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Spectral Method for Three-Dimensional Nonlinear Klein-Gordon Equation by Using Generalized Laguerre and Spherical Harmonic Functions

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Abstract. In this paper, a generalized Laguerre-spherical harmonic spectral method is proposed for the Cauchy problem of three-dimensional nonlinear Klein-Gordon equation. The goal is to make the numerical solutions to preserve the same conservation as that for the exact solution. The stability and convergence of the proposed scheme are proved. Numerical results demonstrate the efficiency of this approach. We also establish some basic results on the generalized Laguerre-spherical harmonic orthogonal approximation, which play an important role in spectral methods for various problems defined on the whole space and unbounded domains with spherical geometry.

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Key words: Generalized Laguerre-spherical harmonic spectral method, Cauchy problem of nonlinear Klein-Gordon equation.

1. Introduction

In this paper, we consider the nonlinear Klein-Gordon equation:

$$\partial_t^2 U(\mathbf{x},t) - \Delta U(\mathbf{x},t) + U(\mathbf{x},t) + U^3(\mathbf{x},t) = f(\mathbf{x},t), \tag{1.1}$$

which plays an important role in several fields, such as quantum mechanics, relativistic scalar field with power interaction, soliton theory and nonlinear meson theory of nuclear forces, see, e.g., [3,15,17,19,20]. The existence, uniqueness and regularity of its solution were studied, see, e.g., [17,20]. On the other hand, many algorithms were proposed for its numerical simulation. In the early work, we usually employed finite difference method

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for periodical problems and non-periodical problems defined on bounded domains, see, e.g., [1, 16, 22] and the references therein. Some authors developed the Legendre spectral method for (1.1) defined on a finite interval, see [8]. In practice, it is also important and interesting to consider non-periodical problems in the whole multiple-dimensional space. We might use finite difference method for such problems. However, in this case, we have to confine our computation to certain bounded subdomain with artificial boundary condition. This treatment induces additional numerical errors. Thereby, it seems reasonable to solve (1.1) directly by using spectral method for the whole space. Whereas, so far, there is no result on the numerical solution of non-periodical Cauchy problem of multiple-dimensional nonlinear Klein-Gordon equation.

This work is concerned with numerical solution of the Cauchy problem of three dimensional nonlinear Klein-Gordon equation. Because of several reasons, we preferable to an alternative formulation in spherical coordinates. Let $\mathbf{x} = (x_1, x_2, x_3)^T$ and

$$x_1 = \rho\cos\lambda\cos\theta, \quad x_2 = \rho\sin\lambda\cos\theta, \quad x_3 = \rho\sin\theta.$$

Then (1.1) becomes

$$\begin{aligned}&\partial_t^2 U(\rho,\lambda,\theta,t) - \frac{1}{\rho^2}\partial_\rho\big(\rho^2\partial_\rho U(\rho,\lambda,\theta,t)\big) - \frac{1}{\rho^2\cos\theta}\partial_\theta\big(\cos\theta\,\partial_\theta U(\rho,\lambda,\theta,t)\big)\\&\quad - \frac{1}{\rho^2\cos^2\theta}\partial_\lambda^2 U(\rho,\lambda,\theta,t) + U(t,\rho,\lambda,\theta) + U^3(\rho,\lambda,\theta,t) = f(\rho,\lambda,\theta,t).\end{aligned} \tag{1.2}$$

Since the longitude λ and the latitude θ vary on the finite intervals $[0,2\pi]$ and $[-\frac{1}{2}\pi,\frac{1}{2}\pi]$, respectively, there left only the variable ρ varying from 0 to ∞. This fact lowers the difficulties of calculation essentially. Moreover, we could adopt the spherical harmonic approximation in the longitude and latitude directions. Thereby, benefiting from the orthogonality of spherical harmonic functions, we simplify actual computation and numerical analysis.

The remaining problem is how to approximate (1.2) in the radial direction properly. It is natural to use certain orthogonal approximation on the half line. As we know, there have been already several kinds of Laguerre-type approximations. For instance, Funaro [5], Iranzo and Falques [14], Maday, Pernaud-Thomas and Vandeven [18], Guo and Shen [10], Guo and Xu [11], and Xu and Guo [23] developed the standard Laguerre approximation with its applications to partial differential equations defined on the half line and an infinite strip. Meanwhile, Guo, Shen and Xu [9], and Guo and Zhang [13] considered the generalized Laguerre approximation with its applications to two-dimensional exterior problems an so on.

In order to solve (1.2) efficiently, we need a specific orthogonal approximation, based on the main feature of (1.2).

• The nonlinear Klein-Gordon equation possesses certain conservation, which plays an important role in both theoretical analysis and numerical simulation. But the weight function $e^{-\rho}$ used in the standard Laguerre approximation destroys such property. Although we may reformulate (1.2) to a well-posed problems in a certain weighted space and then resolve the resulting problem by the standard Laguerre spectral method, it is much simpler to approximate (1.2) directly. As a result, the numerical solution keeps the same conservation as the exact solution.

• In the weak formulation of (1.2), the coefficients of terms

$$\partial_\rho U(\rho,\lambda,\theta,t),\ \ \partial_\theta U(\rho,\lambda,\theta,t),\ \ \partial_\lambda U(\rho,\lambda,\theta,t),\ \ U(\rho,\lambda,\theta),\ \ U^3(\rho,\lambda,\theta,t)$$

involve the factors $\rho^2, 1, 1, \rho^2$ and ρ^2, respectively. Thus, a specific orthogonal approximation with the most appropriate non-uniform weights for the derivatives of different orders, could simulate underlying problem properly and leads to better error estimate of numerical solution.

• To fit the asymptotic behaviors of exact solution well, it is better to modify the generalized Laguerre functions, so that the numerical solution matches the exact solution more closely.

According to the above discussions, we shall take the following functions as the basis functions in actual computation (cf. [12]),

$$\tilde{\mathscr{L}}_l^{(\alpha,\beta)}(\rho) = \frac{1}{l!}\rho^{-\alpha} e^{\frac{1}{2}\beta\rho}\partial_\rho^l(\rho^{l+\alpha}e^{-\beta\rho}), \quad \alpha > -1,\ \ \beta > 0, \quad l = 0,1,2,\cdots. \tag{1.3}$$

It is noted that Shen [21] applied the standard Laguerre functions $\tilde{\mathscr{L}}_l^{(0,1)}(\rho)$ to one-dimensional problems. But that approach is only available for some problems, in which the coefficients neither degenerate as $\rho \to 0$, nor brow up as $\rho \to \infty$. Therefore, it is no longer available for (1.2). Besides, Guo and Zhang [12] used the generalized Laguerre functions $\tilde{\mathscr{L}}_l^{(2,\beta)}(\rho)$ for differential equations of degenerate type on the half line, such as the Black-Scholes type equations. But so far, there has been no work concerning its application to multiple-dimensional problems.

The outline of this paper is as follows. In the next section, we introduce a new orthogonal approximation by using generalized Laguerre and spherical harmonic functions. In Section 3, we propose the generalized Laguerre-spherical harmonic spectral method for the Cauchy problem of three-dimensional nonlinear Klein-Gordon equation, and present the main results on its stability and convergence. In Section 4, we describe the numerical implementation and present some numerical results demonstrating the efficiency of this new approach. The final section is for some concluding remarks. In Appendix of this paper, we establish some results on the generalized Laguerre-spherical harmonic approximation, with which we prove the stability and convergence of suggested scheme.

2. Generalized Laguerre-spherical harmonic orthogonal approximation

We first recall the orthogonal approximation by using the generalized Laguerre functions. Let $\Lambda = \{\rho \mid 0 < \rho < \infty\}$ and $\chi(\rho)$ be a certain weight function. For any integer $r \geq 0$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ as usual, equipped with the inner product $(u,v)_{r,\chi,\Lambda}$, the semi-norm $|v|_{r,\chi,\Lambda}$ and the norm $||v||_{r,\chi,\Lambda}$, respectively. In particular, $H^0_\chi(\Lambda) = L^2_\chi(\Lambda)$ with the inner product $(u,v)_{\chi,\Lambda}$ and the norm $||v||_{\chi,\Lambda}$. For any $r > 0$, the space $H^r_\chi(\Lambda)$ with the norm $\|v\|_{r,\chi,\Lambda}$, is defined by space interpolation as in [2]. We omit the subscript χ in notation whenever $\chi(\rho)\equiv 1$.

The generalized Laguerre functions fulfill the following recurrences (cf. [12]),

$$(l+1)\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l+1}(\rho)+(\beta\rho-2l-\alpha-1)\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l}(\rho)+(l+\alpha)\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l-1}(\rho)=0, \tag{2.1}$$

$$\partial_\rho\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l}(\rho)=-\frac{1}{2}\beta\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l}(\rho)-\beta\tilde{\mathscr{L}}^{(\alpha+1,\beta)}_{l-1}(\rho), \tag{2.2}$$

$$\rho\partial_\rho\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l}(\rho)=\frac{1}{2}(l+1)\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l+1}(\rho)-\frac{1}{2}(\alpha+1)\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l}(\rho)-\frac{1}{2}(l+\alpha)\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l-1}(\rho). \tag{2.3}$$

The generalized Laguerre functions form a complete $L^2_{\rho^\alpha}(\Lambda)$-orthogonal system, namely,

$$(\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l},\ \tilde{\mathscr{L}}^{(\alpha,\beta)}_{m})_{\rho^\alpha,\Lambda}=\gamma^{(\alpha,\beta)}_l\delta_{l,m},\qquad \gamma^{(\alpha,\beta)}_l=\frac{\Gamma(l+\alpha+1)}{\beta^{\alpha+1}l!}, \tag{2.4}$$

where $\delta_{l,m}$ is the Kronecker symbol. Thus, for any $v\in L^2_{\rho^\alpha}(\Lambda)$,

$$v(\rho)=\sum_{l=0}^{\infty}\tilde{\hat{v}}^{(\alpha,\beta)}_l\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l}(\rho),\qquad \tilde{\hat{v}}^{(\alpha,\beta)}_l=\frac{1}{\gamma^{(\alpha,\beta)}_l}\int_0^\infty\rho^\alpha v(\rho)\tilde{\mathscr{L}}^{(\alpha,\beta)}_{l}(\rho)d\rho. \tag{2.5}$$

For any positive integer N, $\mathscr{P}_N(\Lambda)$ stands for the set of all algebraic polynomials of degree at most N. Throughout this paper, we denote by c a generic positive constant independent of N,β and any function.

We set

$$\mathscr{Q}_{N,\beta}(\Lambda)=\{e^{-\frac{1}{2}\beta\rho}\psi\mid\psi\in\mathscr{P}_N(\Lambda)\}.$$

The orthogonal projection $\tilde{P}_{N,\alpha,\beta,\Lambda}:L^2_{\rho^\alpha}(\Lambda)\to\mathscr{Q}_{N,\beta}(\Lambda)$ is defined by

$$(\tilde{P}_{N,\alpha,\beta,\Lambda}v-v,\phi)_{\rho^\alpha,\Lambda}=0,\quad\forall\phi\in\mathscr{Q}_{N,\beta}(\Lambda). \tag{2.6}$$

Let $\omega_{\alpha,\beta}(\rho)=\rho^\alpha e^{-\beta\rho}$. It was shown in Theorem 3.1 of [12] that for any integers $r\geq 0$ and $r\leq N+1$,

$$\|\tilde{P}_{N,\alpha,\beta,\Lambda}v-v\|_{\rho^\alpha,\Lambda}\leq c(\beta N)^{-\frac{r}{2}}||\partial^r_\rho(e^{\frac{1}{2}\beta\rho}v)||_{\omega_{\alpha+r,\beta},\Lambda}, \tag{2.7}$$

provided that $||\partial^r_\rho(e^{\frac{1}{2}\beta\rho}v)||_{\omega_{\alpha+r,\beta},\Lambda}$ is finite.

Remark 2.1. Obviously,

$$||\partial^r_\rho(e^{\frac{1}{2}\beta\rho}v)||_{\omega_{\alpha+r,\beta},\Lambda}\leq c_\beta||v||_{r,\rho^{\alpha+r},\Lambda},$$

where c_β is a positive constant depending on β.

We now turn to the spherical harmonic approximation. Denote by S the unit spherical surface,

$$S=\left\{(\lambda,\theta)\mid 0\le\lambda<2\pi,\ -\frac{\pi}{2}\le\theta<\frac{\pi}{2}\right\}.$$

The differentiations with respect to λ and θ are denoted by $\partial_\lambda v(\lambda,\theta)$ and $\partial_\theta v(\lambda,\theta)$, respectively. Furthermore,

$$\begin{aligned}\nabla_S v(\lambda,\theta)&=\Big(\frac{1}{\cos\theta}\partial_\lambda v(\lambda,\theta),\partial_\theta v(\lambda,\theta)\Big),\\ \Delta_S v(\lambda,\theta)&=\frac{1}{\cos\theta}(\partial_\theta(\cos\theta\,\partial_\theta v(\lambda,\theta))+\frac{1}{\cos^2\theta}\partial_\lambda^2 v(\lambda,\theta).\end{aligned}$$

We define the space $L^2(S)$ as usual, with the following inner product and norm,

$$(u,v)_S=\int_S u(\lambda,\theta)v(\lambda,\theta)dS=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}u(\lambda,\theta)v(\lambda,\theta)\cos\theta\,d\lambda d\theta,\quad \|v\|_S=(v,v)_S^{\frac{1}{2}}.$$

Next, let

$$H^1(S)=\left\{v\mid v,\frac{1}{\cos\theta}\partial_\lambda v,\partial_\theta v\in L^2(S)\right\},$$

equipped with the semi-norm and the norm as

$$|v|_{1,S}=\left(\left\|\frac{1}{\cos\theta}\partial_\lambda v\right\|_S^2+\left\|\partial_\theta v\right\|_S^2\right)^{\frac{1}{2}},\quad \|v\|_{1,S}=\left(\|v\|_S^2+|v|_{1,S}^2\right)^{\frac{1}{2}}.$$

For any integer $r>0$, we define the space $H^r(S)$ inductively, namely,

$$\begin{aligned}H^{2k}(S)&=\left\{v\mid v\in H^{2k-1}(S),\Delta_S^k v\in L^2(S)\right\},\\ H^{2k+1}(S)&=\left\{v\mid v\in H^{2k}(S),\nabla_S(\Delta_S^k v)\in\left[L^2(S)\right]^2\right\},\end{aligned}$$

with the following semi-norms and norms,

$$\begin{aligned}|v|_{2k,S}&=||\Delta_S^k v||_S, & ||v||_{2k,S}&=(||v||_{2k-1,S}^2+|v|_{2k,S}^2)^{\frac{1}{2}},\\ |v|_{2k+1,S}&=||\nabla_S(\Delta_S^k v)||_S, & ||v||_{2k+1,S}&=(||v||_{2k,S}^2+|v|_{2k+1,S}^2)^{\frac{1}{2}}.\end{aligned}$$

In particular, the norm $\|v\|_{2,S}$ is equivalent to $(\|v\|_S^2+\|\Delta_S v\|_S^2)^{\frac{1}{2}}$ (cf. [6,7]). Further, let

$$\tilde{H}^r(S)=\{v\mid v\in H^r(S)\ \text{ and }\ \partial_\lambda^k v(\lambda+2\pi,\theta)=\partial_\lambda^k v(\lambda,\theta),\quad 0\le k\le r-1\}.$$

For any $r>0$, the spaces $H^r(S)$ and $\tilde{H}^r(S)$ are defined by space interpolation as in [2].

Let $L_l(x)$ be the Legendre polynomial of degree l. The normalized associated Legendre functions are given by

$$\begin{cases}L_{l,m}(x)=\sqrt{\dfrac{(2m+1)(m-l)!}{2(m+l)!}}\left(1-x^2\right)^{\frac{l}{2}}\partial_x^l L_m(x), & \text{for}\quad l\ge 0,\ m\ge|l|,\\ L_{l,m}(x)=L_{-l,m}(x), & \text{for}\quad l<0,\ m\ge|l|.\end{cases}$$

The spherical harmonic function are defined by

$$Y_{l,m}(\lambda,\theta)=\frac{1}{\sqrt{2\pi}}e^{il\lambda}L_{l,m}(\sin\theta),\qquad m\geq|l|.$$

The set of $Y_{l,m}(\lambda,\theta)$ is the complete normalized $L^2(S)$-orthogonal system, i.e.,

$$\int_S Y_{l,m}(\lambda,\theta)\bar{Y}_{l',m'}(\lambda,\theta)dS=\delta_{l,l'}\delta_{m,m'}. \tag{2.8}$$

Thus, for any $v\in L^2(S)$,

$$v(\lambda,\theta)=\sum_{l=-\infty}^{\infty}\sum_{m\geq|l|}\hat{v}_{l,m}Y_{l,m}(\lambda,\theta),\qquad \hat{v}_{l,m}=\int_S v(\lambda,\theta)\bar{Y}_{l,m}(\lambda,\theta)dS.$$

Let M be any positive integer, and

$$\tilde{V}_M(S)=\mathrm{span}\left\{Y_{l,m}(\lambda,\theta)\mid |l|\leq M,\ |l|\leq m\leq M\right\}.$$

Denote by $V_M(S)$ the subset of $\tilde{V}_M(S)$ containing all real-valued functions. The $L^2(S)$-orthogonal projection $P_{M,S}:L^2(S)\to V_M(S)$ is defined by

$$(P_{M,S}v-v,\phi)_S=0,\quad \forall\phi\in V_M(S). \tag{2.9}$$

According to a slight modification of Lemma 5 of [6], we have that (cf. (2.13) of [7]) for any $v\in\tilde{H}^r(S)$ and $0\leq\mu\leq r$,

$$\|P_{M,S}v-v\|_{\mu,S}\leq cM^{\mu-r}|v|_{r,S}. \tag{2.10}$$

We now introduce the new approximation by using generalized Laguerre and spherical harmonic functions. To do this, let $\Omega=\Lambda\times S$ and $\chi(\rho)$ be a certain weight function. The weighted Sobolev spaces $L^2_\chi(\Omega)$ and $H^r_\chi(\Omega)$ are defined in the usual way. Their norms are denoted by $\|v\|_{\chi,\Omega}$ and $\|v\|_{r,\chi,\Omega}$, respectively. The inner product and norm of the space $L^2_\chi(\Omega)$ are given by

$$\|v\|_{\chi,\Omega}=(v,v)^{\frac{1}{2}}_{\chi,\Omega},$$

and

$$\begin{aligned}(u,v)_{\chi,\Omega}&=\int_\Omega u(\rho,\lambda,\theta)v(\rho,\lambda,\theta)\chi(\rho)d\Omega\\&=\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}u(\rho,\lambda,\theta)v(\rho,\lambda,\theta)\chi(\rho)\cos\theta d\lambda d\theta d\rho.\end{aligned}$$

We omit the subscript χ in notation whenever $\chi(\rho)\equiv 1$.

In actual computation, we take $V_{N,M,\beta}(\Omega)=\mathcal{Q}_{N,\beta}(\Lambda)\otimes V_M(S)$ as the space of trial functions. The orthogonal projection $P_{N,M,\alpha,\beta}:L^2_{\rho^\alpha}(\Omega)\to V_{N,M,\beta}(\Omega)$ is defined by

$$(P_{N,M,\alpha,\beta}v-v,\phi)_{\rho^\alpha,\Omega}=0,\quad \forall\phi\in V_{N,M,\beta}(\Omega). \tag{2.11}$$

In the forthcoming discussions, we shall use a specific orthogonal approximation. For this purpose, we define the spaces

$$H^1_{\rho^2}(\Omega) = \{v \mid ||v||_{1,\rho^2,\Omega} < \infty\},$$
$$\tilde{H}^1_{\rho^2}(\Omega) = H^1_{\rho^2}(\Omega) \cap \{v \mid v(\rho,\lambda+2\pi,\theta) = v(\rho,\lambda,\theta)\},$$

with the norm

$$||v||_{1,\rho^2,\Omega} = \left(||\partial_\rho v||^2_{\rho^2,\Omega} + \left\| \frac{1}{\cos\theta} \partial_\lambda v \right\|^2_\Omega + ||\partial_\theta v||^2_\Omega + ||v||^2_{\rho^2,\Omega} \right)^{\frac{1}{2}}.$$

It is noted that

$$||\nabla_S v||^2_\Omega = \left\| \frac{1}{\cos\theta} \partial_\lambda v \right\|^2_\Omega + ||\partial_\theta v||^2_\Omega.$$

The orthogonal projection $P^1_{N,M,\beta} : \tilde{H}^1_{\rho^2}(\Omega) \to V_{N,M,\beta}(\Omega)$ is defined by

$$\begin{aligned}&(\partial_\rho\ (P^1_{N,M,\beta} v - v), \partial_\rho \phi)_{\rho^2,\Omega} + (\nabla_S (P^1_{N,M,\beta} v - v), \nabla_S \phi)_\Omega \\ &\quad + (P^1_{N,M,\beta} v - v, \phi)_{\rho^2,\Omega} = 0, \qquad \forall \phi \in V_{N,M,\beta}(\Omega).\end{aligned}$$

We shall estimate the approximation errors in the appendix of this paper.

3. Generalized Laguerre-spherical harmonic spectral method

In this section, we provide the generalized Laguerre-spherical harmonic spectral method for the nonlinear Klein-Gordon equation. We consider the following problem

$$\begin{cases} \partial_t^2 U(\rho,\lambda,\theta,t) - \Delta U(\rho,\lambda,\theta,t) + U(\rho,\lambda,\theta,t) & \\ \quad + U^3(\rho,\lambda,\theta,t) = f(\rho,\lambda,\theta,t), & \text{in} \quad \Omega \times (0,T], \\ U(\rho,\lambda+2\pi,\theta,t) = U(\rho,\lambda,\theta,t), & \text{in} \quad \Omega \times [0,T], \\ \rho^{\frac{3}{2}} U(\rho,\lambda,\theta,t) \to 0 \quad \text{as } \rho \to \infty, & \text{on} \quad S \times [0,T], \ a.e., \\ \partial_t U(\rho,\lambda,\theta,0) = U_1(\rho,\lambda,\theta), & \text{in} \quad \Omega, \\ U(\rho,\lambda,\theta,0) = U_0(\rho,\lambda,\theta), & \text{in} \quad \Omega. \end{cases} \tag{3.1}$$

In addition, we have (cf. [4]):

$$\partial_\lambda U(\rho,\lambda,\theta,t) = 0 \quad \text{for} \ \ |\theta| = \frac{1}{2}\pi.$$

Let $p \geq 1$ and $L^p_{\rho^2}(\Omega)$ be the weighted space with the norm $||v||_{L^p_{\rho^2}(\Omega)}$. Indeed, this norm is equivalent to the norm $||v||_{L^p(\mathscr{R}^3)}$ in the Cartesian coordinates. A similar relation is valid between the norms $||v||_{H^1_{\rho^2}(\Omega)}$ and $||v||_{H^1(\mathscr{R}^3)}$. By the imbedding theory (see, e.g., [2]),

$$H^1(\mathscr{R}^3) \hookrightarrow L^p(\mathscr{R}^3), \quad p \leq 6.$$

Therefore,

$$H^1_{\rho^2}(\Omega) \hookrightarrow L^p_{\rho^2}(\Omega), \quad p \le 6.$$

Multiplying (3.1) by $\rho^2 v$ and integrating the resulting equation by parts, we derive a weak formulation. It is to find $U \in L^\infty(0,T;H^1_{\rho^2}(\Omega)) \cap W^{1,\infty}(0,T;L^2_{\rho^2}(\Omega))$ such that

$$\begin{aligned}&\int_\Omega \rho^2 \partial_t^2 U(t) v d\Omega + \int_\Omega \rho^2 \partial_\rho U(t) \partial_\rho v d\Omega + \int_\Omega (\nabla_S U(t) \cdot \nabla_S v) d\Omega \\ &\quad + \int_\Omega \rho^2 U(t) v d\Omega + \int_\Omega \rho^2 U^3(t) v d\Omega = \int_\Omega \rho^2 f(t) v d\Omega, \quad \forall v \in H^1_{\rho^2}(\Omega), \end{aligned} \tag{3.2}$$

coupled with the initial state as in (3.1).

The solution of (3.2) possesses certain conservation. To show this, we set

$$E(v,t) = ||\partial_t v(t)||^2_{\rho^2,\Omega} + ||\partial_\rho v(t)||^2_{\rho^2,\Omega} + ||\nabla_S v(t)||^2_\Omega + ||v(t)||^2_{\rho^2,\Omega} + \frac{1}{2}||v(t)||^4_{L^4_{\rho^2}(\Omega)}.$$

Taking $\phi = 2\partial_t U(t)$ in (3.2) and integrating the result with respect to t, we obtain

$$E(U,t) = E(U,0) + 2\int_0^t (f(\xi), \partial_t U(\xi))_{\rho^2,\Omega} d\xi. \tag{3.3}$$

This fact leads to that if $U_0 \in H^1_{\rho^2}(\Omega), U_1 \in L^2_{\rho^2}(\Omega)$ and $f \in L^2(0,T;L^2_{\rho^2}(\Omega))$, then

$$||U||_{L^\infty(0,T;H^1_{\rho^2}(\Omega))\cap W^{1,\infty}(0,T;L^2_{\rho^2}(\Omega))} \le d(U_0,U_1,f), \tag{3.4}$$

where the quantity $d(U_0,U_1,f)$ depends only on $||U_0||_{1,\rho^2,\Omega}$, $||U_1||_{\rho^2,\Omega}$ and $||f||_{L^2(0,T;L^2_{\rho^2}(\Omega))}$.

The generalized Laguerre-spherical harmonic spectral scheme for (3.2) is to seek $u_{N,M}(t) \in V_{N,M,\beta}(\Omega)$ such that

$$\begin{cases}\displaystyle\int_\Omega \rho^2 \partial_t^2 u_{N,M}(t) \phi d\Omega + \int_\Omega \rho^2 \partial_\rho u_{N,M}(t) \partial_\rho \phi d\Omega + \int_\Omega (\nabla_S u_{N,M}(t), \nabla_S \phi) d\Omega \\ \displaystyle\quad + \int_\Omega \rho^2 u_{N,M}(t) \phi d\Omega + \int_\Omega \rho^2 u^3_{N,M}(t) \phi d\Omega = \int_\Omega \rho^2 f(t) \phi d\Omega, \quad \forall \phi \in V_{N,M,\beta}(\Omega), \\ \partial_t u_{N,M}(0) = P_{N,M,2,\beta} U_1, \\ u_{N,M}(0) = P^1_{N,M,\beta} U_0. \end{cases} \tag{3.5}$$

Taking $\phi = 2\partial_t u_{N,M}(t)$ in (3.5) and integrating the result with respect to t, we obtain

$$E(u_{N,M},t) = E(u_{N,M},0) + 2\int_0^t (f(\xi), \partial_t u_{N,M}(\xi))_{\rho^2,\Omega} d\xi. \tag{3.6}$$

Clearly, (3.6) simulates (3.3) properly. Thus, the scheme (3.5) is expected to provide reasonable numerical solution of (3.2). In opposite, if we use the standard generalized

Laguerre approximation with the weight $\rho^2 e^{-\rho}$, then we lose such conservation. In fact, this is one of main motivations of this work.

It can be verified that

$$||u_{N,M}||_{L^\infty(0,T;H^1_{\rho^2}(\Omega))\cap W^{1,\infty}(0,T;L^2_{\rho^2}(\Omega))} \le d(U_0, U_1, f). \tag{3.7}$$

We now consider the stability of scheme (3.5). Assume that the data f, $\partial_t u_{N,M}(0)$ and $u_{N,M}(0)$ are disturbed by $\tilde{f}$, $\tilde{\partial}_t u_{N,M}(0)$ and $\tilde{u}_{N,M}(0)$ respectively, which induce the error of numerical solution, denoted by $\tilde{u}_{N,M}$. We measure the average error as

$$R(t) = E(\tilde{u}_{N,M}, 0) + 2\int_0^t ||\tilde{f}(\xi)||^2_{\rho^2,\Omega} d\xi.$$

We have the following result.

Theorem 3.1. *If for a suitably small constant $c_* > 0$, $R(T) \le c_*$, then for all $0 \le t \le T$,*

$$E(\tilde{u}_{N,M}, t) \le d^* e^{d_* t} R(t),$$

where the positive constants d^* and d_* depend only on c_*.

We next deal with the convergence of (3.5). For description of the numerical errors, we introduce the non-isotropic space $\mathscr{B}^{r,s}(\Omega)$, equipped with the norm

$$||v||_{\mathscr{B}^{r,s}(\Omega)} = \left(||v||^2_{H^1(S,H^r_{\rho^{r+1}}(\Lambda))} + ||v||^2_{H^s(S,L^2(\Lambda))} + ||v||^2_{H^{s-1}(S,H^1_{\rho^2}(\Lambda))} \right)^{\frac{1}{2}}.$$

For simplicity of statements, we shall use the following notations:

$$\begin{aligned}
\mathscr{A}_{r,\beta}(v) &= \beta^{1-r}\int_S ||\partial_\rho^{r-1}(e^{\frac{1}{2}\beta\rho}v)||^2_{\omega_{r+1,\beta,\Lambda}} dS,\\
\mathscr{A}^*_{r,\beta}(v) &= ||v||^2_{H^{s-1}(S,L^2_{\rho^2}(\Lambda))},\\
\mathscr{D}_{r,\beta}(v) &= (1+\frac{1}{\beta})^4\beta^{1-r}\int_S \left(||\partial_\rho^r(e^{\frac{\beta\rho}{2}}v)||^2_{\omega_{r+1,\beta},\Lambda} + \left\| \frac{1}{\cos\theta}\partial_\lambda\partial_\rho^r(e^{\frac{\beta\rho}{2}}v)\right\|^2_{\omega_{r+1,\beta},\Lambda} \right.\\
&\quad \left. + ||\partial_\theta\partial_\rho^r(e^{\frac{\beta\rho}{2}}v)||^2_{\omega_{r+1,\beta},\Lambda} \right) dS,\\
\mathscr{D}^*_s(v) &= ||\partial_\rho v||^2_{H^{s-1}(S,L^2_{\rho^2}(\Lambda))} + ||v||^2_{H^s(S,L^2(\Lambda))}.
\end{aligned}$$

Moreover, we define

$$
\begin{aligned}
&d_\beta(U)=\max_{0\le t\le T}\Big(\mathscr{D}_{1,\beta}(U(t))+\mathscr{D}_1^*(U(t))\Big)^2,\\
&R_{N,M,\beta}(U,t)=N^{1-r}\int_0^T\mathscr{D}_{r,\beta}(\partial_t^2U(\xi))d\xi+M^{2-2s}\int_0^T\mathscr{D}_s^*(\partial_t^2U(\xi))d\xi\\
&\qquad+cN^{1-r}\Big(d_\beta^{\frac12}(U)+1\Big)\int_0^T\mathscr{D}_{r,\beta}(U(\xi))d\xi+cM^{2-2s}\Big(d_\beta^{\frac12}(U)+1\Big)\int_0^T\mathscr{D}_s^*(U(\xi))d\xi,\\
&R^*_{N,M,\beta}(U_0,U_1)=N^{1-r}\Big(\mathscr{D}_{r,\beta}(U_1)+\mathscr{A}_{r,\beta}(U_1)\Big)+M^{2-2s}\Big(\mathscr{D}^*_{r,\beta}(U_1)+\mathscr{A}^*_{r,\beta}(U_1)\Big).
\end{aligned}
$$

We have the following result.

Theorem 3.2. *If* $U\in H^2(0,T;\mathscr{B}^{r,s}(\Omega))\cap L^\infty(0,T;H^1_{\rho^2}(\Omega)\cap\mathscr{B}^{1,1}(\Omega))$, $s>1$ *and integers* $1<r\le N+1$, *then*

$$
E(U-u_{N,M},t)\le ce^{c(d_\beta(U)+1)t}\Big(R_{N,M,\beta}(U,t)+R^*_{N,M,\beta}(U_0,U_1)\Big).
$$

We shall prove Theorems 3.1 and 3.2 in the appendix of this paper.

According to the definition of $R_{N,M,\beta}(U,t)$, the result presented in Theorem 3.2 does not seem optimal. It is due to the property of the Laguerre approximation. We will get back to this issue in the end of the appendix of this paper.

4. Numerical results

We first describe the implementation for scheme (3.5). We take

$$
\varphi^{(q)}_{l,j,k}(\rho,\lambda,\theta)=\psi_l(\rho)\mathscr{Z}^{(q)}_{j,k}(\lambda,\theta)
$$

as the basis functions, where $\psi_l(\rho)=\tilde{\mathscr{L}}^{(0,\beta)}_l(\rho)$ and

$$
\mathscr{Z}^{(1)}_{j,k}(\lambda,\theta)=\frac{1}{\sqrt{2\pi}}\sin(j\lambda)L_{j,k}(\sin\theta),\qquad \mathscr{Z}^{(2)}_{j,k}(\lambda,\theta)=\frac{1}{\sqrt{2\pi}}\cos(j\lambda)L_{j,k}(\sin\theta).
$$

Using (2.1) with $\alpha=0$ gives

$$
\beta\rho\partial_\rho\psi_l(\rho)=(2l+1)\psi_l(\rho)-(l+1)\psi_{l+1}(\rho)-l\psi_{l-1}(\rho). \tag{4.1}
$$

Accordingly,

$$
\begin{aligned}
\int_\Lambda\rho^2\psi_l(\rho)\psi_n(\rho)d\rho=\frac{1}{\beta^3}\Big\{&(6l^2+6l+2)\delta_{l,n}-4l^2\delta_{l-1,n}-4(l+1)^2\delta_{l,n-1}\\
&+l(l-1)\delta_{l-2,n}+(l+1)(l+2)\delta_{l,n-2}\Big\}.
\end{aligned} \tag{4.2}
$$

On the other hand, (2.3) with $\alpha=0$ implies

$$2\rho\partial_\rho\psi_l(\rho)=(l+1)\psi_{l+1}(\rho)-\psi_l(\rho)-l\psi_{l-1}(\rho).$$

This leads to

$$\begin{aligned}4\rho^2\partial_\rho^2\psi_l(\rho)=&(l+1)(l+2)\psi_{l+2}(\rho)-4(l+1)\psi_{l+1}(\rho)-2(l^2+l-1)\psi_l(\rho)\\&+4l\psi_{l-1}(\rho)+l(l-1)\psi_{l-2}(\rho).\end{aligned}\tag{4.3}$$

Now, we take $\phi(\rho,\lambda,\theta)=\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)$ in (3.5). Since

$$\Delta_S\mathscr{X}_{j,k}^{(q)}(\rho,\lambda,\theta)=-k(k+1)\mathscr{X}_{j,k}^{(q)}(\rho,\lambda,\theta),$$

we deduce that

$$\begin{aligned}&\int_\Omega\rho^2\partial_t^2u_{N,M}(\rho,\lambda,\theta,t)\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)d\Omega\\&\quad-\int_\Omega u_{N,M}(\rho,\lambda,\theta,t)\Big(2\rho\partial_\rho\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)+\rho^2\partial_\rho^2\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)\Big)d\Omega\\&\quad+\int_\Omega p(p+1)u_{N,M}(\rho,\lambda,\theta,t)\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)d\Omega+\int_\Omega\rho^2u_{N,M}(\rho,\lambda,\theta,t)\\&\quad\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)d\Omega+\int_\Omega\rho^2u_{N,M}^3(\rho,\lambda,\theta,t)\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)d\Omega\\=&\,d(l,q)\int_\Omega\rho^2f(\rho,\lambda,\theta,t)\varphi_{l,m,p}^{(q)}(\rho,\lambda,\theta)d\Omega,\end{aligned}\tag{4.4}$$

where

$$d(l,q)=\begin{cases}1, & l=0,\ q=2,\\2, & \text{otherwise.}\end{cases}$$

We expand the numerical solution as

$$\begin{aligned}u_{N,M}(\rho,\lambda,\theta,t)=&\sum_{n=0}^N\sum_{j=1}^M\sum_{k=j}^M u_{n,j,k}^{(1)}(t)\varphi_{n,j,k}^{(1)}(\rho,\lambda,\theta)\\&+\sum_{n=0}^N\sum_{j=0}^M\sum_{k=j}^M u_{n,j,k}^{(2)}(t)\varphi_{n,j,k}^{(2)}(\rho,\lambda,\theta).\end{aligned}\tag{4.5}$$

For notational convenience, we shall use the following notations:

$$\begin{aligned}\mathscr{Y}^{(q)}_{l,m,p}(u_{N,M}(t)) =& \, l(l-1)u^{(q)}_{l-2,m,p}(t) - 4l^2 u^{(q)}_{l-1,m,p}(t) + 2\left(3l^2+3l+1\right)u^{(q)}_{l,m,p}(t) \\ & - 4(l+1)^2 u^{(q)}_{l+1,m,p}(t) + (l+1)(l+2)u^{(q)}_{l+2,m,p}(t), \end{aligned} \tag{4.6a}$$

$$\begin{aligned}\mathscr{M}^{(q)}_{l,m,p}(u_{N,M}(t)) =& \, \frac{1}{4}l(l-1)(\beta^2-4)u^{(q)}_{l-2,m,p}(t) + 4l^2 u^{(q)}_{l-1,m,p}(t) \\ & - \frac{1}{2}\left(\beta^2(l^2+l+1) + 4(3l^2+3l+1) + 2\beta^2 p(p+1)\right)u^{(q)}_{l,m,p}(t) \\ & + 4(l+1)^2 u^{(q)}_{l+1,m,p}(t) + \frac{1}{4}(l+1)(l+2)(\beta^2-4)u^{(q)}_{l+2,m,p}(t), \end{aligned} \tag{4.6b}$$

$$\mathscr{G}^{(q)}_{l,m,p}(u_{N,M}(t)) = -\beta^3 \int_\Omega \rho^2 u^3_{N,M}(\rho,\lambda,\theta,t)\varphi^{(q)}_{l,m,p}(\rho,\lambda,\theta)d\Omega, \tag{4.6c}$$

$$\mathscr{F}^{(q)}_{l,m,p}(u_{N,M}(t)) = \beta^3 \int_\Omega \rho^2 f(\rho,\lambda,\theta,t)\varphi^{(q)}_{l,m,p}(\rho,\lambda,\theta)d\Omega, \quad q=1,2. \tag{4.6d}$$

With the aid of (4.1)-(4.3), (4.5) and (4.6), we obtain from (4.4) that

$$\begin{aligned}&\partial_t^2 \mathscr{Y}^{(q)}_{l,m,p}(u_{N,M}(t)) \\ &= \mathscr{M}^{(q)}_{l,m,p}(u_{N,M}(t)) + \mathscr{G}^{(q)}_{l,m,p}(u_{N,M}(t)) + \mathscr{F}^{(q)}_{l,m,p}(u_{N,M}(t)), \quad q=1,2.\end{aligned} \tag{4.7}$$

In actual computation, we take the step size τ in time. We approximate $\partial_t^2 u^{(q)}_{l,m,p}(t)$ by

$$\frac{1}{\tau^2}\left(u^{(q)}_{l,m,p}(t+\tau) - 2u^{(q)}_{l,m,p}(t) + u^{(q)}_{l,m,p}(t-\tau)\right).$$

For stability of nonlinear calculation, we approximate the nonlinear term $u^3_{N,M}(\rho,\lambda,\theta,t)$ by

$$\begin{aligned}\frac{1}{4}\Big(& u^3_{N,M}(\rho,\lambda,\theta,t+\tau) + u^2_{N,M}(\rho,\lambda,\theta,t+\tau)u_{N,M}(\rho,\lambda,\theta,t-\tau) \\ & + u_{N,M}(\rho,\lambda,\theta,t+\tau)u^2_{N,M}(\rho,\lambda,\theta,t-\tau) + u^3_{N,M}(\rho,\lambda,\theta,t-\tau)\Big).\end{aligned}$$

We also introduce the notation

$$\begin{aligned}&\mathscr{G}^{(q,j)}_{l,m,p}(u_{N,M}(t+\tau)) \\ &= -\beta^3 \int_\Omega \rho^2 u^{3-j}_{N,M}(\rho,\lambda,\theta,t+\tau)u^j_{N,M}(\rho,\lambda,\theta,t-\tau)\varphi^{(q)}_{l,m,p}(\rho,\lambda,\theta)d\Omega, \quad j=1,2.\end{aligned}$$

Then we have from (4.7) that for $t = k\tau, k \geq 2$,

$$\begin{aligned}
&\mathscr{Y}^{(q)}_{l,m,p}(u_{N,M}(t+\tau)) - \frac{1}{2}\tau^2 \mathscr{M}^{(q)}_{l,m,p}(u_{N,M}(t+\tau)) - \frac{1}{4}\tau^2 \Big(\mathscr{G}^{(q)}_{l,m,p}(u_{N,M}(t+\tau)) \\
&\quad + \mathscr{G}^{(q,1)}_{l,m,p}(u_{N,M}(t+\tau)) + \mathscr{G}^{(q,2)}_{l,m,p}(u_{N,M}(t+\tau)) \Big) \\
&= 2\mathscr{Y}^{(q)}_{l,m,p}(u_{N,M}(t)) - \mathscr{Y}^{(q)}_{l,m,p}(u_{N,M}(t-\tau)) + \frac{1}{2}\tau^2 \mathscr{M}^{(q)}_{l,m,p}(u_{N,M}(t-\tau)) \\
&\quad + \frac{1}{4}\tau^2 \mathscr{G}^{(q)}_{l,m,p}(u_{N,M}(t-\tau)) + \frac{1}{2}\tau^2 \Big(\mathscr{F}^{(q)}_{l,m,p}(u_{N,M}(t+\tau)) + \mathscr{F}^{(q)}_{l,m,p}(u_{N,M}(t-\tau)) \Big), \quad (4.8)
\end{aligned}$$

for $q = 1, 2$. The above algorithm is implicit. Thus, we use the Newton iteration to resolve $\varphi^{(q)}_{l,m,p}$ at each time step. Finally we obtain $u_{N,M}(\rho, \lambda, \theta, t)$ from (4.5). In general, it is difficult to evaluate the quantities

$$\mathscr{G}^{(q)}_{l,m,p}(u_{N,M}(t+\tau)), \quad \mathscr{G}^{(q,1)}_{l,m,p}(u_{N,M}(t+\tau)), \quad \mathscr{G}^{(q,2)}_{l,m,p}(u_{N,M}(t+\tau)),$$

and

$$\mathscr{G}^{(q)}_{l,m,p}(u_{N,M}(t-\tau)), \quad \mathscr{F}^{(q)}_{l,m,p}(u_{N,M}(t))$$

exactly. Therefore, we use the generalized Laguerre-spherical harmonic quadrature for them.

Remark 4.1. The algorithm (4.8) is equivalent to

$$\begin{aligned}
&\frac{1}{\tau^2}\int_\Omega \rho^2 (u_{N,M}(t+\tau) - 2u_{N,M}(t) + u_{N,M}(t-\tau))\phi d\Omega \\
&\quad + \frac{1}{2}\int_\Omega \rho^2 \partial_\rho (u_{N,M}(t+\tau) + u_{N,M}(t-\tau))\partial_\rho \phi d\Omega \\
&\quad + \frac{1}{2}\int_\Omega (\nabla_S (u_{N,M}(t+\tau) + u_{N,M}(t-\tau)), \nabla_S \phi) d\Omega \\
&\quad + \frac{1}{2}\int_\Omega \rho^2 (u_{N,M}(t+\tau) + u_{N,M}(t-\tau))\phi d\Omega \\
&\quad + \frac{1}{4}\int_\Omega \rho^2 \Big(u^3_{N,M}(t+\tau) + u^2_{N,M}(t+\tau) u_{N,M}(t-\tau) \\
&\quad + u_{N,M}(t+\tau) u^2_{N,M}(t-\tau) + u^3_{N,M}(t-\tau) \Big) \phi d\Omega \\
&= \frac{1}{2}\int_\Omega \rho^2 (f(t+\tau) + f(t-\tau))\phi d\Omega, \qquad \forall \phi \in V_{N,M,\beta}(\Omega). \quad (4.9)
\end{aligned}$$

By taking

$$\phi = \frac{1}{2\tau}(u_{N,M}(t+\tau) - u_{N,M}(t-\tau)),$$

we could derive a discrete conservation similar to (3.6), which simulates the continuous conservation (3.3). Furthermore, we may follow the same line as in [8] to derive the error estimate of numerical solution, as long as $\tau = \mathcal{O}(N^{-2})$. For shortening the paper, we leave the detail to the readers.

Now, we denote by $\rho_{N,n}$ and $\omega_{N,n}$ the zeros of $\mathscr{L}_{N+1}^{(0,\beta)}(\rho)$ and the corresponding Christoffel numbers, respectively. Meanwhile, let $\xi_{M,j}$ and $\eta_{M,j}$ be the zeros of $L_{M+1}(z)$ and the corresponding Christoffel numbers, respectively. Furthermore, let $\sigma_M = 2\pi/(2M+1)$, and

$$(\rho_{N,n}, \lambda_{M,k}, \theta_{M,j}) = (\rho_{N,n}, k\sigma_M, \arcsin(\xi_{M,j})), \qquad \omega_{N,M,n,k,j} = \omega_{N,n}\sigma_M\eta_{M,j}.$$

The numerical errors are measured by

$$E_{N,M}(t) = \Big(\sum_{n=0}^{N}\sum_{j=0}^{2M}\sum_{k=0}^{M}|U(\rho_{N,n}, \lambda_{M,k}, \theta_{M,j}, t) - u_{N,M}(\rho_{N,n}, \lambda_{M,k}, \theta_{M,j}, t)|^2\Big)^{\frac{1}{2}}.$$

We take the test solution

$$U(\rho, \lambda, \theta, t) = \Big(e^{bt} - \frac{b^2t^2}{2} - bt - 1\Big)e^{-\rho}\sin 2\lambda \cos^2\theta.$$

In actual computation, we take $b = 1$. In Fig. 1, we plot the values of $E_{N,M}(1)$ with $\beta = 1$ and $\tau = 0.01, 0.001$, vs. the mode $N = 2M$. They indicate the convergence of proposed scheme as M increases and τ decreases. Especially, it provides accurate numerical results even for moderate mode $N = 2M$.

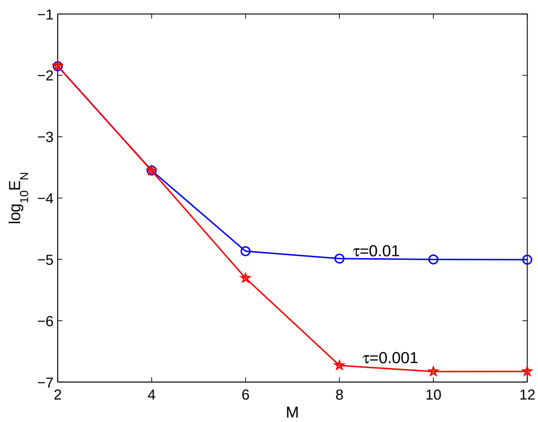

Figure 1: Convergence rates with $N = 2M$ and $\beta = 1$.

In Table 1, we compare the accuracy of numerical results with $\beta = 1, 2, 3$, $M = 6$ and $\tau = 0.01, 0.001$. They indicate that suitable choice of parameter β might lead to better numerical results. But, unfortunately, how to choose the best parameter β is still an open problem. Roughly speaking, it seems better to take bigger β, if the exact solutions decay faster as ρ goes to infinity. In this case, the asymptotic behaviors of numerical solutions simulate the asymptotic behaviors of exact solutions more properly. In our example, the test function decays like $e^{-\rho}$, as ρ increases, while the numerical solutions with $\beta = 2$ also do so. Therefore, we obtain better numerical results when $\beta = 2$, as is shown in Table 1.

Table 1: Convergence rate with $M = 6$.

	$\beta = 1$	$\beta = 2$	$\beta = 3$
$\tau = 0.01$	1.36E-5	9.91E-6	1.02E-5
$\tau = 0.001$	4.92E-6	1.50E-7	1.60E-7

5. Concluding remarks

In this paper, we proposed an efficient spectral method by using the generalized Laguerre and spherical harmonic functions, with its application to the Cauchy problem of three-dimensional nonlinear Klein-Gordon equation. The numerical results demonstrated its efficiency. Although we only considered the Klein-Gordon equation in this paper, the suggested method is also applicable to many other problems.

The new approach has several advantages. Firstly, we do not need any artificial boundary, and so avoid additional errors. Next, there left only one variable ρ, varying on infinite interval. Thus the calculation is much simpler than that with the Cartesian coordinates. Thirdly, benefiting from the orthogonality of spherical harmonic functions, this method saves a lot of work. Moreover, it is suitable for parallel computation, if we approximate nonlinear terms appearing in revolutionary problems explicitly. Another important merit is that the corresponding numerical solutions keep the same conservations as the exact solutions. Consequently, it oftentimes simulates physical processes properly.

In this paper, we established some basic results on the generalized Laguerre-spherical harmonic orthogonal approximation, which serve as the mathematical foundation of spectral methods for various problems defined in the whole three-dimensional space or on unbounded domains with spherical geometry. Besides, we may follow the same line as in this paper, to design new spectral method for exterior problems. We shall report the related results in the future.

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Appendix: Proof of Theorems 3.1 and 3.2

We first establish some results on the generalized Laguerre-spherical harmonic approximation, which will be used in the proof of Theorems 3.1 and 3.2.

Lemma A.1. *For any $s \geq 0$ and integers $1 \leq r \leq N+1$,*

$$\begin{aligned} &||P_{N,M,\alpha,\beta}v - v||^2_{\rho^\alpha,\Omega} \\ \leq\, & c(\beta N)^{-r} \int_S ||\partial^r_\rho (e^{\frac{1}{2}\beta\rho} v)||^2_{\omega_{\alpha+r,\beta,\Lambda}} dS + cM^{-2s} ||v||^2_{H^s(S,L^2_{\rho^\alpha}(\Lambda))}, \end{aligned}$$

provided that the norms appearing at the right side are finite.

Proof. By the projection theorem,

$$\|P_{N,M,\alpha,\beta}v - v\|^2_{\rho^\alpha,\Omega} \le \|\phi - v\|^2_{\rho^\alpha,\Omega}, \quad \forall \phi \in V_{N,M,\beta}(\Omega). \tag{A.1}$$

Let $\tilde{P}_{N,\alpha,\beta,\Lambda}v$ and $P_{M,S}v$ be the same as in (2.6) and (2.9), respectively. We take $\phi = \tilde{P}_{N,\alpha,\beta,\Lambda}(P_{M,S}v)$. Thanks to (2.7) and (2.10), we deduce that

$$\begin{aligned}
&\|\phi - v\|^2_{\rho^\alpha,\Omega} \\
\le& 2\int_\Omega \rho^\alpha(\tilde{P}_{N,\alpha,\beta,\Lambda}(P_{M,S}v) - (P_{M,S}v))^2 d\Omega + 2\int_\Omega (P_{M,S}(\rho^{\frac{\alpha}{2}}v) - \rho^{\frac{\alpha}{2}}v)^2 d\Omega \\
\le& \ c(\beta N)^{-r}\int_S \|\partial_\rho^r(e^{\frac{1}{2}\beta\rho}P_{M,S}v)\|^2_{\omega_{\alpha+r,\beta,\Lambda}} dS + cM^{-2s}\|\rho^{\frac{\alpha}{2}}v\|^2_{H^s(S,L^2(\Lambda))} \\
\le& c(\beta N)^{-r}\int_S \|\partial_\rho^r(e^{\frac{1}{2}\beta\rho}v)\|^2_{\omega_{\alpha+r,\beta,\Lambda}} dS + cM^{-2s}\|v\|^2_{H^s(S,L^2_{\rho^\alpha}(\Lambda))}.
\end{aligned}$$

This completes the proof of the lemma. □

We next estimate $\|P^1_{N,M,\beta}v - v\|_{1,\rho^2,\Omega}$. To do this, we need several preparations. Let

$$\chi_1(\rho) = \rho^2 e^{-\beta\rho}, \quad \chi_2(\rho) = \left(1 + \beta\rho + \left(\eta - \frac{\beta^2}{4}\right)\rho^2\right)e^{-\beta\rho}, \quad \eta > \max(\beta^2/4, 1),$$

and the space

$$\bar{H}^1_{\chi_1,\chi_2}(\Lambda) = \left\{v \mid \partial_\rho v \in L^2_{\chi_1}(\Lambda), v \in L^2_{\chi_2}(\Lambda)\right\},$$

with the norm $(\|\partial_\rho v\|^2_{\chi_1,\Lambda} + \|v\|^2_{\chi_2,\Lambda})^{\frac{1}{2}}$. The orthogonal projection

$$P^1_{N,\chi_1,\chi_2,\Lambda} : \bar{H}^1_{\chi_1,\chi_2}(\Lambda) \to \mathscr{P}_N(\Lambda)$$

is defined by

$$(\partial_\rho(P^1_{N,\chi_1,\chi_2,\Lambda}v - v), \partial_\rho\phi)_{\chi_1,\Lambda} + (P^1_{N,\chi_1,\chi_2,\Lambda}v - v, \phi)_{\chi_2,\Lambda} = 0, \quad \forall\phi \in \mathscr{P}_N(\Lambda).$$

Further, we introduce the space

$$H^1_{\rho^2,1}(\Lambda) = \left\{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{1,\rho^2,1,\Lambda} < \infty\right\}$$

equipped with the norm

$$\|v\|_{1,\rho^2,1,\Lambda} = \left(\|\partial_\rho v\|^2_{\rho^2,\Lambda} + \|v\|^2_{\rho^2,\Lambda} + \|v\|^2_\Lambda\right)^{\frac{1}{2}}.$$

The projection $\bar{P}^1_{N,\beta,\Lambda} : H^1_{\rho^2,1}(\Lambda) \to \mathscr{Q}_{N,\beta}(\Lambda)$ is defined by

$$\bar{P}^1_{N,\beta,\Lambda}v = e^{-\frac{\beta\rho}{2}}\left(P^1_{N,\chi_1,\chi_2,\Lambda}(ve^{\frac{\beta\rho}{2}})\right).$$

According to Theorem 3.3 of [12], we assert that for any $v \in H^1_{\rho^2,1}(\Lambda) \cap H^r_{\rho^{r+1}}(\Lambda)$, $\eta > \max(\beta^2/4, 1)$ and integers $1 \le r \le N+1$,

$$\begin{aligned}&\int_\Lambda \rho^2(\partial_\rho(\bar{P}^1_{N,\beta,\Lambda}v - v))^2 d\rho + \int_\Lambda (1+a\rho^2)(\bar{P}^1_{N,\beta,\Lambda}v - v)^2 d\rho \\ &\le c\Big(1+\frac{1}{\beta}\Big)^4 (\beta N)^{1-r} \big\|\partial^r_\rho(e^{\frac{\beta\rho}{2}}v)\big\|^2_{\omega_{r+1,\beta,\Lambda}}. \end{aligned} \tag{A.2}$$

Lemma A.2. *For any $v \in \mathscr{B}^{r,s}(\Omega)$, $s \ge 1$ and integers $1 \le r \le N+1$,*

$$\|P^1_{N,M,\beta}v - v\|^2_{1,\rho^2,\Omega} \le cN^{1-r}\mathscr{D}_{r,\beta}(v) + cM^{2-2s}\mathscr{D}^*_s(v).$$

Proof. By the projection theorem,

$$\|P^1_{N,M,\beta}v - v\|^2_{1,\rho^2,\Omega} \le \|\phi - v\|^2_{1,\rho^2,\Omega}, \quad \forall \phi \in V_{N,M,\beta}(\Omega). \tag{A.3}$$

We take $\phi = \bar{P}^1_{N,\beta,\Lambda}(P_{M,S}v)$. With the aid of (2.10) and (A.2), we verify that

$$\begin{aligned}&\|\partial_\rho(\phi - v)\|^2_{\rho^2,\Omega} \\ &\le 2\int_\Omega \rho^2\big(\partial_\rho \bar{P}^1_{N,\beta,\Lambda}(P_{M,S}v) - \partial_\rho(P_{M,S}v)\big)^2 d\Omega + 2\int_\Omega (P_{M,S}(\rho\partial_\rho v) - \rho\partial_\rho v)^2 d\Omega \\ &\le c\Big(1+\frac{1}{\beta}\Big)^4(\beta N)^{1-r}\int_S \big\|\partial^r_\rho(e^{\frac{\beta\rho}{2}}P_{M,S}v)\big\|^2_{\omega_{r+1,\beta,\Lambda}}dS + cM^{2-2s}\|\rho\partial_\rho v\|^2_{H^{s-1}(S,L^2(\Lambda))} \\ &\le c\Big(1+\frac{1}{\beta}\Big)^4(\beta N)^{1-r}\int_S \big\|\partial^r_\rho(e^{\frac{\beta\rho}{2}}v)\big\|^2_{\omega_{r+1,\beta,\Lambda}}dS + cM^{2-2s}\|\partial_\rho v\|^2_{H^{s-1}(S,L^2_{\rho^2}(\Lambda))}.\end{aligned}$$

The norm $\|\phi - v\|_{\rho^2,\Omega}$ has the same upper-bound as in the above. On the other hand, using (2.10) and (A.2) again yields that

$$\begin{aligned}&\int_\Omega (\nabla_S(\phi - v))^2 d\Omega \\ &\le 2\int_\Omega \Big(\frac{1}{\cos\theta}\partial_\lambda(\bar{P}^1_{N,\beta,\Lambda}(P_{M,S}v) - P_{M,S}v)\Big)^2 d\Omega + 2\int_\Omega\Big(\frac{1}{\cos\theta}\partial_\lambda(P_{M,S}v - v)\Big)^2 d\Omega \\ &\quad + 2\int_\Omega \Big(\partial_\theta(\bar{P}^1_{N,\beta,\Lambda}(P_{M,S}v) - P_{M,S}v)\Big)^2 d\Omega + 2\int_\Omega(\partial_\theta(P_{M,S}v - v))^2 d\Omega \\ &\le c\Big(1+\frac{1}{\beta}\Big)^4(\beta N)^{1-r}\int_S\Big(\Big\|\frac{1}{\cos\theta}\partial_\lambda\partial^r_\rho(e^{\frac{\beta\rho}{2}}v)\Big\|^2_{\omega_{r+1,\beta,\Lambda}} + \|\partial_\theta v\|^2_{\omega_{r+1,\beta,\Lambda}}\Big)dS \\ &\quad + cM^{2-2s}\|v\|^2_{H^s(S,L^2(\Lambda))}.\end{aligned}$$

The previous statements with (A.3) leads to the desired result. □

Proof of Theorem 3.1. We now turn to prove Theorem 3.1. By (3.5), the errors satisfy the following equation,

$$\begin{aligned}&\int_\Omega \rho^2\partial_t^2\tilde{u}_{N,M}(t)\phi d\Omega+\int_\Omega \rho^2\partial_\rho\tilde{u}_{N,M}(t)\partial_\rho\phi d\Omega+\int_\Omega(\nabla_S\tilde{u}_{N,M}(t))\cdot\nabla_S\phi)d\Omega\\&\quad+\int_\Omega\rho^2\tilde{u}_{N,M}(t)\phi d\Omega+\int_\Omega\rho^2\tilde{u}^3_{N,M}(t)\phi d\Omega+3\int_\Omega\rho^2\tilde{u}^2_{N,M}(t)u_{N,M}(t)\phi d\Omega\\&\quad+3\int_\Omega\rho^2\tilde{u}_{N,M}2(t)u^2_{N,M}(t)\phi d\Omega=\int_\Omega\rho^2\tilde{f}(t)\phi d\Omega,\qquad\forall\phi\in V_{N,M,\beta}(\Omega).\end{aligned}\tag{A.4}$$

Taking $\phi=2\partial_t\tilde{u}_{N,M}$ in (A.4), we obtain

$$\begin{aligned}&\frac{d}{dt}E(\tilde{u}_{N,M},t)+F_1(u_{N,M},\tilde{u}_{N,M},t)+F_2(u_{N,M},\tilde{u}_{N,M},t)\\&\le||\tilde{f}(t)||^2_{\rho^2,\Omega}+||\partial_t\tilde{u}_{N,M}(t)||^2_{\rho^2,\Omega},\end{aligned}\tag{A.5}$$

where

$$\begin{aligned}F_1(u_{N,M},\tilde{u}_{N,M},t)&=6\int_\Omega\rho^2\tilde{u}^2_{N,M}(t)u_{N,M}(t)\partial_t\tilde{u}_{N,M}d\Omega,\\F_2(u_{N,M},\tilde{u}_{N,M},t)&=6\int_\Omega\rho^2\tilde{u}_{N,M}(t)u^2_{N,M}(t)\partial_t\tilde{u}_{N,M}(t)d\Omega.\end{aligned}$$

Using the Hölder inequality, the imbedding inequality and (3.7) successively, we deduce that

$$\begin{aligned}|F_1(u_{N,M},\tilde{u}_{N,M},t)|&\le c||u_{N,M}(t)||_{L^6_{\rho^2}(\Omega)}||\tilde{u}_{N,M}(t)||^2_{L^6_{\rho^2}(\Omega)}||\partial_t\tilde{u}_{N,M}||_{\rho^2,\Omega}\\&\le c\left(||u_{N,M}(t)||^2_{1,\rho^2,\Omega}||\tilde{u}_{N,M}(t)||^4_{1,\rho^2,\Omega}+||\partial_t\tilde{u}_{N,M}||^2_{\rho^2,\Omega}\right)\\&\le d(U_0,U_1,f)\big(E^2(\tilde{u}_{N,M},t)+E(\tilde{u}_{N,M},t)\big).\end{aligned}$$

Similarly,

$$|F_2(u_{N,M},\tilde{u}_{N,M},t)|\le d(U_0,U_1,f)E(\tilde{u}_{N,M},t).$$

Substituting the above two estimates into (A.5) and integrating the resulting inequality with respect to t, we obtain

$$\begin{aligned}E(\tilde{u}_{N,M},t)&\le E(\tilde{u}_{N,M},0)+d(U_0,U_1,f)\\&\quad\times\int_0^t\int_\Omega\left(E^2(\tilde{u}_{N,M},\xi)+E(\tilde{u}_{N,M},\xi)+||\tilde{f}(\xi)||^2_{\rho^2,\Omega}\right)d\xi.\end{aligned}\tag{A.6}$$

We shall use the nonlinear Gronwall inequality stated below.

Lemma A.3. *Let $c_* \geq 0$, and $F(t)$ and $R(t)$ be non-negative functions of t. Moreover, $R(t)$ is non-decreasing for t. If $R(T) \leq e^{-2c_*T}$ and for $0 \leq t \leq T$,*

$$F(t) \leq c_* \int_0^t (F^2(\xi) + F(\xi))d\xi + R(t),$$

then for $0 \leq t \leq T$,

$$F(t) \leq e^{2c_*t} R(t).$$

Applying Lemma A.3 to (A.6), we reach the conclusion as stated in Theorem 3.1. □

Proof of Theorem 3.2. We next prove Theorem 3.2. Let $U_{N,M} = P^1_{N,M,\beta}U$. By the definition of $P^1_{N,M,\beta}U$, we have from (3.2) that

$$\begin{aligned}
&\int_\Omega \rho^2 \partial_t^2 U_{N,M}(t)\phi d\Omega + \int_\Omega \rho^2 \partial_\rho U_{N,M}(t)\partial_\rho \phi d\Omega + \int_\Omega (\nabla_S U_{N,M}(t)\cdot \nabla_S \phi) d\Omega \\
&\quad + \int_\Omega \rho^2 U_{N,M}(t)\phi d\Omega + \int_\Omega \rho^2 U^3_{N,M}(t)\phi d\Omega + G_1(U, U_{N,M}, \phi, t) + G_2(U, U_{N,M}, \phi, t) \\
&= \int_\Omega \rho^2 f(t)\phi d\Omega, \quad \forall \phi \in V_{N,M,\beta}(\Omega),
\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}
&G_1(U, U_{N,M}, \phi, t) = \int_\Omega \rho^2 \partial_t^2 (U(t) - U_{N,M}(t))\phi d\Omega, \\
&G_2(U, U_{N,M}, \phi, t) = \int_\Omega \rho^2 \Big(U(t) - U_{N,M}(t))(U^2(t) + U(t)U_{N,M}(t) + U^2_{N,M}(t) \Big) \phi d\Omega.
\end{aligned}$$

Set $\tilde{U}_{N,M} = u_{N,M} - U_{N,M}$. Subtracting (A.7) from (3.5), gives

$$\begin{aligned}
&\int_\Omega \rho^2 \partial_t^2 \tilde{U}_{N,M}(t)\phi d\Omega + \int_\Omega \rho^2 \partial_\rho \tilde{U}_{N,M}(t)\partial_\rho \phi d\Omega + \int_\Omega \Big(\nabla_S \tilde{U}_{N,M}(t)\cdot \nabla_S \phi\Big) d\Omega \\
&\quad + \int_\Omega \rho^2 \tilde{U}_{N,M}(t)\phi d\Omega + \int_\Omega \rho^2 \tilde{U}^3_{N,M}(t)\phi d\Omega \\
&= \sum_{j=1}^2 G_j(U, U_{N,M}, \phi, t) + \sum_{j=3}^4 G_j(\tilde{U}_{N,M}, U_{N,M}, \phi, t),
\end{aligned} \tag{A.8}$$

where

$$\begin{aligned}
&G_3(\tilde{U}_{N,M}, U_{N,M}, \phi, t) = -3 \int_\Omega \tilde{U}^2_{N,M}(t) U_{N,M}(t)\phi d\Omega, \\
&G_4(\tilde{U}_{N,M}, U_{N,M}, \phi, t) = -3 \int_\Omega \tilde{U}_{N,M}(t) U^2_{N,M}(t)\phi d\Omega.
\end{aligned}$$

Taking $\phi = 2\partial_t \tilde{U}_{N,M}$ in (A.8), yields

$$\frac{d}{dt}E(\tilde{U}_{N,M},t) = 2G_1(U,U_{N,M},\partial_t\tilde{U}_{N,M},t) + 2G_2(U,U_{N,M},\partial_t\tilde{U}_{N,M},t) + 2G_3(\tilde{U}_{N,M},U_{N,M},\partial_t\tilde{U}_{N,M},t) + 2G_4(\tilde{U}_{N,M},U_{N,M},\partial_t\tilde{U}_{N,M},t). \quad \text{(A.9)}$$

We now estimate the terms at the right side of (A.9). Firstly, by Lemma A.2,

$$\begin{aligned} &|G_1(U,U_{N,M},\partial_t\tilde{U}_{N,M},t)| \\ &\le cN^{1-r}\mathscr{D}_{r,\beta}(\partial_t^2 U(t)) + cM^{2-2s}\mathscr{D}_s^*(\partial_t^2 U(t)) + ||\partial_t\tilde{U}_{N,M}(t)||^2_{\rho^2,\Omega}. \end{aligned} \quad \text{(A.10)}$$

Next, by virtue of imbedding inequality and Lemma A.2 with $r = s = 1$,

$$||U_{N,M}(t)||_{L^6_{\rho^2}(\Omega)} \le c||U_{N,M}(t)||_{1,\rho^2,\Omega} \le c\left(\mathscr{D}_{1,\beta}(U(t)) + \mathscr{D}_1^*(U(t))\right)^{\frac{1}{2}}. \quad \text{(A.11)}$$

Therefore, we use the Hölder inequality, (A.11) and Lemma A.2 to obtain that

$$\begin{aligned} &|G_2(U,U_{N,M},\partial_t\tilde{U}_{N,M},t)| \\ &\le ||U(t) - U_{N,M}(t)||_{L^6_{\rho^2}(\Omega)}||\partial_t\tilde{U}_{N,M}(t)||_{\rho^2,\Omega}\Big(||U(t)||^2_{L^6_{\rho^2}(\Omega)} + ||U_{N,M}(t)||^2_{L^6_{\rho^2}(\Omega)}\Big) \\ &\le c||U(t) - U_{N,M}(t)||_{1,\rho^2,\Omega}||\partial_t\tilde{U}_{N,M}(t)||_{\rho^2,\Omega}\left(||U(t)||^2_{1,\rho^2,\Omega} + ||U_{N,M}(t)||^2_{1,\rho^2,\Omega}\right) \\ &\le c\left(\mathscr{D}_{1,\beta}(U(t)) + \mathscr{D}_1^*(U(t))\right)\left(N^{1-r}\mathscr{D}_{r,\beta,}(U(t)) + M^{2-2s}\mathscr{D}_s^*(U(t))\right) \\ &\quad + ||\partial_t\tilde{U}_{N,M}(t)||^2_{\rho^2,\Omega}. \end{aligned} \quad \text{(A.12)}$$

Furthermore,

$$\begin{aligned} &|G_3(\tilde{U}_{N,M},U_{N,M},\partial_t\tilde{U}_{N,M},t)| \\ &\le ||U_{N,M}(t)||_{L^6_{\rho^2}(\Omega)}||\tilde{U}_{N,M}(t)||^2_{L^6_{\rho^2}(\Omega)}||\partial_t\tilde{U}_{N,M}(t)||_{\rho^2,\Omega} \\ &\le c\left(\mathscr{D}_{1,\beta}(U(t)) + \mathscr{D}_1^*(U(t))\right)\left(||\tilde{U}_{N,M}(t)||^4_{1,\rho^2,\Omega} + ||\partial_t\tilde{U}_{N,M}(t)||^2_{\rho^2,\Omega}\right). \end{aligned} \quad \text{(A.13)}$$

Similarly,

$$\begin{aligned} &|G_4(\tilde{U}_{N,M},U_{N,M},\partial_t\tilde{U}_{N,M},t))| \\ &\le c\left(\mathscr{D}_{1,\beta}(U(t)) + \mathscr{D}_1^*(U(t))\right)^2\left(||\tilde{U}_{N,M}(t)||^2_{1,\rho^2,\Omega} + ||\partial_t\tilde{U}_{N,M}(t)||^2_{\rho^2,\Omega}\right). \end{aligned} \quad \text{(A.14)}$$

Substituting (A.10) and (A.12)-(A.14) in to (A.9) and integrating the result with respect to t, we reach that

$$E(\tilde{U}_{N,M},t) \le c\left(d_\beta(U)+1\right)\int_0^t \left(E^2(\tilde{U}_{N,M},\xi) + E(\tilde{U}_{N,M},\xi)\right)d\xi + \tilde{R}_{N,M,\beta}(U,t), \quad \text{(A.15)}$$

where

$$\tilde{R}_{N,M,\beta}(U,t) = E(\tilde{U}_{N,M},0) + cR_{N,M,\beta}(U,t).$$

Thus, it remains to estimate $E(\tilde{U}_{N,M},0)$. Since

$$\partial_t U_{N,M}(0) = P^1_{N,M,\beta}U_1, \quad U_{N,M}(0) = P^1_{N,M,\beta}U_0,$$

we use Lemmas A.1 and A.2 to obtain

$$\begin{aligned} E(\tilde{U}_{N,M},0) &\le c\left(||P_{N,M,2,\beta}U_1 - U_1||^2_{\rho^2,\Omega} + ||P^1_{N,M,\beta}U_1 - U_1||^2_{\rho^2,\Omega}\right) \\ &\le cR^*_{N,M,\beta}(U_0,U_1). \end{aligned}$$

Due to $r,s>1$, we have $\tilde{R}_{N,M,\beta}(t) \to 0$ as $N,M \to \infty$. Thereby, we can apply Lemma A.3 to (A.15). Finally, we obtain

$$E(\tilde{U}_{N,M},t) \le ce^{c(d_\beta(U)+1)t}\left(R_{N,M,\beta}(U,t) + R^*_{N,M,\beta}(U_0,U_1)\right).$$

The above estimate, along with Lemma A.2, leads to the conclusion as stated in Theorem 3.2. □

As mentioned in Section 3, the result stated in Theorem 3.2 does not seem optimal. The reason of this matter is that there is no optimal estimate in $L^2_{\rho^2}(\Lambda)$-norm for the orthogonal approximation $\bar{P}^1_{N,\beta,\Lambda}$. Consequently, we lack the optimal estimate in $L^2_{\rho^2}(\Omega)$-norm for the orthogonal approximation $P^1_{N,M,\beta}$, see Lemma A.2. As a result, we could not derive a result better than that as in Theorem 3.2. Indeed, so far, there is no optimal result on the error estimate in the weighted L^2-norm for any weighted H^1-orthogonal approximation by using the Laguerre polynomials or the Laguerre functions. It is still an open and difficult problem, see, e.g., [10,12,13,21].

References

[1] J. Ablowitz, M. D. Kruskal and J. F. Ladik, *SIAM Appl. Math.,* **36**(1979), 428-437.

[2] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Spinger-Verlag, Berlin, 1976.

[3] R. K. Bullough and P. J. Caudrey, *The solution and its history in solitons,* ed. by R. K. Bullough and P. J. Caudrey, Springer-Verlag, Berlin, 1980.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol. 1,* Interscience, New York, 1953.

[5] D. Funaro, *Polynomial Approximations of Differential Equations,* Springer-Verlag, Berlin, 1992.

[6] B.-Y. Guo, A spectral method for the vorticity equation on the surface, *Math. Comp.,* **64**(1995), 1067-1079.

[7] B.-Y. Guo and W. Huang, Mixed Jacobi-spherical harmonic spectral method for Navier-Stokes equations, *Appl. Numer. Math.,* **57**(2007), 939-961.

[8] B.-Y. Guo, X. Li and L. Vazquez, A Legendre spectral method for solving the nonlinear Klein-Gordon equation, *Comp. Appl. Math.,* **15**(1996), 19-36.

[9] B.-Y. Guo, J. Shen and C.-L. Xu, Generalized Laguerre approximation and its applications to exterior problems, *J. Comp. Math.,* **23**(2005), 113-130.

[10] B.-Y. Guo and J. Shen, Laguerre-Galerkin spectral method for nonlinear differential equation on a semi-infinite interval, *Numer. Math.,* **86**(2000), 635-654.

[11] B.-Y. Guo and C.-L. Xu, Mixed Laguerre-Legendre pseudospectral method for incompressible flow in an infinite strip, *Math. Comp.*, **73**(2003), 95-125.

[12] B.-Y. Guo and X.-Y. Zhang, Spectral method for differential equations of degenerate type on unbounded domains by using generalized Laguerre functions, *Appl. Numer. Math.,* **57** (2007), 455-471.

[13] B.-Y. Guo and X.-Y. Zhang, A new generalized Laguerre approximation and its applications, *J. Comp. Appl. Math.,* **181**(2005), 342-363.

[14] V. Iranzo and A. Falquès, Some spectral approximations for differential equations in unbounded domains, *Comput. Methods Appl. Mech. Engrg.,* **98**(1992), 105-126.

[15] K. Jorgens, Das Aufangswertproblem in Grossen fur eine Klasse nichtlinearer Wallengleichungen, *Math. Zeitschr.,* **77**(1961), 295-308.

[16] P.-Y. Kuo, Numerical solutions of nonlinear wave equations, *Numerical Mathematics,* **4**(1982), 46-56.

[17] J. L. Lions, *Quelques Methods de Resolution des Problems aux Limites Non Lineaires,* Dunod, Paris, 1969.

[18] Y. Maday, B. Pernaud-Thomas and H.Vandeven, Reappraisal of Laguerre type spectral methods, *La Recherche Aerospatiale,* **6** (1985), 13-35.

[19] L. I. Schiff, Nonlinear meson theory of nuclear forces, I, *Phys. Rev.,* **84**(1981), 1-9.

[20] I. E. Segal, The global Cauchy problem for a relativistic scalar field with power interaction, *Bull. Soc. Math. France,* **91**(1963), 129-135.

[21] J. Shen, Stable and efficient spectral methods in unbounded domains using Laguerre functions, *SIAM Numer. Anal.,* **38**(2002), 1113-1133.

[22] W. Strauss and L. Vazquez, Numerical solution of a non linear Klein-Gordon equation, *J. Comp. Phys.,* **28** (1978), 271-278.

[23] C.-L. Xu and B.-Y. Guo, Laguerre pseudospectral method for nonlinear partial differential equations, *J. Comp. Math.,* **20**(2002), 77-96.