

The following Lemma 2.1 shows that the extension of f in (1.13) is reasonable.

Lemma 2.1. *For any real sequence $\{a_k\}_{k=1}^\infty$ satisfying $\sum_{k=1}^\infty a_k^2[J_1(\mu_k)]^2 < \infty$, the series $\sum_{k=1}^\infty a_k\varphi_k(x)$ converges in $L^2([1,3],x)$ norm.*

Proof. In this proof, the notation $f(x) = \mathcal{O}(g(x))$ means $A_1 \le |\frac{f(x)}{g(x)}| \le A_2$ as $x \to x_0$, where x_0 is a fix number, and A_1, A_2 are two positive constants.

By [23], for $x \ge 1$, $J_0(x)$ can be written as follows:

$$J_0(x) = \sqrt{\frac{2}{\pi x}}\Big[P(x)\cos(x - \frac{\pi}{4}) + Q(x)\sin(x - \frac{\pi}{4})\Big], \tag{2.5}$$

where

$$P(x) = 1 + \sum_{k=1}^{\infty}(-1)^k \frac{[(4k-1)!!]^2}{(2k)!(8x)^{2k}}, \quad Q(x) = \frac{1}{8x} + \sum_{k=1}^{\infty}(-1)^k \frac{[(4k+1)!!]^2}{(2k+1)!(8x)^{2k+1}}.$$

And the k-th root μ_k of $J_0(x)$ satisfies

$$\mu_k = k\pi - \frac{\pi}{4} + \mathcal{O}(\frac{1}{k}), \tag{2.6}$$

which implies $\mu_k \to \infty$ as $k \to \infty$. Then we have the following asymptotic relation

$$\varphi_k(x) = J_0(\mu_k x) = \sqrt{\frac{2}{\pi \mu_k x}}\cos(\mu_k x - \frac{\pi}{4}) + \mathcal{O}(\frac{1}{(\mu_k x)^{\frac{3}{2}}}), \tag{2.7}$$

for $x \ge 1$, along with (2.6), we can obtain that for a sufficiently large $k \in \mathbb{N}$,

$$\varphi_k(x) = \sqrt{\frac{2}{\pi \mu_k x}}\cos((k\pi - \frac{\pi}{4})x - \frac{\pi}{4}) + \mathcal{O}(\frac{1}{(\mu_k x)^{\frac{3}{2}}}). \tag{2.8}$$

For any $n \in \mathbb{N}$, write

$$R_n := \Big(\int_1^3 \Big(\sum_{k=n+1}^{\infty} a_k \varphi_k(x)\Big)^2 x\,dx\Big)^{1/2}.$$

Then by (2.8) and the Minkowski inequality, we have

$$\begin{aligned} R_n \le & \Big(\int_1^3 \Big[\sum_{k=n+1}^{\infty} a_k \sqrt{\frac{2}{\pi \mu_k x}}\cos((k\pi - \frac{\pi}{4})x - \frac{\pi}{4})\Big]^2 x\,dx\Big)^{1/2} \\ & + \Big(\int_1^3 \Big[\sum_{k=n+1}^{\infty} a_k \mathcal{O}(\frac{1}{(\mu_k x)^{\frac{3}{2}}})\Big]^2 x\,dx\Big)^{1/2} \\ =: & I_1 + I_2. \end{aligned}$$

Notice that

$$\int_1^3 \cos\Big((k\pi - \frac{\pi}{4})x - \frac{\pi}{4}\Big)\cos\Big((l\pi - \frac{\pi}{4})x - \frac{\pi}{4}\Big)\,dx = \begin{cases} 1, & k = l, \\ 0, & k \ne l, \end{cases}$$

for all $k,l \in \mathbb{N}$. Then we have

$$I_1 = \Big(\sum_{k=n+1}^{\infty} \frac{2}{\pi\mu_k} a_k^2 \Big)^{1/2}.$$

By Hölder's inequality, we see that

$$\begin{aligned} I_2 &\le \Big\{ \int_1^3 \Big[\sum_{k=n+1}^{\infty} |a_k| \Big| \mathcal{O}\big(\frac{1}{(\mu_k x)^{\frac{3}{2}}}\big) \Big| \Big]^2 x\,dx \Big\}^{\frac{1}{2}} \\ &\le \sqrt{2} \Big(\sum_{k=n+1}^{\infty} \frac{a_k^2}{\mu_k} \Big)^{\frac{1}{2}} \Big(\sum_{k=n+1}^{\infty} \Big| \mathcal{O}\Big(\frac{1}{\mu_k^2}\Big) \Big| \Big)^{\frac{1}{2}}. \end{aligned}$$

Thus, by the assumption of the lemma, we have $R_n \to 0$, as $n \to \infty$, which completes the proof of the lemma. □

Remark 2.1. According as the Parseval's formula in the theory of Fourier series, for any function $f \in L^2([0,1],x)$, from (1.12) we have

$$\|f\| = \frac{1}{2} \sum_{k=1}^{\infty} a_k^2 [J_1(\mu_k)]^2,$$

where $a_k = (f,\varphi_k)/(\varphi_k,\varphi_k)$. On the other hand, for any sequence $\{a_k\}_{k=1}^{\infty}$ satisfying

$$\frac{1}{2} \sum_{k=1}^{\infty} a_k^2 [J_1(\mu_k)]^2 < \infty,$$

there exists a function f belonging to $L^2([0,1],x)$, such that $f(x) = \sum_{k=1}^{\infty} a_k \varphi_k(x)$. From this, it is nature to consider the convergence as stated in Lemma 2.1.

In order to estimate the lower bound of Theorem 2.1, we need the following statement proved by Arestov ([4, Theorem 1], [5, Lemma 4.2]). We used the standard notation $\|f\|_{C[a,b]} = \max_{x\in[a,b]} |f(x)|$ for the uniform norm of a continuous function f on $[a,b]$.

Lemma 2.2. (Arestov) *Assume that continuous functions ψ_k, $k \in \mathbb{N}$, defined on $[0,b]$, $b > 0$, satisfy the following conditions:*

(1) $\psi_k(0) = 0, \quad k \in \mathbb{N}$;

(2) $\sup_{k\in\mathbb{N}} \|\psi_k\|_{C[0,b]} < \infty$;

(3) $\liminf_{k\to\infty} \Big\{ \max_{t\in[\xi,b]} \psi_k(t) \Big\} \le 1$, *for any* $\xi \in (0,b]$.

Then for any $\varepsilon \in (0,1)$, there exists a function

$$F(t)=\sum_{k=1}^{\infty}\rho_k\psi_k(t), \quad \text{with} \quad \sum_{k=1}^{\infty}\rho_k=1, \quad \rho_k\ge 0,$$

such that

$$F(t)\le 1+\varepsilon, \quad t\in[0,b].$$

Lemma 2.3. *Let $r\in(0,\infty)$ and $t\in[0,1]$ and let $(I-T(t))^{r/2}$ be as (1.15). Then for any $f\in L^2([0,1],x)$, we have*

$$\|(I-T(t))^{r/2}f\|^2=\frac{1}{2}\sum_{m=1}^{\infty}a_m^2[J_1(\mu_m)]^2(1-\varphi_m(t))^r, \tag{2.9}$$

$$E^2(f,\mathscr{L}_n)=\frac{1}{2}\sum_{m=n+1}^{\infty}a_m^2[J_1(\mu_m)]^2, \tag{2.10}$$

where $a_m=(f,\varphi_m)/(\varphi_m,\varphi_m)$.

Proof. Since

$$T^k(t)f(x)=\sum_{m=1}^{\infty}a_m[\varphi_m(t)]^k\varphi_m(x),$$

by the definition of difference operator Δ_t^r, we obtain

$$(I-T(t))^{r/2}f(x)=\sum_{m=1}^{\infty}a_m\varphi_m(x)(1-\varphi_m(t))^{r/2}.$$

Using this and the orthogonality of the system $\{\varphi_m(x)\}_{m\in\mathbb{N}}$ with weight x, we have

$$\begin{aligned}\|(I-T(t))^{r/2}f\|^2&=\int_0^1\sum_{m=1}^{\infty}a_m^2(1-\varphi_m(t))^r[\varphi_m(x)]^2x\,dx\\&=\frac{1}{2}\sum_{m=1}^{\infty}a_m^2[J_1(\mu_m)]^2(1-\varphi_m(t))^r,\end{aligned}$$

which proves (2.9). Obviously,

$$\begin{aligned}E^2(f,\mathscr{L}_n)&=\left\|\sum_{m=n+1}^{\infty}a_m\varphi_m\right\|^2=\int_0^1\sum_{m=n+1}^{\infty}a_m^2[\varphi_m(x)]^2x\,dx\\&=\frac{1}{2}\sum_{m=n+1}^{\infty}a_m^2[J_1(\mu_m)]^2.\end{aligned}$$

The second equality follows from the orthogonality of $\{\varphi_k\}_{k\in\mathbb{N}}$ with weight x, and the last equality follows from the formula (1.9). This completes the proof of the lemma. □

From the formula (1.17), (2.9) and (2.10), the problem (1.17) can be reduced to the following extremal problem for the Bessel functions: given $n \in \mathbb{N}$, find the best constant $K = K_n(\tau, r)$ in the inequality

$$\sum_{k=n+1}^{\infty} \rho_k \leq K \max_{0\leq t\leq \tau} \sum_{k=1}^{\infty} \rho_k (1 - \varphi_k(t))^r,$$

where $\sum_{k=1}^{\infty} \rho_k < \infty$, and $\rho_k \geq 0$ for $k \in \mathbb{N}$. Moreover, it is easy to prove that

$$K_n(\tau, r) = \sup \left\{ \frac{\sum_{k=n+1}^{\infty} \rho_k}{\max_{0\leq t\leq \tau} \sum_{k=n+1}^{\infty} \rho_k (1 - \varphi_k(t))^r} : \sum_{k=n+1}^{\infty} \rho_k < \infty; \rho_k \geq 0 \right\}. \tag{2.11}$$

It is also obvious that

$$\{\mathscr{K}_n(\tau, r)\}^2 = K_n(\tau, r) \tag{2.12}$$

with $n \in \mathbb{N}$, $\tau > 0$ and $r > 0$. Problem (2.11) can be transformed as follows:

$$K_n(\tau, r) = \sup \left\{ \sum_{k=n+1}^{\infty} \rho_k : \sum_{k=n+1}^{\infty} \rho_k (1 - \varphi_k(t))^r \leq 1,\ t \in [0, \tau];\ \rho_k \geq 0,\ k \geq n+1 \right\}.$$

3. The proofs of Theorems 1.1 and 1.2

3.1. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following theorem for the lower bound estimation.

Theorem 3.1. *Let $\tau > 0$, $r > 0$ and $n \in \mathbb{N}$. Then the exact constant $\mathscr{K} = \mathscr{K}_n(\tau, r)$ (see* (1.18)) *in the Jackson inequality* (1.17) *in $L^2([0,1], x)$ has the following lower bound:*

$$\mathscr{K}_n(\tau, r) \geq 1. \tag{3.1}$$

Proof. Recall that $\varphi_k(x) = J_0(\mu_k x)$, $|J_0(x)| \leq 1$ and $J_0(0) = 1$. We have $\varphi_k(0) = J_0(0) = 1$ for $k \in \mathbb{N}$. Fix $\xi \in (0, \tau]$. Since μ_{n+k} tends to infinite, as $k \to \infty$, then for all $x \in [\xi, \tau]$, we have

$$\mu_{n+k} x \longrightarrow \infty, \quad \text{as} \quad k \longrightarrow \infty.$$

The asymptotic property (2.7) of $J_0(x)$ (see [18, p.397]) implies that for any sufficiently small $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that the inequality

$$-\varepsilon < J_0(\mu_{n+k} x) < \varepsilon$$

is valid for all $k \geq k_0$ and $x \in [\xi, \tau]$. Furthermore, for all $k \geq k_0$ and $x \in [\xi, \tau]$,

$$1 - \varepsilon < 1 - \varphi_{n+k}(x) < 1 + \varepsilon,$$

which leads to

$$1-c_0\varepsilon<\max_{x\in[\xi,\tau]}[1-\varphi_{n+k}(x)]^r<1+c_0\varepsilon, \tag{3.2}$$

where c_0 is a positive constant depending only on r. From the arbitrariness of ε and (3.2), we obtain that

$$\lim_{k\to\infty}\max_{x\in[\xi,\tau]}[1-\varphi_{n+k}(x)]^r\le 1. \tag{3.3}$$

Set

$$\psi_k(x)=[1-\varphi_{n+k}(x)]^r.$$

From the above discussion and the inequality (3.3), we obtain that $\psi_k(x)$ satisfies the conditions of Lemma 2.2 for $b=\tau$. Hence, for any $\epsilon\in(0,1)$, there is a function $f\in L^2([0,1],x)$ such that

$$\frac{E^2(f,\mathscr{L}_n)}{\omega_r^2(f,\tau)}=\frac{\sum\limits_{k=n+1}^{\infty}a_k^2[J_1(\mu_k)]^2}{\max\limits_{0\le t\le\tau}\sum\limits_{k=n+1}^{\infty}a_k^2[J_1(\mu_k)]^2[1-\varphi_k(t)]^r}\ge\frac{1}{1+\epsilon},\quad a_k=\frac{(f,\varphi_k)}{(\varphi_k,\varphi_k)},$$

which implies that (3.1) is valid. Thus the proof of Theorem 3.1 is complete. □

The following theorem shows the upper bound estimation of the Jackson constant.

Theorem 3.2. *Let $r\ge 1$ and $n\in\mathbb{N}$. Then the exact constant $\mathscr{K}=\mathscr{K}_n(\tau,r)$(see (1.18)) in the Jackson inequality (1.17) in $L^2([0,1],x)$ has the following upper bound:*

$$\mathscr{K}_n(\tau,r)\le 1,$$

for $\tau\ge 2\mu_1/\mu_{n+1}$.

Proof. Set $\tau_n=\mu_1/\mu_{n+1}$. Since $\mathscr{K}_n(\tau,r)\le\mathscr{K}_n(2\tau_n,r)$ for $\tau\ge 2\tau_n$, it is sufficient to prove the inequality $\mathscr{K}_n(2\tau_n,r)\le 1$.

Note that $\tau_n<\frac{1}{2}$ if $n\in\mathbb{N}$ (see [18], pp. 417). Then we can define a function $V(x)$ as follows,

$$V(x)=\begin{cases}\varphi_{n+1}(x), & x\in[0,\tau_n],\\ 0, & x\in(\tau_n,1].\end{cases}$$

Obviously, $V(x)\in L^2([0,1],x)$ and $V(x)\ge 0$ on $[0,1]$. By (1.12), the function $V(x)$ can be expanded as the following Fourier-Bessel series

$$V(x)=\sum_{k=1}^{\infty}a_k\varphi_k(x),$$

where

$$\begin{aligned}a_k=\frac{(V,\varphi_k)}{(\varphi_k,\varphi_k)}&=\frac{2}{[J_1(\mu_k)]^2}\int_0^1 V(x)\varphi_k(x)xdx\\&=\frac{2}{[J_1(\mu_k)]^2}\int_0^{\tau_n}\varphi_{n+1}(x)\varphi_k(x)xdx.\end{aligned}$$

Now, we turn our attention to $a_k\varphi_k(\tau_n)$. Write

$$\begin{aligned} a_k\varphi_k(\tau_n) &= \frac{2\varphi_k(\tau_n)}{[J_1(\mu_k)]^2}\int_0^{\tau_n}\varphi_{n+1}(x)\varphi_k(x)x\,dx \\ &= \frac{2\varphi_k(\tau_n)}{[J_1(\mu_k)]^2}\int_0^{\tau_n}\psi_{n+1}(x)\psi_k(x)dx, \end{aligned} \tag{3.4}$$

where

$$\psi_k(x)=\varphi_k(x)x^{1/2}. \tag{3.5}$$

Since $\varphi_k(x)=J_0(\mu_k x)$, $\varphi_k(x)$ satisfies the following differential equation

$$x^2\varphi_k''(x)+x\varphi_k'(x)+\mu_k^2x^2\varphi_k(x)=0. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\psi_k''(x)+\left(\mu_k^2+\frac{1}{4x^2}\right)\psi_k(x)=0.$$

Hence,

$$\psi_k''(x)\psi_{n+1}(x)-\psi_{n+1}''(x)\psi_k(x)=(\mu_{n+1}^2-\mu_k^2)\psi_k(x)\psi_{n+1}(x). \tag{3.7}$$

Denote

$$c_k=\left(\mu_{n+1}^2-\mu_k^2\right)\int_0^{\tau_n}\psi_k(x)\psi_{n+1}(x)dx. \tag{3.8}$$

From (3.7), $\psi_{n+1}(\tau_n)=0$ and $\psi_k(0)=0$ for any $k\in\mathbb{N}$. Consequently,

$$\begin{aligned} c_k &= \int_0^{\tau_n}\left(\psi_k''(x)\psi_{n+1}(x)-\psi_{n+1}''(x)\psi_k(x)\right)dx \\ &= \left[\psi_k'(x)\psi_{n+1}(x)-\psi_{n+1}'(x)\psi_k(x)\right]\Big|_0^{\tau_n} \\ &= -\psi_{n+1}'(\tau_n)\psi_k(\tau_n)=-\varphi_{n+1}'(\tau_n)\varphi_k(\tau_n)\tau_n. \end{aligned} \tag{3.9}$$

If $k\neq n+1$, from (3.4), (3.8) and (3.9), we have

$$a_k\varphi_k(\tau_n)=\frac{2\varphi_k(\tau_n)}{[J_1(\mu_k)]^2}\frac{c_k}{\mu_{n+1}^2-\mu_k^2}=-\frac{2\varphi_{n+1}'(\tau_n)[\varphi_k(\tau_n)]^2\tau_n}{(\mu_{n+1}^2-\mu_k^2)[J_1(\mu_k)]^2}.$$

Since $\varphi_{n+1}'(\tau_n)<0$ and $\mu_k>\mu_j$ for $k>j$, it is easy to see that

$$a_k\varphi_k(\tau_n)<0,\quad k>n+1;\quad a_k\varphi_k(\tau_n)>0,\quad k=1,2,\cdots,n.$$

If $k=n+1$, then $\varphi_{n+1}(\tau_n)=0$, which leads to $a_{n+1}\varphi_{n+1}(\tau_n)=0$. Hence, consider the following set of functions:

$$\mathscr{D}_{n+1}=\left\{\sum_{k=n+1}^{\infty}\rho_k\varphi_k:\sum_{k=n+1}^{\infty}\rho_k=1,\quad \rho_k\geq 0,\quad k\geq n+1\right\},$$

where $n \in \mathbb{N}$ and $\varphi_k(x) = J_0(\mu_k x)$. Then for any $F \in \mathscr{D}_{n+1}$, one can obtain

$$\begin{aligned}\int_0^{\tau_n} T(\tau_n)F(x)V(x)x\,dx &= \int_0^1 T(\tau_n)F(x)V(x)x\,dx \\ &= \int_0^1 \Big(\sum_{k=n+1}^{\infty} \rho_k \varphi_k(\tau_n)\varphi_k(x)\Big)\Big(\sum_{m=1}^{\infty} a_m \varphi_m(x)\Big)x\,dx \\ &= \frac{1}{2}\sum_{k=n+1}^{\infty} \rho_k a_k \varphi_k(\tau_n)[J_1(\mu_k)]^2 < 0.\end{aligned}$$

Therefore, there exists a point $\tilde{x} \in [0, \tau_n]$, such that $T(\tau_n)F(\tilde{x}) < 0$. Then by the definition of $T(\tau_n)F(\tilde{x})$, there exists a point $x^* \in [0, 2\tau_n]$, such that $F(x^*) < 0$. So we have

$$\inf_{0 \le t \le 2\tau_n} \sum_{k=n+1}^{\infty} \rho_k \varphi_k(t) < 0, \tag{3.10}$$

where $\sum_{k=n+1}^{\infty} \rho_k = 1$, and $\rho_k \geq 0$ for $k \geq n+1$. Thus, for all $r > 0$,

$$\max_{0 \le t \le 2\tau_n} \sum_{k=n+1}^{\infty} \rho_k [1 - r\varphi_k(t)] > 1.$$

With the help of the inequality $1 - ru \leq (1-u)^r$ for $|u| \leq 1$ and $r \geq 1$, we have

$$\begin{aligned}\sum_{k=n+1}^{\infty} \rho_k = 1 &< \max_{0 \le t \le 2\tau_n} \sum_{k=n+1}^{\infty} \rho_k [1 - r\varphi_k(t)] \\ &\leq \max_{0 \le t \le 2\tau_n} \sum_{k=n+1}^{\infty} \rho_k [1 - \varphi_k(t)]^r,\end{aligned}$$

which together with (2.11) and (2.12) yields that $\mathscr{K}_n(2\tau_n, r) \leq 1$. We complete the proof of Theorem 3.2. $\square$

Theorem 1.1 is an easy corollary of Theorems 3.1 and 3.2.

3.2. Proof of Theorem 1.2

Using the inequality (3.10) in the proof of Theorem 3.2, we have

$$\max_{0 \le t \le 2\tau_n} \sum_{k=n+1}^{\infty} \rho_k [1 - \varphi_k(t)] > 1.$$

Using the fact that $2^{r-1}(1-u) \leq (1-u)^r$ for $0 < r < 1$, $|u| \leq 1$, we obtain

$$\begin{aligned}2^{r-1}\sum_{k=n+1}^{\infty} \rho_k &= 2^{r-1} < 2^{r-1} \max_{0 \le t \le 2\tau_n} \sum_{k=n+1}^{\infty} \rho_k [1 - \varphi_k(t)] \\ &\leq \max_{0 \le t \le 2\tau_n} \sum_{k=n+1}^{\infty} \rho_k [1 - \varphi_k(t)]^r.\end{aligned}$$

The above inequality and the formula (2.11), (2.12) give us the desired conclusion. □

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