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Cubature Formula and Interpolation on the Cubic Domain

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Abstract. Several cubature formulas on the cubic domains are derived using the discrete Fourier analysis associated with lattice tiling, as developed in [10]. The main results consist of a new derivation of the Gaussian type cubature for the product Chebyshev weight functions and associated interpolation polynomials on $[-1,1]^2$, as well as new results on $[-1,1]^3$. In particular, compact formulas for the fundamental interpolation polynomials are derived, based on $n^3/4+\mathcal{O}(n^2)$ nodes of a cubature formula on $[-1,1]^3$.

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1. Introduction

For a given weight function W supported on a set $\Omega \in \mathbb{R}^d$, a cubature formula of degree $2n-1$ is a finite sum, $L_n f$, that provides an approximation to the integral and preserves polynomials of degree up to $2n-1$; that is,

$$\int_\Omega f(x)W(x)dx = \sum_{k=1}^{N} \lambda_k f(x_k) =: L_n f, \quad \forall f \in \Pi_{2n-1}^d,$$

where Π_n^d denotes the space of polynomials of total degree at most n in d variables. The points $x_k \in \mathbb{R}^d$ are called *nodes* and the numbers $\lambda_k \in \mathbb{R}\setminus\{0\}$ are called *weights* of the cubature.

Our primary interests are Gaussian type cubature, which has minimal or nearer minimal number of nodes. For $d=1$, it is well known that Gaussian quadrature of degree $2n-1$ needs merely $N=n$ nodes and these nodes are precisely the zeros of the orthogonal polynomial of degree n with respect to W. The situation for $d>1$, however, is much more

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complicated and not well understood in general. As in the case of $d=1$ for which a cubature of degree $2n-1$ needs at least n nodes, the cubature of degree $2n-1$ for $d \ge 1$ needs $N \ge \dim \Pi_{n-1}^d$ number of nodes, but few formulas are known to attain this lower bound (see, e.g., [1, 10]). In fact, for the centrally symmetric weight function (symmetric with respect to the origin), it is known that the number of nodes, N, of a cubature of degree $2n-1$ in two dimension satisfies the lower bound

$$N \ge \dim \Pi_{n-1}^2 + \left\lfloor \frac{n}{2} \right\rfloor, \tag{1.1}$$

known as Möller's lower bound [11]. It is also known that the nodes of a cubature that attains the lower bound (1.1), if it exists, are necessarily the common zeros of $n+1-\lfloor \frac{n}{2} \rfloor$ orthogonal polynomials of degree n with respect to W. Similar statements on the nodes hold for cubature formulas that have number of nodes slightly above Möller's lower bound, which we shall call cubature of *Gaussian type*. These definitions also hold in d-dimension, where the lower bound for the number of nodes for the centrally symmetric weight function is given in [12].

There are, however, only a few examples of such formulas that are explicitly constructed and fewer still can be useful for practical computation. The best known example is $\Omega = [-1,1]^d$ with the weight function

$$W_0(x) := \prod_{i=1}^d \frac{1}{\sqrt{1-x_i^2}} \qquad \text{or} \qquad W_1(x) := \prod_{i=1}^d \sqrt{1-x_i^2} \tag{1.2}$$

and only when $d=2$. In this case, several families of Gaussian type cubature are explicitly known, they were constructed ([13, 17]) by studying the common zeros of corresponding orthogonal polynomials, which are product Chebyshev polynomials of the first kind and the second kind, respectively. Furthermore, interpolation polynomial bases on the nodes of these cubature formulas turn out to possess several desirable features ([18], and also [5]). On the other hand, studying common zeros of orthogonal polynomials of several variables is in general notoriously difficult. In the case of (1.2), the product Chebyshev polynomials have the simplest structure among all orthogonal polynomials, which permits us to study their common zeros and construct cubature formulas in the case $d=2$, but not yet for the case $d=3$ or higher.

The purpose of the present paper is to provide a completely different method for constructing cubature formulas with respect to W_0 and W_1. It uses the discrete Fourier analysis associated with lattice tiling, developed recently in [10]. This method has been used in [10] to establish cubature for *trigonometric functions* on the regular hexagon and triangle in $\mathbb{R}^2$, a topic that has been studied in [15, 16], and on the rhombic dodecahedron and tetrahedron of $\mathbb{R}^3$ in [9]. The cubature on the hexagon can be transformed, by symmetry, to a cubature on the equilateral triangle that generates the hexagon by reflection, which can in turn be further transformed, by a nontrivial change of variables, to Gaussian cubature formula for algebraic polynomials on the domain bounded by Steiner's hypercycloid. The theory developed in [10] uses two lattices, one determines the domain of integral and the points that defined the discrete inner product, the other determines the space of exponentials or trigonometric functions that are integrated exactly by the cubature. In [9,10] the two lattices are taken to be the same. In this paper we shall choose one as $\mathbb{Z}^d$ itself, so that the integral domain is fixed as the cube, while we choose the other one differently. In $d = 2$, we choose the second lattice so that its spectral set is a rhombus, which allows us to establish cubature formulas for trigonometric functions that are equivalent to Gaussian type cubature formulas for W_0 and W_1. In the case of $d = 3$, we choose the rhombic dodecahedron as a tiling set and obtain a cubature of degree $2n-1$ that uses $n^3/4 + \mathcal{O}(n^2)$ nodes, worse than the expected lower bound of $n^3/6 + \mathcal{O}(n^2)$ but far better than the product Gaussian cubature of n^3 nodes. This cubature with $n^3/4 + \mathcal{O}(n^2)$ nodes has appeared recently and tested numerically in [7]. We will further study the Lagrange interpolation based on its nodes, for which the first task is to identify the subspace that the interpolation polynomials belongs. We will not only identify the interpolation space, but also give the compact formulas for the fundamental interpolation polynomials.

One immediate question arising from this study is if there exist cubature formulas of degree $2n-1$ with $n^3/6 + \mathcal{O}(n^2)$ nodes on the cube. Although examples of cubature formulas of degree $2n-1$ with

$$N = \dim \Pi_{n-1}^d = \frac{n^d}{d! + \mathcal{O}\left(n^{d-1}\right)}$$

nodes are known to exist for special non-centrally symmetric regions ([1]), we are not aware of any examples for symmetric domains that use $N = n^d/d! + \mathcal{O}(n^{d-1})$ nodes. From our approach of tiling and discrete Fourier analysis, it appears that the rhombic dodecahedron gives the smallest number of nodes among all other fundamental domains that tile $\mathbb{R}^3$ by translation. Giving the fact that this approach yields the cubature formulas with optimal order for the number of nodes, it is tempting to make the conjecture that a cubature formula of degree $2n-1$ on $[-1,1]^3$ needs at least $n^3/4 + \mathcal{O}(n^2)$ nodes.

The paper is organized as follows. In the following section we recall the result on discrete Fourier analysis and lattice tiling in [10]. Cubature and interpolation for $d = 2$ are developed in Section 3 and those for $d = 3$ are discussed in Section 4, both the latter two sections are divided into several subsections.

2. Discrete Fourier analysis with lattice tiling

We recall basic results in [10] on the discrete Fourier analysis associated with a lattice. A lattice of $\mathbb{R}^d$ is a discrete subgroup that can be written as $A\mathbb{Z}^d = \{Ak : k \in \mathbb{Z}^d\}$, where A is a $d \times d$ invertible matrix, called the generator of the lattice. A bounded set $\Omega_A \subset \mathbb{R}^d$ is said to tile $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ if

$$\sum_{k \in \mathbb{Z}^d} \chi_{\Omega_A}(x + Ak) = 1, \quad \text{for almost all } x \in \mathbb{R}^d,$$

where χ_E denotes the characteristic function of the set E. Such a set is called a spectral set of the lattice. We further request, in this paper, that the spectral set Ω is chosen so that it tiles $\mathbb{R}^d$ without overlapping. The simplest lattice is $\mathbb{Z}^d$ itself, for which the set that tiles $\mathbb{R}^d$ is

$$\Omega := \big[-\tfrac{1}{2}, \tfrac{1}{2}\big)^d.$$

We reserve the notation Ω as above throughout the rest of this paper. The set Ω is chosen as half open so that its translations by $\mathbb{Z}^d$ tile $\mathbb{R}^d$ without overlapping. It is well known that the exponential functions

$$\mathrm{e}_k(x) := \mathrm{e}^{2\pi i k\cdot x}, \quad k \in \mathbb{Z}^d,\ x \in \mathbb{R}^d,$$

form an orthonormal basis for $L^2(\Omega)$. These functions are periodic with respect to $\mathbb{Z}^d$; that is, they satisfy

$$f(x+k) = f(x), \quad \forall k \in \mathbb{Z}^d.$$

Let B be a $d \times d$ matrix such that all entries of B are integers. Denote

$$\begin{aligned}\Lambda_B &= \left\{k \in \mathbb{Z}^d : B^{-\mathrm{tr}} k \in \Omega\right\},\\ \Lambda_B^{\dagger} &= \left\{k \in \mathbb{Z}^d : k \in \Omega_B\right\},\end{aligned} \tag{2.1}$$

where Ω_B is the spectral set of the lattice $B\mathbb{Z}^d$. It is known that $|\Lambda_B| = |\Lambda_B^{\dagger}| = |\det B|$, where $|E|$ denotes the cardinality of the set E. We need the following theorem [10, Theorem 2.5].

Theorem 2.1. *Let B be a $d \times d$ matrix with integer entries. Define the discrete inner product*

$$\langle f, g\rangle_B := \frac{1}{|\det(B)|} \sum_{j\in\Lambda_B} f\left(B^{-\mathrm{tr}} j\right) \overline{g\left(B^{-\mathrm{tr}} j\right)}$$

for $f,\ g \in C(\Omega)$, the space of continuous functions on Ω. Then,

$$\langle f,\ g\rangle_B = \langle f,\ g\rangle := \int_\Omega f(x)\overline{g(x)}\,dx, \tag{2.2}$$

for all $f,\ g$ in the finite dimensional subspace

$$\mathscr{T}_B := \operatorname{span}\left\{\mathrm{e}^{2\pi i\, k\cdot x} : k \in \Lambda_B^{\dagger}\right\}.$$

The dimension of $\mathscr{T}_B$ is $|\Lambda_B^{\dagger}| = |\det B|$.

This result is a special case of a general result in [10], in which Ω is replaced by Ω_A for an invertible matrix A, and the set Λ_B is replaced by Λ_N with $N = B^{\mathrm{tr}}A$ and N is assumed to have integer entries. Since we are interested only at the cube $\big[-\frac{1}{2}, \frac{1}{2}\big]^d$ in this paper, we have chosen A as the identity matrix.

We can also use the discrete Fourier analysis to study interpolation based on the points in Λ_B. We say two points $x, y \in \mathbb{R}^d$ congruent with respect to the lattice $B\mathbb{Z}^d$, if $x - y \in B\mathbb{Z}^d$, and we write $x \equiv y \mod B$. We then have the following result:

Theorem 2.2. *For a generic function f defined in $C(\Omega)$, the unique interpolation function $\mathscr{I}_B f$ in $\mathscr{T}_B$ that satisfies*

$$\mathscr{I}_B f\left(B^{-\mathrm{tr}} j\right) = f\left(B^{-\mathrm{tr}} j\right), \quad \forall j \in \Lambda_B$$

is given by

$$\mathscr{I}_B f(x) = \sum_{k \in \Lambda_B^\dagger} \langle f, \mathrm{e}_k \rangle \mathrm{e}_k(x) = \sum_{k \in \Lambda_B} f\left(B^{-\mathrm{tr}} k\right) \Psi_{\Omega_B}\left(x - B^{-\mathrm{tr}} k\right), \tag{2.3}$$

where

$$\Psi_{\Omega_B}(x) = \frac{1}{|\det(B)|} \sum_{j \in \Lambda_{B^\dagger}} \mathrm{e}^{2\pi i j^{\mathrm{tr}} x}. \tag{2.4}$$

The proof of this result is based on the second one of the following two relations that are of independent interests:

$$\frac{1}{|\det(B)|} \sum_{j \in \Lambda_B} \mathrm{e}^{2\pi i k^{\mathrm{tr}} B^{-\mathrm{tr}} j} = \begin{cases} 1, & \text{if } k \equiv 0 \mod B, \\ 0, & \text{otherwise}, \end{cases} \tag{2.5}$$

$$\frac{1}{|\det(B)|} \sum_{k \in \Lambda_B^\dagger} \mathrm{e}^{-2\pi i k^{\mathrm{tr}} B^{-\mathrm{tr}} j} = \begin{cases} 1, & \text{if } j \equiv 0 \mod B^{\mathrm{tr}}, \\ 0, & \text{otherwise}. \end{cases} \tag{2.6}$$

For proofs and further results we refer to [9,10]. Throughout this paper we will write, for $k \in \mathbb{Z}^d$, $2k = (2k_1, \cdots, 2k_d)$ and $2k+1 = (2k_1+1, \cdots, 2k_d+1)$.

3. Cubature and interpolation on the square

In this section we consider the case $d = 2$. In the first subsection, the general results in the previous section is specialized to a special case and cubature formulas are derived for a class of trigonometric functions. These results are converted to results for algebraic polynomials in the second subsection. Results on polynomial interpolation are derived in the third subsection.

3.1. Discrete Fourier analysis and cubature formulas on the plane

We choose the matrix B as

$$B = n \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad B^{-1} = \frac{1}{2n} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since B is a rotation, by 45 degree, of a constant multiple of the diagonal matrix, we choose the domain Ω_B defined by

$$\Omega_B = \{x \in \mathbb{R}^2 : -n \le x_1 + x_2 < n, -n \le x_2 - x_1 < n\},$$

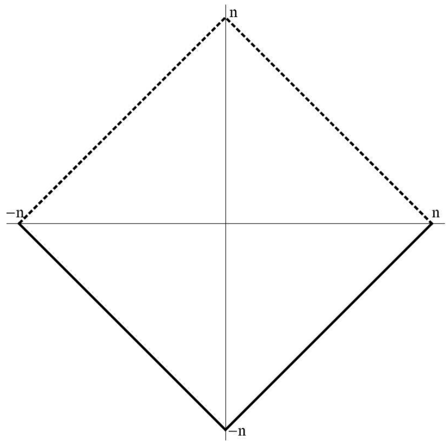

Figure 1: Rhombus Ω_B.

so that it tiles $\mathbb{R}^2$ without overlapping. This domain is depicted in Fig. 1 above.

From the expression of $B^{-\mathrm{tr}}$, it follows readily that $\Lambda_B = \Lambda_B^{\dagger} =: \Lambda_n$, where

$$\Lambda_n = \{ j \in \mathbb{Z}^2 : -n \le j_1 + j_2 < n,\ -n \le j_2 - j_1 < n \}.$$

The cardinality of Λ_n is $|\Lambda_n| = 2n^2$. We further denote the space $\mathscr{T}_B$ by $\mathscr{T}_n$, which is given by

$$\mathscr{T}_n := \operatorname{span}\left\{ \mathrm{e}^{2\pi i\, k\cdot x} :\ k \in \Lambda_n \right\}.$$

Theorem 3.1. *Define the set*

$$X_n := \left\{ 2k \in \mathbb{Z}^2 : -\tfrac{n}{2} \le k_1, k_2 < \tfrac{n}{2} \right\} \cup \left\{ 2k + 1 \in \mathbb{Z}^2 : -\tfrac{n+1}{2} \le k_1, k_2 < \tfrac{n-1}{2} \right\}.$$

Then, for all $f, g \in \mathscr{T}_n$,

$$\langle f, g \rangle_n := \frac{1}{2n^2} \sum_{k \in X_n} f\left(\tfrac{k}{2n}\right) \overline{g\left(\tfrac{k}{2n}\right)} = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} f(x) \overline{g(x)} dx.$$

Proof. Changing variables from j to $k = 2nB^{-\mathrm{tr}} j$, or $k_1 = j_1 + j_2$ and $k_2 = j_2 - j_1$, then, as j_1 and j_2 need to be integers and $j_1 = \frac{k_1 - k_2}{2}$, $j_2 = \frac{k_1 + k_2}{2}$, we see that

$$j \in \Lambda_n \quad \Longleftrightarrow \quad k = 2nB^{-\mathrm{tr}} j \in X_n. \tag{3.1}$$

Hence, as $\det(B) = 2n^2$, we conclude that $\langle f, g \rangle_n = \langle f, g \rangle_B$ and this theorem follows as a special case of Theorem 2.1. $\square$

The set Λ_n lacks symmetry as the inequalities in its definition are half open and half closed. We denote its symmetric counterpart by Λ_n^*, which is defined by

$$\Lambda_n^* := \{ j \in \mathbb{Z}^2 : -n \le j_1 + j_2 \le n,\ -n \le j_1 - j_2 \le n \}.$$

We also denote the counterpart of $\mathscr{T}_n$ by $\mathscr{T}_n^*$, which is defined by

$$\mathscr{T}_n^* := \operatorname{span}\left\{ e^{2\pi i\, k\cdot x} : \; k \in \Lambda_n^* \right\}.$$

Along the same line, we also define the counterpart of X_n as

$$X_n^* := \left\{ 2k : -\tfrac{n}{2} \le k_1, k_2 \le \tfrac{n}{2} \right\} \cup \left\{ 2k+1 : -\tfrac{n+1}{2} \le k_1, k_2 \le \tfrac{n-1}{2} \right\}.$$

It is easy to see that $|X_n| = |\Lambda_n| = 2n^2$, whereas $|X_n^*| = 2n^2 + 2n + 1$. We further partition the set X_n^* into three parts,

$$X_n^* = X_n^\circ \cup X_n^e \cup X_n^v,$$

where $X_n^\circ = X_n^* \cap (-n,n)^2$ is the set of interior points of X_n^*, X_n^e consists of those points in X_n^* that are on the edges of $[-n,n]^2$ but not at the 4 vertices or corners, while X_n^v consists of those points of X_n^* at the vertices of $[-n,n]^2$.

Theorem 3.2. *Define the inner product*

$$\langle f, g\rangle_n^* := \frac{1}{2n^2} \sum_{k\in X_n^*} c_k^{(n)} f\left(\tfrac{k}{2n}\right) \overline{g\left(\tfrac{k}{2n}\right)}, \qquad \textit{where} \quad c_k^{(n)} = \begin{cases} 1, & k \in X_n^\circ \\ \frac{1}{2}, & k \in X_n^e \\ \frac{1}{4}, & k \in X_n^v \end{cases}. \tag{3.2}$$

Then, for all $f, g \in \mathscr{T}_n$,

$$\int_{[-\frac12,\frac12]^2} f(x)\overline{g(x)}dx = \langle f, g\rangle_n = \langle f, g\rangle_n^*.$$

Proof. Evidently we only need to show that $\langle f,g\rangle_n = \langle f,g\rangle_n^*$. Since $c_k^{(n)} = 1$ for $k \in X_n^\circ$, the partial sums over interior points of the two sums agree. The set X_n^e of boundary points can be divided into two parts, $X_n^e = X_n^{e,1} \cup X_n^{e,2}$, where $X_n^{e,1}$ consists of points in X_n that are on the edges of $[-n,n)^2$, but not equal to $(-n,-n)$, and $X_n^{e,2}$ is the complementary of $X_n^{e,1}$ in X_n^e. Evidently, if $x \in X_n^{e,1}$, then either $x + (2n, 0)$ or $x + (0, 2n)$ belongs to $X_n^{e,2}$. Hence, if f is a periodic function, $f(x+k) = f(x)$ for $k \in \mathbb{Z}^2$, then

$$\sum_{k\in X_n^e} c_k^{(n)} f\left(\tfrac{k}{2n}\right) = \frac{1}{2}\sum_{k\in X_n^e} f\left(\tfrac{k}{2n}\right) = \sum_{k\in X_n^{e,1}} f\left(\tfrac{k}{2n}\right).$$

Furthermore, for $(-n,-n) \in X_n$, X_n^* contains all four vertices $(\pm n, \pm n)$. Since a periodic function takes the same value on all four points,

$$\sum_{k\in X_n^v} c_k^{(n)} f\left(\tfrac{k}{2n}\right) = f\left(-\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Consequently, we have proved that $\langle f,g\rangle_n = \langle f,g\rangle_n^*$ if f, g are periodic functions. ◻

As a consequence of the above two theorems, we deduce the following two cubature formulas:

Theorem 3.3. *For $n \ge 2$, the cubature formulas*

$$\int_{[-\frac{1}{2},\frac{1}{2}]^2} f(x)dx = \frac{1}{2n^2} \sum_{k \in X_n^*} c_k^{(n)} f\left(\tfrac{k}{2n}\right),$$
$$\int_{[-\frac{1}{2},\frac{1}{2}]^2} f(x)dx = \frac{1}{2n^2} \sum_{k \in X_n} f\left(\tfrac{k}{2n}\right), \tag{3.3}$$

are exact for $f \in \mathscr{T}_{2n-1}^*$.

Proof. It suffices to prove that both cubature formulas in (3.3) are exact for every e_j with $j \in \Lambda_{2n-1}^*$. For this purpose, we first claim that for any $j \in \mathbb{Z}^2$, there exist $v \in \Lambda_n$ and $l \in \mathbb{Z}^2$ such that $j = v + Bl$. Indeed, the translations of Ω_B by $B\mathbb{Z}^2$ tile $\mathbb{R}^2$, thus we have $j = x + Bl$ for certain $x \in \Omega_B$ and $l \in \mathbb{Z}^2$. Since all entries of the matrix B are integers, we further deduce that $v := x = j - Bl \in \mathbb{Z}^2 \cap \Omega_B = \Lambda_n$.

Next assume $j \in \Lambda_{2n-1}^*$. Clearly the integral of e_j over Ω is $\delta_{j,0}$. On the other hand, let us suppose $j = v + Bl$ with $v \in \Lambda_n$ and $l \in \mathbb{Z}^2$. Then, it is easy to see that $\mathrm{e}_j\left(\frac{k}{2n}\right) = \mathrm{e}_v\left(\frac{k}{2n}\right)$ for each $k \in X_n^*$. Consequently, we obtain from Theorem 3.2 that

$$\begin{aligned} \sum_{k \in X_n^*} c_k^{(n)} \mathrm{e}_j\left(\tfrac{k}{2n}\right) &= \sum_{k \in X_n^*} c_k^{(n)} \mathrm{e}_v\left(\tfrac{k}{2n}\right) = \sum_{k \in X_n} \mathrm{e}_v\left(\tfrac{k}{2n}\right) \\ &= \sum_{k \in X_n} \mathrm{e}_j\left(\tfrac{k}{2n}\right) = \int_\Omega \mathrm{e}_v(x)dx = \delta_{v,0}. \end{aligned}$$

Since $v = 0$ implies $j = Bl \in \mathbb{Z}^2$ which gives $j = l = 0$, we further obtain that $\delta_{v,0} = \delta_{j,0}$. This completes the proof of (3.3). □

We note that the second cubature in (3.3) is a so-called Chebyshev cubature; that is, all its weights are equal.

3.2. Cubature for algebraic polynomials

The set Λ_n^* is symmetric with respect to the mappings $(x_1, x_2) \mapsto (-x_1, x_2)$ and $(x_1, x_2) \mapsto (x_1, -x_2)$. It follows that both the spaces

$$\mathscr{T}_n^{\text{even}} := \operatorname{span}\{\cos 2\pi j_1 x_1 \cos 2\pi j_2 x_2 : 0 \le j_1 + j_2 \le n\},$$
$$\mathscr{T}_n^{\text{odd}} := \operatorname{span}\{\sin 2\pi j_1 x_1 \sin 2\pi j_2 x_2 : 1 \le j_1 + j_2 \le n\},$$

are subspaces of $\mathscr{T}_n^*$. Recall that Chebyshev polynomials of the first kind, $T_n(t)$, and the second kind, $U_n(t)$, are defined, respectively, by

$$T_n(t) = \cos n\theta, \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad t = \cos\theta.$$

They are orthogonal with respect to $1/\sqrt{1-t^2}$ and $\sqrt{1-t^2}$ over $[-1,1]$, respectively. Both are algebraic polynomials of degree n in t. Recall the definition of W_0 and W_1 in (1.2). Under the changing of variables

$$t_1 = \cos 2\pi x_1, \quad t_2 = \cos 2\pi x_2, \quad (x_1, x_2) \in \big[-\tfrac{1}{2}, \tfrac{1}{2}\big]^2, \tag{3.4}$$

the subspace $\mathscr{T}_n^{\text{even}}$ becomes the space Π_n^2 of polynomials of degree n in the variables (t_1, t_2),

$$\Pi_n^2 = \operatorname{span}\{T_j(t_1)T_{k-j}(t_2) : 0 \le j \le k \le n\}$$

and the orthogonality of e_k over Ω implies that $T_j^k(t) := T_j(t_1)T_{k-j}(t_2)$ are orthogonal polynomials of two variables,

$$\frac{1}{\pi^2}\int_{[-1,1]^2} T_j^k(t)T_{j'}^{k'}(t)W_0(t)\,dt = \begin{cases} 1, & k = k' = j = j' = 0, \\ \frac{1}{2}, & k = k' = j = j' > 0 \text{ or } k = k' > j = j' = 0, \\ \frac{1}{4}, & k = k' > j = j' > 0, \\ 0, & (k,j) \ne (k',j'). \end{cases}$$

We note also that the subspace $\mathscr{T}_n^{\text{odd}}$ becomes the space $\{\sqrt{1-t_1^2}\sqrt{1-t_2^2}\,p(t) : p \in \Pi_{n-1}^2\}$ in the variables $t = (t_1, t_2)$, and the orthogonality of e_k also implies that $U_j^k(t) := U_j(t_1)U_{k-j}(t_2)$ are orthogonal polynomials of two variables,

$$\frac{1}{\pi^2}\int_{[-1,1]^2} U_j^k(t)U_{j'}^{k'}(t)W_1(t)\,dt = \frac{1}{4}\delta_{j,j'}\delta_{k,k'}.$$

The symmetry allows us to translate the results in the previous subsection to algebraic polynomials. Since $\cos 2\pi j_1 x_1 \cos 2\pi j_2 x_2$ are even in both variables, we only need to consider their values over $X_n^* \cap \{x : x_1 \ge 0, x_2 \ge 0\}$. Hence, we define

$$\Xi_n := \Big\{(2k_1, 2k_2) : 0 \le k_1, k_2 \le \tfrac{n}{2}\Big\} \cup \Big\{(2k_1+1, 2k_2+1) : 0 \le k_1, k_2 \le \tfrac{n-1}{2}\Big\}, \tag{3.5}$$

and, under the change of variables (3.4),

$$\Gamma_n := \{(z_{k_1}, z_{k_2}) : (k_1, k_2) \in \Xi_n\}, \quad \text{where } z_k = \cos\tfrac{k\pi}{n}. \tag{3.6}$$

Furthermore, we denote by $\Gamma_n^\circ := \Gamma_n \cap (-1,1)^2$ the subset of interior points of Γ_n, by Γ_n^e the set of points in Γ_n that are on the boundary of $[-1,1]^2$ but not at the four corners, and by Γ_n^v the set of points in Γ_n that are at the corners of $[-1,1]^2$. The sets Ξ_n°, Ξ_n^e and Ξ_n^v are defined accordingly. A simple counting shows that

$$|\Xi_n| = \big(\lfloor\tfrac{n}{2}\rfloor + 1\big)^2 + \big(\lfloor\tfrac{n-1}{2}\rfloor + 1\big)^2 = \frac{n(n+1)}{2} + \Big\lfloor\frac{n}{2}\Big\rfloor + 1. \tag{3.7}$$

Theorem 3.4. *The cubature formula*

$$\frac{1}{\pi^2}\int_{[-1,1]^2} f(t)W_0(t)dt = \frac{1}{2n^2}\sum_{k\in\Xi_n}\lambda_k^{(n)} f(z_{k_1},z_{k_2}), \quad \lambda_k^{(n)} := \begin{cases} 4, & k\in\Xi_n^\circ, \\ 2, & k\in\Xi_n^e, \\ 1, & k\in\Xi_n^v, \end{cases} \tag{3.8}$$

is exact for Π^2_{2n-1}.

Proof. We note that X_n^* is symmetric in the sense that $k\in X_n^*$ implies that $(-k_1,k_2)\in X_n^*$ and $(k_1,-k_2)\in X_n^*$. Let $g(x) = f(\cos 2\pi x_1, \cos 2\pi x_2)$. Then, g is even in each of its variables and $g(\frac{k}{2n}) = f(z_{k_1},z_{k_2})$. Notice that $f\in\Pi^2_{2n-1}$ implies $g\in\mathscr{T}^*_{2n-1}$. Applying the first cubature formula (3.3) to $g(x)$, we see that (3.8) follows from the following identity,

$$\sum_{k\in X_n^*} c_k^{(n)} g\big(\tfrac{k}{2n}\big) = \sum_{k\in\Xi_n}\lambda_k^{(n)} f\big(z_{k_1},z_{k_2}\big).$$

To prove this identity, let $k\sigma$ denote the set of distinct elements in $\{(\pm k_1,\pm k_2)\}$; then $g(\frac{k}{2n})$ takes the same value on all points in $k\sigma$. If $k\in X_n^*$, $k_1\neq 0$ and $k_2\neq 0$, then $k\sigma$ contains 4 points;

$$\begin{aligned} &\sum_{j\in k\sigma} c_k^{(n)} g\big(\tfrac{j}{2n}\big) = 4g\big(\tfrac{k}{2n}\big) \quad \text{if } k\in X_n^\circ, \\ &\sum_{j\in k\sigma} c_k^{(n)} g\big(\tfrac{j}{2n}\big) = 2g\big(\tfrac{k}{2n}\big) \quad \text{if } k\in X_n^e, \\ &\sum_{j\in k\sigma} c_k^{(n)} g\big(\tfrac{j}{2n}\big) = g\big(\tfrac{k}{2n}\big) \quad\ \ \text{if } k\in X_n^v. \end{aligned}$$

If $k_1=0$ and $k_2\neq 0$ or $k_2=0$ and $k_1\neq 0$, then $k\sigma$ contains 2 points;

$$\begin{aligned} &\sum_{j\in k\sigma} c_k^{(n)} g\big(\tfrac{j}{2n}\big) = 2g\big(\tfrac{k}{2n}\big) \quad \text{if } k\in X_n^\circ, \\ &\sum_{j\in k\sigma} c_k^{(n)} g\big(\tfrac{j}{2n}\big) = g\big(\tfrac{k}{2n}\big) \quad\ \ \text{if } k\in X_n^e. \end{aligned}$$

Finally, if $k=(0,0)$ then $k\sigma$ contains 1 point and $g(0,0)$ has coefficient 1. Putting these together proves the identity. □

By (3.7), the number of nodes of the cubature formula (3.8) is just one more than the lower bound (1.1). We can also write (3.8) into a form that is more explicit. Indeed, if $n=2m$, then (3.8) can be written as

$$\begin{aligned} &\frac{1}{\pi^2}\int_{[-1,1]^2} f(t)W_0(t)\,dt \\ =&\frac{2}{n^2}\sum_{i=0}^{m}{}'' \sum_{j=0}^{m}{}'' f(z_{2i},z_{2j}) + \frac{2}{n^2}\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} f(z_{2i+1},z_{2j+1}), \end{aligned} \tag{3.9}$$

where $\sum''$ means that the first and the last terms in the summation are halved. If $n = 2m+1$, then (3.8) can be written as

$$\begin{aligned}&\frac{1}{\pi^2}\int_{[-1,1]^2} f(t)W_0(t)\,dt\\ =&\frac{2}{n^2}\sum_{i=0}^{m}{}'\sum_{j=0}^{m}{}' f(z_{2i},z_{2j}) + \frac{2}{n^2}\sum_{i=0}^{m}{}'\sum_{j=0}^{m}{}' f(z_{n-2i},z_{n-2j}),\end{aligned} \tag{3.10}$$

where $\sum'$ means that the first term in the sum is divided by 2. The formula (3.10) appeared in [17], where it was constructed by considering the common zeros of orthogonal polynomials of two variables.

From the cubature formula (3.3), we can also derive cubature formulas for the Chebyshev weight W_1 of the second kind.

Theorem 3.5. *The cubature formula*

$$\frac{1}{\pi^2}\int_{[-1,1]^2} f(t)W_1(t)dt = \frac{2}{n^2}\sum_{k\in\Xi_n^\circ}\sin^2\tfrac{k_1\pi}{n}\sin^2\tfrac{k_2\pi}{n}f(z_{k_1},z_{k_2}) \tag{3.11}$$

is exact for Π^2_{2n-5}.

Proof. We apply the first cubature formula in (3.3) on the functions

$$\sin\big(2\pi(k_1+1)x_1\big)\sin\big(2\pi(k_2+1)x_2\big)\sin 2\pi x_1\sin 2\pi x_2$$

for $0 \le k_1+k_2 \le 2n-5$, where $t_1 = \cos 2\pi x_1$ and $t_2 = \cos 2\pi x_2$ as in (3.4). Clearly these functions are even in both x_1 and x_2 and they are functions in $\mathcal{T}^*_{2n-1}$. Furthermore, they are zero when $x_1 = 0$ or $x_2 = 0$, or when (x_1,x_2) are on the boundary of X^*_n. Hence, the change of variables (3.4) shows that the first cubature in (3.3) becomes (3.11) for $U_{k_1}(t_1)U_{k_2}(t_2)$. □

A simple counting shows that

$$|\Xi_n^\circ| = \lfloor\frac{n}{2}\rfloor^2 + \lfloor\frac{n-1}{2}\rfloor^2 = \frac{(n-1)(n-2)}{2} + \lfloor\frac{n}{2}\rfloor.$$

The number of nodes of the cubature formula (3.11) is also one more than the lower bound (1.1). In this case, this formula appeared already in [13].

3.3. Interpolation by polynomials

As shown in [10], there is a close relation between interpolation and discrete Fourier transform. We start with a simple result on interpolation by trigonometric functions in $\mathcal{T}_n$.

Proposition 3.6. *For $n \ge 1$ define*

$$I_n f(x) := \sum_{k \in X_n} f\left(\tfrac{k}{2n}\right) \Phi_n\left(x - \tfrac{k}{2n}\right), \quad \Phi_n(x) := \frac{1}{2n^2} \sum_{\nu \in \Lambda_n} \mathrm{e}_\nu(x). \tag{3.12}$$

Then, $I_n f\left(\frac{k}{2n}\right) = f\left(\frac{k}{2n}\right)$ for all $k \in X_n$.

Proof. For $j \in \Lambda_n$ define $k = 2nB^{-\mathrm{tr}} j$. From the relation (3.1), $j \in \Lambda_n$ is equivalent to $k \in X_n$ with $k = 2nB^{-\mathrm{tr}} j$. As a result, we can write $I_n f(x)$ as

$$I_n f(x) = \sum_{j \in \Lambda_n} f\left(B^{-\mathrm{tr}} j\right) \Phi_n\left(x - B^{-\mathrm{tr}} j\right)$$

and the interpolation means $I_n f(B^{-\mathrm{tr}} j) = f(B^{-\mathrm{tr}} j)$ for $j \in \Lambda_n$. For $k, j \in \Lambda_n$,

$$\Phi_n\left(B^{-\mathrm{tr}}(j-k)\right) = \frac{1}{2n^2} \sum_{\nu \in \Lambda_n} \mathrm{e}_\nu\left(B^{-\mathrm{tr}}(j-k)\right) = \delta_{k,j}$$

by (2.6). □

For our main result, we need a lemma on the symmetric set X_n^* and Λ_n^*. Recall that $c_k^{(n)}$ is defined for $k \in X_n^*$. Since the relation (3.1) clearly extends to

$$j \in \Lambda_n^* \quad \Longleftrightarrow \quad k = 2nB^{-\mathrm{tr}} j \in X_n^*, \tag{3.13}$$

we define $\widetilde{c}_j^{(n)} = c_k^{(n)}$ whenever k and j are so related. Comparing to (3.12), we then define

$$I_n^* f(x) := \sum_{k \in X_n^*} f\left(\tfrac{k}{2n}\right) \Phi_n^*\left(x - \tfrac{k}{2n}\right), \quad \text{where } \Phi_n^*(x) := \frac{1}{2n^2} \sum_{\nu \in \Lambda_n^*} \widetilde{c}_\nu^{(n)} \mathrm{e}_\nu(x). \tag{3.14}$$

We also introduce the following notation: for $k \in X_n^e$, we denote by k' the point on the opposite edge of X_n^*; that is, $k' \in X_n^e$ and $k' = k \pm (2n, 0)$ or $k' = k \pm (0, 2n)$. Furthermore, we denote by j' the index corresponding to k' under (3.13).

Lemma 3.7. *The function $I_n^* f \in \mathscr{T}_n^*$ satisfies*

$$I_n^* f\left(\tfrac{k}{2n}\right) = \begin{cases} f\left(\frac{k}{2n}\right), & k \in X_n^\circ, \\ f\left(\frac{k}{2n}\right) + f\left(\frac{k'}{2n}\right), & k \in X_n^e, \\ f\left(\frac{k}{2n}\right) + f\left(\frac{(-k_1, k_2)}{2n}\right) + f\left(\frac{(k_1, -k_2)}{2n}\right) + f\left(\frac{-k}{2n}\right), & k \in X_n^\nu. \end{cases}$$

Proof. As in the proof of the previous theorem, we can write $I_n^* f$ as

$$I_n^* f(x) = \sum_{j \in \Lambda_n^*} f(B^{-\mathrm{tr}} j) \Phi_n^*(x - B^{-\mathrm{tr}} j)$$

by using (3.13). Let $S_k(x) = \Phi_n^*(B^{-\mathrm{tr}} j)$. For all $k, j \in \Lambda_n^*$,

$$S_k(B^{-\mathrm{tr}} j) = \frac{1}{2n^2} \sum_{\nu \in \Lambda_n^*} \widetilde{c}_\nu^{(n)} \mathrm{e}_\nu \left(B^{-\mathrm{tr}}(j-k)\right).$$

Since $\mathrm{e}_\nu(B^{-\mathrm{tr}} j) = \mathrm{e}_\mu(B^{-\mathrm{tr}} j)$ for any $\mu \equiv \nu \mod B$, we derive by using a similar argument as in Theorem 3.2 that

$$S_k(B^{-\mathrm{tr}} j) = \frac{1}{2n^2} \sum_{\nu \in \Lambda_n} \mathrm{e}_\nu \left(B^{-\mathrm{tr}}(j-k)\right).$$

By (2.6), $S_k(B^{-\mathrm{tr}} j) = \delta_{k,j}$ if $k, j \in \Lambda_n$. If $j \in \Lambda_n^* \setminus \Lambda_n$ then $j' \in \Lambda_n$, so that if $k \in \Lambda_n$ then $S_k(B^{-tr} j) = \delta_{k,j'}$. The same holds for the case of $j \in \Lambda_n$ and $k \in \Lambda_n^* \setminus \Lambda_n$. If both $k, j \in \Lambda_n^* \setminus \Lambda_n$, then $S_k(B^{-\mathrm{tr}} j) = \delta_{k',j'}$. Using the relation (3.13), we have shown that

$$\Phi_n^*(\frac{j-k}{2n}) = 1 \quad \text{when } k \equiv j \mod 2n\mathbb{Z}^2 \text{ and 0 otherwise,}$$

from which the stated result follows. □

It turns out that the function Φ_n^* satisfies a compact formula. Let us define an operator $\mathscr{P}$ by

$$(\mathscr{P} f)(x) = \frac{1}{4} \left[f(x_1, x_2) + f(-x_1, x_2) + f(x_1, -x_2) + f(-x_1, -x_2)\right].$$

For $\mathrm{e}_k(x) = \mathrm{e}^{2\pi i k \cdot x}$, it follows immediately that

$$(\mathscr{P} \mathrm{e}_k)(x) = \cos(2\pi k_1 x_1) \cos(2\pi k_2 x_2), \quad \forall k \in \mathbb{Z}^2. \tag{3.15}$$

Lemma 3.8. *For $n \geq 0$,*

$$\Phi_n^*(x) = 2\left[D_n(x) + D_{n-1}(x)\right] - \frac{1}{4}(\cos 2\pi n x_1 + \cos 2\pi n x_2), \tag{3.16}$$

where

$$D_n(x) := \frac{1}{4} \sum_{\nu \in \Lambda_n^*} \mathrm{e}_\nu(x) = \frac{1}{2} \frac{\cos \pi(2n+1)x_1 \cos \pi x_1 - \cos \pi(2n+1)x_2 \cos \pi x_2}{\cos 2\pi x_1 - \cos 2\pi x_2}. \tag{3.17}$$

Proof. Using the values of $\widetilde{c}_\nu^{(n)}$ and the definition of D_n, it is easy to see that

$$\Phi_n^*(x) = 2\left[D_n(x) + D_{n-1}(x)\right] - \sum_{\nu \in \Lambda^\nu} \mathrm{e}_\nu(x).$$

Since Λ_n^ν contains four terms, $(\pm n, 0)$ and $(0, \pm n)$, the sum over Λ_n^ν becomes the second term in (3.16). On the other hand, using the symmetry of Λ_n^* and (3.15),

$$D_n(x) = \frac{1}{4} \sum_{\nu \in \Lambda_n^*} (\mathscr{P} \mathrm{e}_\nu)(x) = \sum_{0 \leq j_1 + j_2 \leq n}{}' \cos 2\pi j_1 x_1 \cos 2\pi j_2 x_2,$$

where $\sum'$ means that the terms in the sum are halved whenever either $j_1 = 0$ or $j_2 = 0$, from which the second equal sign in (3.17) follows from [18, (4.2.1) and (4.2.7)]. $\square$

Our main result in this section is interpolation over points in $\{\frac{k}{2n} : k \in \Xi_n\}$ with Ξ_n defined in (3.5).

Theorem 3.9. *For $n \ge 0$, define*

$$\mathscr{L}_n f(x) = \sum_{k \in \Xi_n} f\left(\tfrac{k}{2n}\right) \ell_k(x), \quad \ell_k(x) := \lambda_k^{(n)} \mathscr{P}\left[\Phi_n^*(\cdot - \tfrac{k}{2n})\right](x),$$

with $\lambda_k^{(n)}$ given in (3.8). *Then, $\mathscr{L}_n f \in \mathscr{T}_n$ is even in both variables and it satisfies*

$$\mathscr{L}_n f\left(\tfrac{j}{2n}\right) = f\left(\tfrac{j}{2n}\right), \quad \forall j \in \Xi_n.$$

Proof. As shown in the proof of Proposition 3.7, $R_k(x) := \Phi_n^*(x - \frac{k}{2n})$ satisfies $R_k(\frac{j}{2n}) = 1$ when $k \equiv j \mod 2n\mathbb{Z}^2$ and 0 otherwise. Hence, if $j \in \Xi_n^\circ$ then

$$(\mathscr{P}R_k)\left(\tfrac{j}{2n}\right) = \frac{1}{4} R_k\left(\tfrac{j}{2n}\right) = [\lambda_k^{(n)}]^{-1} \delta_{k,j}.$$

If $j \in \Xi_n^e$ then the number of terms in the sum of $(\mathscr{P}R_k)(\frac{j}{2n})$ depends on whether $j_1 j_2$ is zero; if $j_1 j_2 \neq 0$ then

$$(\mathscr{P}R_k)\left(\tfrac{j}{2n}\right) = \frac{1}{4}\left[R_k\left(\tfrac{j}{2n}\right) + R_k\left(\tfrac{j'}{2n}\right)\right] = \frac{1}{2}\delta_{k,j} = [\lambda_k^{(n)}]^{-1} \delta_{k,j},$$

whereas if $j_1 j_2 = 0$ then

$$(\mathscr{P}R_k)\left(\tfrac{j}{2n}\right) = \frac{1}{2} R_k\left(\tfrac{j}{2n}\right) = [\lambda_k^{(n)}]^{-1} \delta_{k,j}.$$

For $j = (n, 0)$ or $(0, n)$ in Ξ_n^v, we have

$$(\mathscr{P}R_k)\left(\tfrac{j}{2n}\right) = \frac{1}{2}\left[R_k\left(\tfrac{j}{2n}\right) + R_k\left(\tfrac{j'}{2n}\right)\right] = \delta_{k,j};$$

for $j = (n, n) \in \Xi_n^v$, we have

$$(\mathscr{P}R_k)\left(\tfrac{j}{2n}\right) = \frac{1}{4}\left[R_k\left(\tfrac{(n,n)}{2n}\right) + R_k\left(\tfrac{(-n,n)}{2n}\right) + R_k\left(\tfrac{(n,-n)}{2n}\right) + R_k\left(\tfrac{(-n,-n)}{2n}\right)\right] = \delta_{k,j};$$

finally for $j = 0 \in \Xi_n^v$, it is evident that $(\mathscr{P}R_k)(0) = \delta_{k,0}$. Putting these together, we have verified that

$$\ell_k\left(\frac{j}{2n}\right) = \delta_{k,j} \quad \forall j, k \in \Xi_n^*,$$

which verifies the interpolation of $\mathscr{L}_n f$. $\square$

As in the case of cubature, we can translate the above theorem to interpolation by algebraic polynomials by applying the change of variables (3.4). Recall Γ_n defined in (3.6).

Theorem 3.10. *For $n \ge 0$, let*

$$\mathscr{L}_n f(t) = \sum_{z_k \in \Gamma_n} f(z_k) \ell_k^*(t), \quad \ell_k^*(t) = \ell_k(x),$$

with $t_i = \cos 2\pi x_i, i = 1,2$. Then, $\mathscr{L}_n f \in \Pi_n^2$ and it satisfies $\mathscr{L}_n f(z_k) = f(z_k)$ for all $z_k \in \Gamma_n$. Furthermore, under the change of variables (3.4)*, the fundamental polynomial $\ell_k^*(t)$ satisfies*

$$\ell_k^*(t) = \frac{1}{2}\mathscr{P}\left[D_n\left(\cdot - \tfrac{k}{2n}\right) + D_{n-1}\left(\cdot - \tfrac{k}{2n}\right)\right](x) - \frac{1}{4}\left[(-1)^{k_1} T_{k_1}(t_1) + (-1)^{k_2} T_{k_2}(t_2)\right].$$

Proof. That $\mathscr{L}_n f$ interpolates at $z_k \in \Gamma_n$ is an immediate consequence of the change of variables, which also shows that $\mathscr{L}_n f \in \Pi_n^2$. Moreover,

$$\cos 2\pi n(x_1 - \frac{k_1}{2n}) = (-1)^{k_1} \cos 2\pi n x_1 = (-1)^{k_1} T_n(x_1),$$

which verifies the formula of $\ell_k^*(t)$. $\square$

The polynomial $\mathscr{L}_n f$ belongs, in fact, to a subspace $\Pi_n^* \subset \Pi_n^2$ of dimension $|\Xi_n| = \dim \Pi_{n-1}^2 + \lfloor \frac{n}{2} \rfloor + 1$, and it is the unique interpolation polynomial in Π_n^*. In the case of n is odd, this interpolation polynomial was defined and studied in [19], where a slightly different scheme with one point less was studied in the case of even n. Recently the interpolation polynomials in [19] have been tested and studied numerically in [3,4]; the results show that these polynomials can be evaluated efficiently and provide valuable tools for numerical computation.

4. Cubature and interpolation on the cube

For $d = 2$, the choice of our spectral set Ω_B and lattice in the previous section ensures that we end up with a space close to the polynomial subspace Π_n^2; indeed, monomials in Π_n^2 are indexed by $0 \le j_1 + j_2 \le n$, a quarter of Λ_n^*. For $d = 3$, the same consideration indicates that we should choose the spectral set as the octahedron $\{x : -n \le x_1 \pm x_2 \pm x_3 \le n\}$. The octahedron, however, does not tile $\mathbb{R}^3$ by lattice translation (see, e.g., [6, p. 452]). As an alternative, we choose the spectral set as *rhombic dodecahedron*, which tiles $\mathbb{R}^3$ by lattice translation with *face centered cubic (fcc) lattice*. In [9], a discrete Fourier analysis on the rhombic dodecahedron is developed and used to study cubature and interpolation on the rhombic dodecahedron, which also leads to results on tetrahedron. In contrast, our results will be established on the cube $[-\frac{1}{2}, \frac{1}{2}]^3$, but our set Ω_B is chosen to be a rhombic dodecahedron.

4.1. Discrete Fourier analysis and cubature formula on the cube

We choose our matrix B as the generator matrix of *fcc* lattice,

$$B = n\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B^{-1} = \frac{1}{2n}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

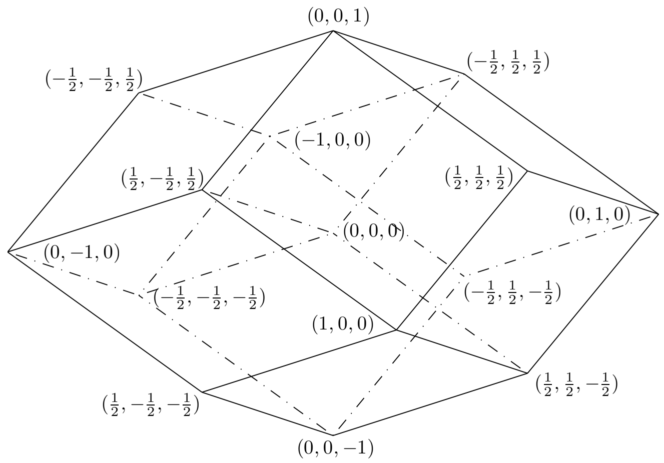

Figure 2: Rhombic dodecahedron.

The spectral set of the *fcc* lattice is the rhombic dodecahedron (see Fig. 2). Thus,

$$\Omega_B = \left\{x \in \mathbb{R}^3 : -n \le x_\nu \pm x_\mu < n, 1 \le \nu < \mu \le 3\right\}.$$

The strict inequality in the definition of Ω_B reflects our requirement that the tiling of the spectral set has no overlapping. From the expression of $B^{-\mathrm{tr}}$, it follows that $\Lambda_B =: \Lambda_n$ is given by

$$\Lambda_n := \left\{j \in \mathbb{Z}^3 : -n \le -j_1 + j_2 + j_3, j_1 - j_2 + j_3, j_1 + j_2 - j_3 < n\right\}.$$

It is known that $|\Lambda_n| = \det(B) = 2n^3$. Furthermore, $\Lambda_B^\dagger =: \Lambda_n^\dagger$ is given by

$$\Lambda_n^\dagger = \mathbb{Z}^3 \cap \Omega_B = \left\{k \in \mathbb{Z}^3 : -n \le k_\nu \pm k_\mu < n, 1 \le \nu < \mu \le 3\right\}.$$

We denote the space $\mathscr{T}_B$ by $\mathscr{T}_n$, which is given by

$$\mathscr{T}_n := \operatorname{span}\left\{\mathrm{e}^{2\pi i\, k\cdot x} : \ k \in \Lambda_n^\dagger\right\}.$$

Then, $\dim \mathscr{T}_n = |\Lambda_n^\dagger| = \det(B) = 2n^3$.

Theorem 4.1. *Define the set*

$$X_n := \left\{2k : -\tfrac{n}{2} \le k_1, k_2, k_3 < \tfrac{n}{2}\right\} \cup \left\{2k + 1 : -\tfrac{n+1}{2} \le k_1, k_2, k_3 < \tfrac{n-1}{2}\right\}.$$

Then, for all $f, g \in \mathscr{T}_n$,

$$\langle f, g\rangle_n := \frac{1}{2n^3} \sum_{k\in X_n} f\left(\tfrac{k}{2n}\right)\overline{g\left(\tfrac{k}{2n}\right)} = \int_{[-\frac{1}{2},\frac{1}{2}]^3} f(x)\overline{g(x)}dx.$$

Proof. Changing variables from j to $k = 2nB^{-\mathrm{tr}}j$, or $j = B^{\mathrm{tr}}k/(2n)$, then, as j_1, j_2, j_3 are integers and $j_1 = \frac{k_2+k_3}{2}$, $j_2 = \frac{k_1+k_3}{2}$, $j_3 = \frac{k_1+k_2}{2}$, we see that

$$j \in \Lambda_n \Longleftrightarrow 2nB^{-\mathrm{tr}}j \in X_n, \qquad \sum_{j\in\Lambda_n} f(B^{-tr}j) = \sum_{k\in X_n} f\left(\tfrac{k}{2n}\right), \tag{4.1}$$

from which we conclude that $\langle f,g\rangle_n = \langle f,g\rangle_B$. Consequently, this theorem is a special case of Theorem 2.1. $\square$

Just like the case of $d = 2$, we denote the symmetric counterpart of X_n by X_n^* which is defined by

$$X_n^* := \left\{2k : -\tfrac{n}{2} \le k_1, k_2, k_3 \le \tfrac{n}{2}\right\} \cup \left\{2k+1 : -\tfrac{n+1}{2} \le k_1, k_2, k_3 \le \tfrac{n-1}{2}\right\}.$$

A simple counting shows that $|X_n^*| = n^3 + (n+1)^3$. The set X_n^* is further partitioned into four parts,

$$X_n^* = X_n^\circ \cup X_n^f \cup X_n^e \cup X_n^v,$$

where $X_n^\circ = X_n^* \cap (-n,n)^2$ is the set of interior points, X_n^f contains the points in X_n^* that are on the faces of $[-n,n]^3$ but not on the edges or vertices, X_n^e contains the points in X_n^* that are on the edges of $[-n,n]^3$ but not on the corners or vertices, while X_n^v denotes the points of X_n^* at the vertices of $[-n,n]^3$.

Theorem 4.2. *Define the inner product*

$$\langle f,g\rangle_n^* := \frac{1}{2n^3}\sum_{k\in X_n^*} c_k^{(n)} f\left(\tfrac{k}{2n}\right)\overline{g\left(\tfrac{k}{2n}\right)}, \qquad \text{where} \quad c_k^{(n)} = \begin{cases} 1, & k\in X_n^\circ \\ \frac12, & k\in X_n^f \\ \frac14, & k \in X_n^e \\ \frac18, & k\in X_n^v \end{cases}. \tag{4.2}$$

Then, for all $f,g \in \mathcal{T}_n$,

$$\int_{[-\frac12,\frac12]^3} f(x)\overline{g(x)}dx = \langle f,g\rangle_n = \langle f,g\rangle_n^*.$$

Proof. The proof follows along the same line as the proof of Theorem 3.2. We only need to show $\langle f,g\rangle_n = \langle f,g\rangle_n^*$ if $f\overline{g}$ is periodic. The interior points of X_n and X_n^* are the same, so that $c_k^{(n)} = 1$ for $k \in X_n^\circ$. Let $\varepsilon_1 = (1,0,0)$, $\varepsilon_2 = (0,1,0)$, and $\varepsilon_3 = (0,0,1)$. Each point k in X_n^f has exactly one opposite point k^* in X_n^f under translation by $\pm n\varepsilon_i$ and only one of them is in X_n, so that

$$f(x_k) = \frac12\left[f(x_k) + f(x_k^*)\right]$$

if f is periodic, which is why we define $c_k^{(n)} = \frac12$ for $k \in X_n^f$. Evidently, only three edges of X_n^* are in $X_n^* \setminus X_n$. Each point in X_n^e corresponds to exactly four points in X_n^e under integer translations $\pm n\varepsilon_i$ and only one among the four is in X_n, so we define $c_k^{(n)} = \frac{1}{4}$ for $k \in X_n^e$. Finally, all eight corner points can be derived from translations $n\varepsilon_i$ points, used repeatedly, and exactly one, $(-n,-n,-n)$, is in $X_n^* \setminus X_n$, so that we define $c_k^{(n)} = \frac{1}{8}$ for $k \in X_n^v$. □

We also denote the symmetric counterpart of $\Lambda_n^\dagger$ by $\Lambda_n^{\dagger *}$,

$$\Lambda_n^{\dagger *} := \{ j \in \mathbb{Z}^3 : -n \le j_\nu \pm j_\mu \le n, 1 \le \nu < \mu \le 3 \}, \tag{4.3}$$

and the counterpart of $\mathscr{T}_n$ by $\mathscr{T}_n^*$, which is defined accordingly by

$$\mathscr{T}_n^* := \operatorname{span}\left\{ \mathrm{e}^{2\pi i\, k\cdot x} :\ k \in \Lambda_n^{\dagger *} \right\}.$$

Theorem 4.3. *For $n \ge 2$, the cubature formulas*

$$\begin{aligned} \int_{[-\frac{1}{2},\frac{1}{2}]^3} f(x)dx &= \frac{1}{2n^3} \sum_{k \in X_n^*} c_k^{(n)} f\left(\tfrac{k}{2n}\right), \\ \int_{[-\frac{1}{2},\frac{1}{2}]^3} f(x)dx &= \frac{1}{2n^3} \sum_{k \in X_n} f\left(\tfrac{k}{2n}\right), \end{aligned} \tag{4.4}$$

are exact for $f \in \mathscr{T}_{2n-1}^*$.

Proof. As in the proof of Theorem 3.3, for any $j \in \mathbb{Z}^3$, there exist $\nu \in \Lambda_n^\dagger$ and $l \in \mathbb{Z}^3$ such that $j = \nu + Bl$.

Assume now $j \in \Lambda_{2n-1}^{\dagger *}$. Clearly the integral of e_j over Ω is $\delta_{j,0}$. On the other hand, let us suppose $j = \nu + Bl$ with $\nu \in \Lambda_n$ and $l \in \mathbb{Z}^3$. Then, it is easy to see that $\mathrm{e}_j(\frac{k}{2n}) = \mathrm{e}_\nu(\frac{k}{2n})$ for each $k \in X_n^*$. Consequently, we get from Theorem 4.2 that

$$\begin{aligned} \sum_{k \in X_n^*} c_k^{(n)} \mathrm{e}_j\left(\tfrac{k}{2n}\right) &= \sum_{k \in X_n^*} c_k^{(n)} \mathrm{e}_\nu\left(\tfrac{k}{2n}\right) = \sum_{k \in X_n} \mathrm{e}_\nu\left(\tfrac{k}{2n}\right) \\ &= \sum_{k \in X_n} \mathrm{e}_j\left(\tfrac{k}{2n}\right) = \int_\Omega \mathrm{e}_\nu(x)dx = \delta_{\nu,0}. \end{aligned}$$

Since $\nu = 0$ implies $j = l = 0$, we further obtain that $\delta_{\nu,0} = \delta_{j,0}$. This states that the cubature formulas (4.4) are exact for each e_j with $j \in \Lambda_{2n-1}^{\dagger *}$, which completes the proof. □

4.2. Cubature formula for algebraic polynomials

We can also translate the cubature in Theorem 4.3 into one for algebraic polynomials. For this we use the change of variables

$$t_1 = \cos 2\pi x_1, \quad t_2 = \cos 2\pi x_2, \quad t_3 = \cos 2\pi x_3, \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^3. \tag{4.5}$$

Under (4.5), the functions $\cos 2\pi k_1 x_1 \cos 2\pi k_2 x_2 \cos 2\pi k_3 x_3$ become algebraic polynomials $T_{k_1}(t_1)T_{k_2}(t_2)T_{k_3}(t_3)$, which are even in each of its variables. The subspace of $\mathscr{T}_n^*$ that consists of functions that are even in each of its variables corresponds to the polynomial subspace

$$\Pi_n^* := \operatorname{span}\{T_{k_1}(x_1)T_{k_2}(x_2)T_{k_3}(x_3) : k_1, k_2, k_3 \geq 0,\, k_\nu + k_\mu \leq n,\, 1 \leq \nu < \mu \leq 3\}.$$

Notice that X_n^* is symmetric in the sense that if $x \in X_n^*$, then $\sigma x \in X_n^*$ for all $\sigma \in \{-1,1\}^3$, where $(\sigma x)_i = \sigma_i x_i$. In order to evaluate functions that are even in each of its variables on X_n^*, we only need to consider $X_n^* \cap \{x : x_1, x_2, x_3 \geq 0\}$. Hence, we define,

$$\Xi_n := \{2k : 0 \leq k_1, k_2, k_3 \leq \tfrac{n}{2}\} \cup \{2k+1 : 0 \leq k_1, k_2, k_3 \leq \tfrac{n-1}{2}\} \tag{4.6}$$

and under the change of variables (4.5), define

$$\Gamma_n := \{(z_{k_1}, z_{k_2}, z_{k_3}) : k \in \Xi_n\}, \quad z_k = \tfrac{k}{2n}. \tag{4.7}$$

Moreover, we denote by Γ_n°, Γ_n^f, Γ_n^e and Γ_n^v the subsets of Γ_n that contains interior points, points on the faces but not on the edges, points on the edges but not on the vertices, and points on the vertices, of $[-1,1]^3$, respectively, and we define Ξ_n°, Ξ_n^f, Ξ_n^e and Ξ_n^v accordingly. A simple counting shows that

$$|\Xi_n| = \left(\lfloor\tfrac{n}{2}\rfloor + 1\right)^3 + \left(\lfloor\tfrac{n-1}{2}\rfloor + 1\right)^3 = \begin{cases} \frac{(n+1)^3}{4} + \frac{3(n+1)}{4}, & n \text{ is even}, \\ \frac{(n+1)^3}{4}, & n \text{ is odd}. \end{cases} \tag{4.8}$$

Theorem 4.4. *Write $z_k = (z_{k_1}, z_{k_2}, z_{k_3})$. The cubature formula*

$$\frac{1}{\pi^3}\int_{[-1,1]^3} f(t)W_0(t)dt = \frac{1}{2n^2}\sum_{k \in \Xi_n} \lambda_k^{(n)} f(z_k), \quad \lambda_k^{(n)} := \begin{cases} 8, & k \in \Xi_n^\circ, \\ 4, & k \in \Xi_n^f, \\ 2, & k \in \Xi_n^e, \\ 1, & k \in \Xi_n^v, \end{cases} \tag{4.9}$$

is exact for Π_{2n-1}^*. In particular, it is exact for Π_{2n-1}^3.

Proof. Let $g(x) = f(\cos 2\pi x_1, \cos 2\pi x_2, \cos 2\pi x_3)$. Then, g is even in each of its variables and $g(\frac{k}{2n}) = f(z_k)$. Applying the first cubature formula in (3.3) to $g(x)$, we see that (3.8) follows from the following identity,

$$\sum_{k \in X_n^*} c_k^{(n)} g\left(\tfrac{k}{2n}\right) = \sum_{k \in \Xi_n} \lambda_k^{(n)} f(z_k).$$

This identity is proved in the same way that the corresponding identity in Theorem 3.4 is proved. Let $k\sigma$ denote the set of distinct elements in $\{k\sigma : \sigma \in \{-1,1\}^3\}$; then $g(\frac{k}{2n})$ takes the same value on all points in $k\sigma$. If $k \in X_n^*$, $k_i \neq 0$ for $i = 1,2,3$, then $k\sigma$ contains 8 points; if exactly one k_i is zero then $k\sigma$ contains 4 points; if exactly two k_i are zero, then $k\sigma$ contains one point; and, finally, if $k = (0,0,0)$, then $k\sigma$ contains one point. In the case of $k_i \neq 0$ for $i = 1,2,3$,

$$\sum_{j \in k\sigma} c_k^{(n)} g(\frac{j}{2n}) = 8g(\frac{k}{2n}) \quad \text{if} \quad k \in X_n^\circ,$$

$$\sum_{j \in k\sigma} c_k^{(n)} g(\frac{j}{2n}) = 4g(\frac{k}{2n}) \quad \text{if} \quad k \in X_n^f,$$

$$\sum_{j \in k\sigma} c_k^{(n)} g(\frac{j}{2n}) = 2g(\frac{k}{2n}) \quad \text{if} \quad k \in X_n^e.$$

The other cases are treated similarly. Thus, (4.9) holds for Π_{2n-1}^*. □

Finally, the definition of Π_n^* shows readily that it contains

$$\Pi_n^3 = \operatorname{span}\{T_{k_1}(x_1)T_{k_2}(x_2)T_{k_3}(x_3) : k_1, k_2, k_3 \geq 0,\ 0 \leq k_1 + k_2 + k_3 \leq n\}$$

as a subspace. In particular, Π_{2n-1}^* contains Π_{2n-1}^3 as a subset.

We note that Π_{2n-1}^* contains Π_{2n-1}^3 as a subspace, but it does not contain Π_{2n}^3 since $T_n(x_1)T_n(x_2)$ is in Π_{2n}^3 but not in Π_{2n-1}^3. Hence, the cubature (4.9) is of degree $2n-1$. A trivial cubature formula of degree $2n-1$ for W_0 can be derived by taking the product of Gaussian quadrature of degree $2n-1$ in one variable, which has exactly n^3 nodes. In contrast, according to (4.8), the number of nodes of our cubature (3.8) is in the order of $n^3/4 + \mathcal{O}(n^2)$, about a quarter of the product formula. As far as we know, this is the best that is available at the present time. On the other hand, the lower bound for the number of nodes states that a cubature formula of degree $2n-1$ needs at least $n^3/6 + \mathcal{O}(n^2)$ nodes. It is, however, an open question if there exist formulas with number of nodes attaining this theoretic lower bound.

Recall the cubature (4.9) is derived by choosing the spectral set as a rhombic dodecahedron. One natural question is how to choose a spectral set that tiles $\mathbb{R}^3$ by translation so that the resulted cubature formula is of degree $2n-1$ and has the smallest number of nodes possible. Among the regular lattice tiling, the rhombic dodecahedron appears to lead to the smallest number of nodes.

Just as Theorem 3.5, we can also derive a cubature formula of degree $2n-5$ for W_1 from Theorem 4.3. We omit the proof as it follows exactly as in Theorem 3.5.

Theorem 4.5. *The cubature formula*

$$\frac{1}{\pi^3}\int_{[-1,1]^3} f(t)W_1(t)dt = \frac{4}{n^3}\sum_{k \in \Xi_n^\circ} \sin^2\tfrac{k_1\pi}{n}\sin^2\tfrac{k_2\pi}{n}\sin^2\tfrac{k_3\pi}{n} f(z_k) \tag{4.10}$$

is exact for Π_{2n-5}^*. *In particular, it is exact for* Π_{2n-5}^3.

4.3. A compact formula for a partial sum

In order to obtain the compact formula for the interpolation function, we follow [9] and use homogeneous coordinates and embed the rhombic dodecahedron into the plane $t_1+t_2+t_3+t_4=0$ of $\mathbb{R}^4$. Throughout the rest of this paper, we adopt the convention of using bold letters, such as $\mathbf{t}$, to denote the points in the space

$$\mathbb{R}^4_H := \left\{\mathbf{t}=(t_1,t_2,t_3,t_4)\in \mathbb{R}^4 : t_1+t_2+t_3+t_4=0\right\}.$$

In other words, the bold letters such as $\mathbf{t}$ and $\mathbf{k}$ will always mean homogeneous coordinates. The transformation between $x\in\mathbb{R}^3$ and $\mathbf{t}\in\mathbb{R}^4_H$ is defined by

$$\begin{cases} x_1=t_2+t_3, \\ x_2=t_1+t_3, \\ x_3=t_2+t_1, \end{cases} \Longleftrightarrow \begin{cases} t_1=\frac{1}{2}(-x_1+x_2+x_3), \\ t_2=\frac{1}{2}(x_1-x_2+x_3), \\ t_3=\frac{1}{2}(x_1+x_2-x_3), \\ t_4=\frac{1}{2}(-x_1-x_2-x_3). \end{cases} \tag{4.11}$$

In this homogenous coordinates, the spectral set Ω_B becomes

$$\Omega_B = \left\{\mathbf{t}\in\mathbb{R}^4_H : -1<t_i-t_j\le 1, \quad 1\le i<j\le 4\right\}. \tag{4.12}$$

We now use homogeneous coordinates to describe $\Lambda_n^{\dagger *}$ defined in (4.3). Let $\mathbb{Z}^4_H := \mathbb{Z}^4\cap\mathbb{R}^4_H$ and

$$\mathbb{H} := \{\mathbf{j}\in\mathbb{Z}^4_H : j_1\equiv j_2\equiv j_3\equiv j_4 \mod 4\}.$$

In order to keep the elements as integers, we make the change of variables

$$\begin{aligned} j_1&=2(-k_1+k_2+k_3), & j_2&=2(k_1-k_2+k_3),\\ j_3&=2(k_1+k_2-k_3), & j_4&=2(-k_1-k_2-k_3), \end{aligned} \tag{4.13}$$

for $k=(k_1,k_2,k_3)\in\Lambda_n^{\dagger *}$. It then follows that $\Lambda_n^{\dagger *}$ in homogeneous coordinates becomes

$$\mathbb{G}_n := \{j\in\mathbb{H} : j_1\equiv j_2\equiv j_3\equiv j_4\equiv 0 \mod 2, \ \ -4n\le j_\nu-j_\mu\le 4n, \ 1\le\nu, \ \mu\le 4\}.$$

We could have changed variables without the factor 2, setting $j_1=-k_1+k_2+k_3$ etc. We choose the current change of variables so that we can use some of the computations in [9]. In fact, the set

$$\mathbb{H}^*_n := \{j\in\mathbb{H} : -4n\le j_\nu-j_\mu\le 4n, \ 1\le\nu, \ \mu\le 4\} \tag{4.14}$$

is used in [9]. The main result of this subsection is a compact formula for the partial sum

$$D_n(x) := \sum_{k\in\Lambda_n^{\dagger *}} \mathrm{e}_k(x) = \sum_{\mathbf{j}\in\mathbb{G}_n} \mathrm{e}_{\mathbf{j}}(\mathbf{t}) =: D_n^*(\mathbf{t}), \quad \mathrm{e}_{\mathbf{j}}(\mathbf{t}) := \mathrm{e}^{\frac{\pi i}{2}\mathbf{j}\cdot\mathbf{t}}, \tag{4.15}$$

where x and $\mathbf{t}$ are related by (4.11) and the middle equality follows from the fact that $\Lambda_n^{\dagger *} = \mathbb{G}_n$ under this change of variables. In fact, by (4.11) and (4.13), we have

$$\begin{aligned} k \cdot x &= k_1(t_2+t_3) + k_2(t_1+t_3) + k_3(t_1+t_2) \\ &= (k_2+k_3)t_1 + (k_1+k_3)t_2 + (k_1+k_2)t_3 \\ &= \frac{1}{4}\left[(j_1-j_4)t_1 + (j_2-j_4)t_2 + (j_3-j_4)t_3\right] = \frac{1}{4}\mathbf{j}\cdot\mathbf{t}, \end{aligned}$$

where in the last step we have used the fact that $\mathbf{t} \in \mathbb{R}_H^4$. The compact formula of $D_n(\mathbf{t})$ is an essential part of the compact formula for the interpolation function.

Theorem 4.6. *For $n \geq 1$,*

$$D_n^*(\mathbf{t}) = \Theta_{n+1}(\mathbf{t}) - \Theta_n(\mathbf{t}) - \left(\Theta_n^{\mathrm{odd}}(\mathbf{t}) - \Theta_{n-2}^{\mathrm{odd}}(\mathbf{t})\right),$$

where

$$\Theta_n(\mathbf{t}) = \prod_{i=1}^{4} \frac{\sin \pi n t_i}{\sin \pi t_i},$$

and for $n \geq 1$,

$$\Theta_n^{\mathrm{odd}}(\mathbf{t}) = \prod_{i=1}^{4} \frac{\sin(n+2)\pi t_i}{\sin 2\pi t_i} \sum_{j=1}^{4} \frac{\sin n\pi t_j}{\sin(n+2)\pi t_j}, \quad \textit{if } n \textit{ is even},$$

$$\Theta_n^{\mathrm{odd}}(t) = \prod_{i=1}^{4} \frac{\sin(n+1)\pi t_i}{\sin 2\pi t_i} \sum_{j=1}^{4} \frac{\sin(n+3)\pi t_j}{\sin(n+1)\pi t_j}, \quad \textit{if } n \textit{ is odd}.$$

Proof. By definition, $\mathbb{G}_n$ is a subset of $\mathbb{H}_n^*$ that contains elements with all indices being even integers. For technical reasons, it turns out to be easier to work with $\mathbb{H}_n^* \setminus \mathbb{G}_n$. In fact, the sum over $\mathbb{H}_n^*$ has already been worked out in [9], which is

$$\sum_{\mathbf{j}\in\mathbb{H}_n^*} \phi_{\mathbf{j}}(\mathbf{t}) = \prod_{i=1}^{4} \frac{\sin(n+1)\pi t_i}{\sin \pi t_i} - \prod_{i=1}^{4} \frac{\sin n\pi t_i}{\sin \pi t_i} = \Theta_{n+1}(\mathbf{t}) - \Theta_n(\mathbf{t}).$$

Thus, we need to find only the sum over odd indices, that is, the sum

$$D_n^{\mathrm{odd}}(\mathbf{t}) := \sum_{\mathbf{j}\in\mathbb{H}_n^{\mathrm{odd}}} \mathrm{e}_{\mathbf{j}}(\mathbf{t}), \quad \mathbb{H}_n^{\mathrm{odd}} := \mathbb{H}_n^* \setminus \mathbb{G}_n.$$

Just as in [9], the index set $\mathbb{H}_n^{\mathrm{odd}}$ can be partitioned into four congruent parts, each within a parallelepiped, defined by

$$\mathbb{H}_n^{(k)} := \left\{\mathbf{j} \in \mathbb{H}_n^{\mathrm{odd}} : 0 \leq j_l - j_k \leq 4n, \quad l \in \mathbb{N}_4\right\}$$

for $k \in \mathbb{N}_4$. Furthermore, for each index set J, $\emptyset \subset J \subseteq \mathbb{N}_4$, define

$$\mathbb{H}_n^J := \left\{ \mathbf{k} \in \mathbb{H}_n^{\text{odd}} : k_i = k_j, \ \forall i,j \in J, \text{ and } 0 \le k_i - k_j \le 4n, \ \forall j \in J, \ \forall i \in \mathbb{N}_4 \setminus J \right\}.$$

Then, we have

$$\mathbb{H}_n^{\text{odd}} = \bigcup_{j\in\mathbb{N}_4} \mathbb{H}_n^{(j)}, \qquad \mathbb{H}_n^J = \bigcap_{j\in J} \mathbb{H}_n^{(j)}.$$

Using the inclusion-exclusion relation of subsets, we have

$$D_n^{\text{odd}}(\mathbf{t}) = \sum_{\emptyset\subset J\subseteq\mathbb{N}_4} (-1)^{|J|+1} \sum_{\mathbf{k}\in\mathbb{H}_n^J} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}}.$$

Fix $j \in J$, using the fact that $t_j = -\sum_{i\neq j} t_i$, we have

$$\sum_{\mathbf{k}\in\mathbb{H}_n^J} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}} = \sum_{\mathbf{k}\in\mathbb{H}_n^J} \mathrm{e}^{\frac{\pi i}{2}\sum_{l\in\mathbb{N}_4\setminus J}(k_l-k_j)t_l} = \sum_{\mathbf{k}\in\mathbb{H}_n^J}\prod_{l\in\mathbb{N}_4\setminus J} \mathrm{e}^{\frac{\pi i}{2}(k_l-k_j)t_l}.$$

Since $\mathbf{k} \in \mathbb{H}_n^J$ implies, in particular, $k_i \equiv k_j \mod 4$, we obtain

$$\sum_{\mathbf{k}\in\mathbb{H}_n^J} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}} = \prod_{l\in\mathbb{N}_4\setminus J} \sum_{\substack{0\le k_l-k_j\le 4n \\ \mathbf{k}\in\mathbb{H}_n^J}} \mathrm{e}^{\frac{\pi i}{2}(k_l-k_j)t_l} = \prod_{l\in\mathbb{N}_4\setminus J} \sum_{\substack{0\le k_l\le n \\ |k|_J \text{ odd}}} \mathrm{e}^{2\pi i\, k_l t_l},$$

where $|k|_J := \sum_{l\in\mathbb{N}_4\setminus J} k_l$. The last equation needs a few words of explanation: if $4k_l' = k_l - k_j$, then using the fact that $k_i = k_j, \ \forall i,j \in J$ for $k \in \mathbb{H}_n^J$ and $k_1 + k_2 + k_3 + k_4 = 0$, we see that

$$\frac{1}{4}\sum_{l\in\mathbb{N}_4\setminus J}(k_l - k_j) = -k_j,$$

which is odd by the definition of $\mathbb{H}_n^J$. On the other hand, assume that $\sum_{l\in\mathbb{N}_4\setminus J} k_l'$ is odd, then we define $k_j = -\sum_{l\in\mathbb{N}_4\setminus J} k_l' \quad \forall j \in J$ and define $k_l = 4kl' + k_j$, so that all components of k are odd and $k \in \mathbb{H}_n^J$.

The condition that $|k|_J$ is an odd integer means that the last term is not a simple product of sums. Setting

$$D_n^O(t) := \sum_{j=0, j\,\text{odd}}^{n} \mathrm{e}^{2\pi i j t} = \frac{\mathrm{e}^{2\pi i t}(1 - \mathrm{e}^{4\pi i\lfloor\frac{n+1}{2}\rfloor t})}{1 - \mathrm{e}^{4\pi i t}},$$

$$D_n^E(t) := \sum_{i=0, i\,\text{even}}^{n} \mathrm{e}^{2\pi i j t} = \frac{1 - \mathrm{e}^{4\pi i\lfloor\frac{n+2}{2}\rfloor t}}{1 - \mathrm{e}^{4\pi i t}},$$

we see that, up to a permutation, only products $D_n^O D_n^O D_n^O$ and $D_n^O D_n^E D_n^E$ are possible for triple products ($|J| = 3$). Only $D_n^O D_n^E$ is possible for double products ($|J| = 2$) while only D_n^O is possible ($|J| = 1$), and there is a constant term. Thus, using the fact that

$$abc - (a-1)(b-1)(c-1) = ab + ac + bc - a - b - c + 1,$$

we conclude that

$$
\begin{aligned}
D_n^{\rm odd}(\mathbf{t}) = \textstyle\sum_{(i_1,i_2,i_3)\in\mathbb{N}_4} & D_n^O(t_{i_1})D_n^O(t_{i_2})D_n^O(t_{i_3}) \\
& + D_n^O(t_1)\left[D_n^E(t_2)D_n^E(t_3)D_n^E(t_4) - \big(D_n^E(t_2)-1\big)\big(D_n^E(t_3)-1\big)\big(D_n^E(t_4)-1\big)\right] \\
& + D_n^O(t_2)\left[D_n^E(t_1)D_n^E(t_3)D_n^E(t_4) - \big(D_n^E(t_1)-1\big)\big(D_n^E(t_3)-1\big)\big(D_n^E(t_4)-1\big)\right] \\
& + D_n^O(t_3)\left[D_n^E(t_1)D_n^E(t_2)D_n^E(t_4) - \big(D_n^E(t_1)-1\big)\big(D_n^E(t_2)-1\big)\big(D_n^E(t_4)-1\big)\right] \\
& + D_n^O(t_4)\left[D_n^E(t_1)D_n^E(t_2)D_n^E(t_3) - \big(D_n^E(t_1)-1\big)\big(D_n^E(t_2)-1\big)\big(D_n^E(t_3)-1\big)\right],
\end{aligned}
$$

where the first sum is over all distinct triple integers in $\mathbb{N}_4$.

Assume that n is an even integer. A quick computation shows that

$$
D_n^O(t_1)D_n^E(t_2)D_n^E(t_2)D_n^E(t_4) = \prod_{j=2}^{4} \frac{\sin\pi(n+2)t_i}{\sin 2\pi t_i}\frac{\sin\pi n t_1}{\sin 2\pi t_1}.
$$

Furthermore, we see that

$$
\begin{aligned}
& D_n^O(t_2)D_n^O(t_3)D_n^O(t_4) - D_n^O(t_1)(D_n^E(t_2)-1)(D_n^E(t_3)-1)(D_n^E(t_4)-1) \\
&= \prod_{j=2}^{4}\frac{\sin\pi n t_i}{\sin 2\pi t_i}\left[\mathrm{e}^{i\pi n t_1} - \frac{\mathrm{e}^{-2\pi i n t_1} - \mathrm{e}i\pi n t_1}{1-\mathrm{e}^{4\pi i t_1}}\right] = -\prod_{j=2}^{4}\frac{\sin\pi n t_i}{\sin 2\pi t_i}\frac{\sin\pi(n-2)t_1}{\sin 2\pi t_1}.
\end{aligned}
$$

Adding the above two terms together and then summing over the permutation of the sum, we end up the formula for $D_n^{\rm odd}(\mathbf{t})$ when n is even. The case of n odd can be handled similarly. □

Let us write down explicitly the function $D_n(x)$ defined in (4.15) in x-variables. Using the elementary trigonometric identity and (4.11), we see that

$$
\begin{aligned}
4\prod_{i=1}^{4}\sin\alpha\pi t_i &= \big(\cos\alpha\pi(x_2-x_1) - \cos\alpha\pi x_3\big)\big(\cos\alpha\pi(x_2+x_1) - \cos\alpha\pi x_3\big) \\
&= \cos^2\alpha x_1 + \cos^2\alpha x_2 + \cos^2\alpha x_3 - 2\cos\alpha x_1\cos\alpha x_2\cos\alpha x_3 - 1,
\end{aligned}
$$

so that we end up with the compact formula

$$
D_n(x) = \widetilde{\Theta}_{n+1}(x) - \widetilde{\Theta}_n(x) - \left(\widetilde{\Theta}_n^{\rm odd}(x) - \widetilde{\Theta}_{n-2}^{\rm odd}(x)\right), \tag{4.16}
$$

where

$$
\widetilde{\Theta}_n(x) = \frac{\cos^2 n\pi x_1 + \cos^2 n\pi x_2 + \cos^2 n\pi x_3 - 2\cos n\pi x_1\cos n\pi x_2\cos n\pi x_3 - 1}{\cos^2\pi x_1 + \cos^2\pi x_2 + \cos^2\pi x_3 - 2\cos\pi x_1\cos\pi x_2\cos\pi x_3 - 1},
$$

and

$$
\begin{aligned}
\widetilde{\Theta}_n^{\rm odd}(x) &= \widetilde{\Theta}_{\frac{n+2}{2}}(2x)\sum_{j=1}^{4}\frac{\sin n\pi t_j}{\sin(n+2)\pi t_j}, \quad \text{if } n \text{ is even}, \\
\widetilde{\Theta}_n^{\rm odd}(t) &= \widetilde{\Theta}_{\frac{n+1}{2}}(2x)\sum_{j=1}^{4}\frac{\sin(n+3)\pi t_j}{\sin(n+1)\pi t_j}, \quad \text{if } n \text{ is odd},
\end{aligned}
$$

in which t_i is given in terms of x_j in (4.11). As a result of this explicit expression, we see that $D_n(x)$ is an even function in each x_i.

4.4. Boundary of the rhombic dodecahedron

In order to develop the interpolation on the set X_n^*, we will need to understand the structure of the points on the boundary of $\Lambda_n^\dagger = \mathbb{Z}^3 \cap \Omega_B$. As Ω_B is a rhombic dodecahedron, we need to understand the boundary of this 12-face polyhedron, which has been studied in detail in [9]. In this subsection, we state the necessary definitions and notations on the boundary of Ω_B, so that the exposition is self-contained. We refer to further details and proofs to [9].

Again we use homogeneous coordinates. For $i,j \in \mathbb{N}_4 := \{1,2,3,4\}$ and $i \neq j$, the (closed) faces of Ω_B are

$$F_{i,j} = \{\mathbf{t} \in \overline{\Omega}_H : t_i - t_j = 1\}.$$

There are a total $2\binom{4}{2} = 12$ distinct $F_{i,j}$, each represents one face of the rhombic dodecahedron. For nonempty subsets I, J of $\mathbb{N}_4$, define

$$\Omega_{I,J} := \bigcap_{i \in I, j \in J} F_{i,j} = \left\{\mathbf{t} \in \overline{\Omega}_H : \ t_j = t_i - 1, \quad \forall i \in I, j \in J\right\}.$$

It is shown in [9] that $\Omega_{I,J} = \emptyset$ if and only if $I \cap J \neq \emptyset$, and $\Omega_{I_1,J_1} \cap \Omega_{I_2,J_2} = \Omega_{I,J}$ if $I_1 \cup I_2 = I$ and $J_1 \cup J_2 = J$. These sets describe the intersections of faces, which can then be used to describe the edges, which are intersections of faces, and vertices, which are intersections of edges. Let

$$\mathcal{K} := \{(I,J) : I,\ J \subset \mathbb{N}_4;\ I \cap J = \emptyset\},$$
$$\mathcal{K}_0 := \{(I,J) \in \mathcal{K} :\ i < j,\ \forall\ (i,j) \in (I,J)\ \}.$$

We now define, for each $(I,J) \in \mathcal{K}$, the boundary element $\mathbb{B}_{I,J}$ of the dodecahedron,

$$\mathbb{B}_{I,J} := \left\{\mathbf{t} \in \Omega_{I,J} :\ \mathbf{t} \notin \Omega_{I_1,J_1}, \quad \forall (I_1,J_1) \in \mathcal{K} \text{ with } |I| + |J| < |I_1| + |J_1|\right\}.$$

It is called a face if $|I| + |J| = 2$, an edge if $|I| + |J| = 3$, and a vertex if $|I| + |J| = 4$. By definition, the elements for faces and edges are without boundary, which implies that $\mathbb{B}_{I,J} \cap \mathbb{B}_{I',J'} = \emptyset$ if $I \neq I_1$ and $J \neq J_1$. In particular, it follows that $\mathbb{B}_{\{i\},\{j\}} = F_{i,j}^\circ$ and, for example, $\mathbb{B}_{\{i\},\{j,k\}} = (F_{i,j} \cap F_{i,k})^\circ$ for distinct integers $i,j,k \in \mathbb{N}_4$.

Let $\mathcal{G} = S_4$ denote the permutation group of four elements and let σ_{ij} denote the element in $\mathcal{G}$ that interchanges i and j; then $\mathbf{t}\sigma_{ij} = \mathbf{t} - (t_i - t_j)\mathrm{e}_{i,j}$. For a nonempty set $I \subset \mathbb{N}_4$, define $\mathcal{G}_I := \left\{\sigma_{ij} : i,j \in I\right\}$, where we take $\sigma_{ij} = \sigma_{ji}$ and take σ_{jj} as the identity element. It follows that $\mathcal{G}_I$ forms a subgroup of $\mathcal{G}$ of order $|I|$. For $(I,J) \in \mathcal{K}$, we then define

$$[\mathbb{B}_{I,J}] := \bigcup_{\sigma \in \mathcal{G}_{I \cup J}} \mathbb{B}_{I,J}\sigma. \tag{4.17}$$

It turns out that $[\mathbb{B}_{I,J}]$ consists of exactly those boundary elements that can be obtained from $\mathbb{B}_{I,J}$ by congruent modulus B, and

$$[\mathbb{B}_{I,J}] \cap [\mathbb{B}_{I_1,J_1}] = \emptyset \quad \text{if } (I,J) \neq (I_1,J_1) \quad \text{for} \quad (I,J) \in \mathscr{K}_0 \quad \text{and} \quad (I_1,J_1) \in \mathscr{K}_0.$$

More importantly, we define, for $0 < i,\ j < i + j \leq 4$,

$$\mathbb{B}^{i,j} := \bigcup_{(I,J)\in\mathscr{K}_0^{i,j}} [\mathbb{B}_{I,J}] \quad \text{with} \quad \mathscr{K}_0^{i,j} := \left\{(I,J) \in \mathscr{K}_0 :\ |I| = i,\ |J| = j\right\}. \tag{4.18}$$

Then, the boundary of $\overline{\Omega}_B$ can be decomposed as

$$\overline{\Omega}_H \setminus \Omega_H^\circ = \bigcup_{(I,J)\in\mathscr{K}} \mathbb{B}_{I,J} = \bigcup_{0<i,j<i+j\leq 4} \mathbb{B}^{i,j}.$$

The main complication is the case of $|I| + |J| = 2$, for which we have, for example,

$$[\mathbb{B}_{\{1\},\{2,3\}}] = \mathbb{B}_{\{1\},\{2,3\}} \cup \mathbb{B}_{\{2\},\{1,3\}} \cup \mathbb{B}_{\{3\},\{1,2\}}. \tag{4.19}$$

The other cases can be written down similarly. Furthermore, we have

$$\begin{aligned}
&\mathbb{B}_{\{1\},\{2,4\}} = \mathbb{B}_{\{1\},\{2,3\}}\sigma_{34}, && \mathbb{B}_{\{1,2\},\{4\}} = \mathbb{B}_{\{1,2\},\{3\}}\sigma_{34},\\
&\mathbb{B}_{\{1\},\{3,4\}} = \mathbb{B}_{\{1\},\{2,3\}}\sigma_{24}, && \mathbb{B}_{\{1,3\},\{4\}} = \mathbb{B}_{\{1,2\},\{3\}}\sigma_{23}\sigma_{34},\\
&\mathbb{B}_{\{2\},\{3,4\}} = \mathbb{B}_{\{1,2\},\{3\}}\sigma_{12}\sigma_{24}, && \mathbb{B}_{\{2,3\},\{4\}} = \mathbb{B}_{\{1,2\},\{3\}}\sigma_{13}\sigma_{34},
\end{aligned} \tag{4.20}$$

with

$$\begin{aligned}
\mathbb{B}_{\{1\},\{2,3\}} &= \left\{(t, t-1, t-1, 2-3t) : \tfrac{1}{2} < t < \tfrac{3}{4}\right\},\\
\mathbb{B}_{\{1,2\},\{3\}} &= \left\{(1-t, 1-t, -t, 3t-2) :\ \tfrac{1}{2} < t < \tfrac{3}{4}\right\}.
\end{aligned} \tag{4.21}$$

If $|I| + |J| = 2$, then $\mathbb{B}_{I,J} = \mathbb{B}_{\{i\},\{j\}}$ is a face and

$$\mathbb{B}^{1,1} = [\mathbb{B}_{\{1\},\{2\}}] \cup [\mathbb{B}_{\{1\},\{3\}}] \cup [\mathbb{B}_{\{1\},\{4\}}] \cup [\mathbb{B}_{\{2\},\{3\}}] \cup [\mathbb{B}_{\{2\},\{4\}}] \cup [\mathbb{B}_{\{3\},\{4\}}].$$

If $|I| + |J| = 3$, then $\mathbb{B}_{I,J}$ is an edge and we have

$$\begin{aligned}
\mathbb{B}^{1,2} &= [\mathbb{B}_{\{1\},\{2,3\}}] \cup [\mathbb{B}_{\{1\},\{2,4\}}] \cup [\mathbb{B}_{\{1\},\{3,4\}}] \cup [\mathbb{B}_{\{2\},\{3,4\}}],\\
\mathbb{B}^{2,1} &= [\mathbb{B}_{\{1,2\},\{3\}}] \cup [\mathbb{B}_{\{1,2\},\{4\}}] \cup [\mathbb{B}_{\{1,3\},\{4\}}] \cup [\mathbb{B}_{\{2,3\},\{4\}}].
\end{aligned} \tag{4.22}$$

If $|I| + |J| = 4$, then

$$\begin{aligned}
&\mathbb{B}^{1,3} = \left[\{(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{3}{4})\}\right], \quad \mathbb{B}^{2,2} = \left[\{(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2})\}\right],\\
&\mathbb{B}^{3,1} = \left[\{(\tfrac{3}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4})\}\right].
\end{aligned} \tag{4.23}$$

Recall that $\mathbb{G}_n$ is $\Lambda_n^{\dagger *} = \mathbb{Z}^3 \cap \overline{\Omega}_B$ in homogeneous coordinates. We now consider the decomposition of the boundary of $\mathbb{G}_n$ according to the boundary elements of the rhombic dodecahedron. First we denote by $\mathbb{G}_n^\circ$ the points inside $\mathbb{G}_n$,

$$\mathbb{G}_n^\circ := \left\{\mathbf{j} \in \mathbb{G}_n : -4n < j_\nu - j_\mu < 4n,\ 1 \le \nu, \mu \le 4\right\} = \left\{\mathbf{j} \in \mathbb{G}_n : \tfrac{\mathbf{j}}{4n} \in \Omega_B^\circ\right\}.$$

We further define, for $0 < i,\ j < i + j \le 4$,

$$\mathbb{G}_n^{i,j} := \left\{\mathbf{k} \in \mathbb{G}_n : \tfrac{\mathbf{k}}{4n} \in \mathbb{B}^{i,j}\right\}. \tag{4.24}$$

The set $\mathbb{G}_n^{i,j}$ describes those points $\mathbf{j}$ in $\mathbb{G}_n$ such that $\frac{\mathbf{j}}{4n}$ are in $B^{i,j}$ of $\partial\Omega_B$. It is easy to see that $\mathbb{G}_n^{i,j} \cap \mathbb{G}_n^{k,l} = \emptyset$ if $i \ne k, j \ne l$ and

$$\bigcup_{0<i,j<i+j\le 4} \mathbb{G}_n^{i,j} = \mathbb{G}_n \setminus \mathbb{G}_n^\circ.$$

4.5. Interpolation by trigonometric polynomials

We first apply the general theory from Section 2 to our set up with Ω_B as a rhombic dodecahedron.

Theorem 4.7. *For $n \ge 1$ define*

$$I_n f(x) := \sum_{k \in X_n} f\left(\tfrac{k}{2n}\right)\Phi_n\left(x - \tfrac{k}{2n}\right), \quad \Phi_n(x) := \frac{1}{2n^3}\sum_{\nu \in \Lambda_n^\dagger} \mathrm{e}_\nu(x). \tag{4.25}$$

Then, for each $j \in X_n$, $I_n f\left(\frac{j}{2n}\right) = f\left(\frac{j}{2n}\right)$.

Proof. By (4.1), $I_n f(\frac{j}{2n}) = f(\frac{j}{2n})$ for $j \in X_n$ is equivalent to $I_n f(B^{-\mathrm{tr}} l) = f(B^{-\mathrm{tr}} l)$ for $l \in \Lambda_n$. Moreover, $I_n f$ can be rewritten as

$$I_n f(x) = \sum_{j \in \Lambda_n} f(B^{-\mathrm{tr}} j)\Phi_n(x - B^{-\mathrm{tr}} j).$$

Hence, this theorem is a special case of Theorem 2.2. ☐

Next we consider interpolation on the symmetric set of points X_n^*. For this we need to modify the kernel function Φ_n. Recall that, under the change of variables (4.13), $\Lambda_n^{\dagger *}$ becomes $\mathbb{G}_n$ in homogeneous coordinates. We define

$$\Phi_n^*(x) := \frac{1}{2n^3}\sum_{\nu \in \Lambda_n^{\dagger *}} \widetilde{\mu}_\nu^{(n)} \mathrm{e}_\nu(x) = \frac{1}{2n^3}\sum_{\mathbf{j} \in \mathbb{G}_n} \mu_{\mathbf{j}}^{(n)} \mathrm{e}_{\mathbf{j}}(\mathbf{t}),$$

where x and $\mathbf{t}$ are related by (4.11). $\widetilde{\mu}_k^{(n)}$ is defined by $\mu_k^{(n)}$ under the change of indices (4.13), and

$$\mu_{\mathbf{j}}^{(n)} = 1 \quad \text{if} \quad \mathbf{j} \in \mathbb{G}_n^\circ, \qquad \mu_{\mathbf{j}}^{(n)} = \frac{1}{\binom{i+j}{i}} \quad \text{if} \quad \mathbf{j} \in \mathbb{G}_n^{i,j}.$$

More explicitly,

$$\mu_{\mathbf{j}}^{(n)} := \begin{cases} 1, & \mathbf{j} \in \mathbb{G}_n^{\circ}, \\ \frac{1}{2}, & \mathbf{j} \in \mathbb{G}_n^{1,1}, \\ \frac{1}{3}, & \mathbf{j} \in \mathbb{G}_n^{1,2} \cup \mathbb{G}_n^{2,1}, \\ \frac{1}{4}, & \mathbf{j} \in \mathbb{G}_n^{1,3} \cup \mathbb{G}_n^{3,1}, \\ \frac{1}{6}, & \mathbf{j} \in \mathbb{G}_n^{2,2}. \end{cases}$$

For each k on the boundary of X_n^*, that is, $\frac{k}{2n}$ on the boundary of $[-\frac{1}{2}, \frac{1}{2}]^3$, let

$$\mathscr{S}_k := \{j \in X_n^* : \tfrac{j}{2n} \equiv \tfrac{k}{2n} \mod \mathbb{Z}^3\}, \tag{4.26}$$

which contains the points on the boundary of X_n^* that are congruent to k under integer translations.

Theorem 4.8. *For $n \geq 1$, define*

$$I_n^* f(x) := \sum_{k \in X_n^*} f\left(\tfrac{k}{2n}\right) R_k(x), \quad R_k(x) := \Phi_n^*\left(x - \tfrac{k}{2n}\right). \tag{4.27}$$

Then, for each $j \in X_n^*$,

$$I_n^* f\left(\tfrac{j}{2n}\right) = \begin{cases} f\left(\tfrac{j}{2n}\right), & j \in X_n^{\circ}, \\ \displaystyle\sum_{k \in S_j} f\left(\tfrac{k}{2n}\right), & j \in X_n^* \setminus X_n^{\circ}. \end{cases} \tag{4.28}$$

In homogeneous coordinates, the function $\Phi_n^*(x) = \widetilde{\Phi}_n^*(\mathbf{t})$ is a real function and it satisfies

$$\begin{aligned} \widetilde{\Phi}_n^*(\mathbf{t}) = \frac{1}{4n^3} \Bigg[& \frac{1}{2}\left(D_n^*(\mathbf{t}) + D_{n-1}^*(\mathbf{t})\right) - \frac{1}{3} \sum_{\nu=1}^{4} \frac{\sin 2\pi \lfloor \frac{n-1}{2} \rfloor t_\nu}{\sin 2\pi t_\nu} \sum_{\substack{j=1 \\ j \neq \nu}}^{4} \cos 2\pi\left(n t_j + \lfloor \tfrac{n}{2} \rfloor t_\nu\right) \\ & - \frac{1}{3} \sum_{1 \leq \mu < \nu \leq 4} \cos 2\pi n (t_\mu + t_\nu) - \frac{1}{2} \begin{cases} \sum_{j=1}^{4} \cos 2\pi n t_j & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \Bigg], \end{aligned} \tag{4.29}$$

from which the formula for $\Phi_n^*(x)$ follows from (4.11) *and* (4.16).

Proof. By (4.1), we need to verify the interpolation at the points $B^{-\mathsf{tr}} l$ for $l \in \Lambda_n^*$. By definition, we can write

$$R_k(B^{-\mathsf{tr}} l) = \frac{1}{2n^3} \sum_{\nu \in \Lambda_n^{\dagger *}} \widetilde{\mu}_\nu^{(n)} \mathrm{e}_\nu\left(B^{-\mathsf{tr}}(l - k)\right).$$

It is easy to see that

$$\nu^{\mathrm{tr}} B^{-\mathrm{tr}} l = \frac{1}{4n}\big(j_1 l_1 + j_2 l_2 + j_3 l_3\big)$$

if ν is related to $\mathbf{j}$ by (4.13). Hence, as in the proof of Theorem 3.15 in [9], we conclude that

$$R_k(B^{-\mathrm{tr}} l) = \frac{1}{2n^3} \sum_{\nu \in \Lambda_n^{\dagger}} \mathrm{e}_\nu\left(B^{-\mathrm{tr}}(l-k)\right),$$

Now, for $l, k \in \Lambda_n^*$, there exist $p \in \Lambda_n$ and $q \in \mathbb{Z}^3$ such that $l - k \equiv p \pm B^{\mathrm{tr}} q$. Consequently, it follows from (2.6) that

$$R_k(B^{-\mathrm{tr}} l) = \frac{1}{2n^3} \sum_{\nu \in \Lambda_n^{\dagger}} \mathrm{e}_\nu(B^{-\mathrm{tr}} p) = \delta_{p,0}.$$

By (4.1), we have verified that

$$R_k\big(\tfrac{j}{2n}\big) = \begin{cases} 1, & \frac{j}{2n} \equiv \frac{k}{2n} \mod \mathbb{Z}^3, \\ 0, & \text{otherwise}, \end{cases} \tag{4.30}$$

which proves the interpolation part of the theorem.

In order to prove the compact formula, we start with the following formula that can be established exactly as in the proof of Theorem 3.15 in [9]:

$$\begin{aligned} \widetilde{\Phi}_n^*(\mathbf{t}) = \frac{1}{4n^3}\Bigg[& \frac{1}{2}\big(D_n^*(\mathbf{t}) + D_{n-1}^*(\mathbf{t})\big) - \frac{1}{6} \sum_{k \in \mathbb{G}_n^{1,2} \cup \mathbb{G}_n^{2,1}} \phi_{\mathbf{k}}(\mathbf{t}) \\ & - \frac{1}{4} \sum_{k \in \mathbb{G}_n^{1,3} \cup \mathbb{G}_n^{3,1}} \phi_{\mathbf{k}}(\mathbf{t}) - \frac{1}{3} \sum_{k \in \mathbb{G}_n^{2,2}} \phi_{\mathbf{k}}(\mathbf{t}) \Bigg]. \end{aligned} \tag{4.31}$$

Let us define $\mathbb{G}_n^{I,J} := \{\mathbf{k} \in \mathbb{G}_n : \frac{\mathbf{k}}{4n} \in \mathbb{B}_{I,J}\}$ for $I, J \subset \mathbb{N}_4$ and also define

$$\left[\mathbb{G}_n^{I,J}\right] := \{\mathbf{k} \in \mathbb{G}_n : \tfrac{\mathbf{k}}{4n} \in [\mathbb{B}_{I,J}]\}.$$

It follows from (4.18), and (4.24) that

$$\mathbb{G}_n^{i,j} = \bigcup_{I,J \in \mathscr{K}_0^{i,j}} \left[\mathbb{G}_n^{I,J}\right], \qquad \left[\mathbb{G}_n^{I,J}\right] = \bigcup_{\sigma \in \mathscr{G}_{I \cup J}} \mathbb{G}_n^{I,J} \sigma.$$

In order to compute the sums in (4.31), we need to use the detail description of the boundary elements of Ω_B in the previous subsection. The computation is parallel to the proof of Theorem 3.15 in [9], in which the similar computation with $\mathbb{G}_n$ replaced by $\mathbb{H}_n$ is carried out. Thus, we shall be brief.

Using $t_1+t_2+t_3+t_4=0$ and the explicit description of $\mathbb{B}^{\{1\},\{2,3\}}$, we get

$$\begin{aligned}\sum_{\mathbf{k}\in[\mathbb{G}_n^{\{1\},\{2,3\}}]}\phi_{\mathbf{k}}(\mathbf{t}) &= \sum_{\mathbf{k}\in\mathbb{G}_n^{\{1\},\{2,3\}}} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}} + \sum_{\mathbf{k}\in\mathbb{G}_n^{\{2\},\{1,3\}}} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}} + \sum_{\mathbf{k}\in\mathbb{G}_n^{\{3\},\{1,2\}}} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}}\\
&= \sum_{j=1, j\text{even}}^{n-1} \mathrm{e}^{-2\pi i j t_4}\left(\mathrm{e}^{2n\pi i(t_1+t_4)}+\mathrm{e}^{2n\pi i(t_2+t_4)}+\mathrm{e}^{2n\pi i(t_3+t_4)}\right)\\
&=\frac{\sin 2\pi\lfloor\frac{n-1}{2}\rfloor t_4}{\sin 2\pi t_4}\mathrm{e}^{-2\pi i\lfloor\frac{n+1}{2}\rfloor t_4}\left(\mathrm{e}^{2\pi i n(t_1+t_4)}+\mathrm{e}^{2\pi i n(t_2+t_4)}+\mathrm{e}^{2\pi i n(t_3+t_4)}\right),\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\sum_{\mathbf{k}\in[\mathbb{G}_n^{\{1,2\},\{3\}}]}\phi_{\mathbf{k}}(\mathbf{t}) &= \sum_{\mathbf{k}\in\mathbb{G}_n^{\{1\},\{2,3\}}} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}} + \sum_{\mathbf{k}\in\mathbb{G}_n^{\{2\},\{1,3\}}} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}} + \sum_{\mathbf{k}\in\mathbb{G}_n^{\{3\},\{1,2\}}} \mathrm{e}^{\frac{\pi i}{2}\mathbf{k}\cdot\mathbf{t}}\\
&=\frac{\sin 2\pi\lfloor\frac{n-1}{2}\rfloor t_4}{\sin 2\pi t_4}\mathrm{e}^{2\pi i\lfloor\frac{n+1}{2}\rfloor t_4}\left(\mathrm{e}^{-2\pi i n(t_1+t_4)}+\mathrm{e}^{-2\pi i n(t_2+t_4)}+\mathrm{e}^{-2\pi i n(t_3+t_4)}\right).\end{aligned}$$

From these and their permutations, we can compute the sum over $\mathbb{G}_n^{1,2}$ and $\mathbb{G}_n^{2,1}$. Putting them together, we obtain

$$\sum_{\mathbf{k}\in\mathbb{G}_n^{1,2}\cup\mathbb{G}_n^{2,1}}\phi_k(\mathbf{t}) = 2\sum_{\nu=1}^{4}\frac{\sin 2\pi\lfloor\frac{n-1}{2}\rfloor t_\nu}{\sin 2\pi t_\nu}\sum_{\substack{j=1\\ j\neq\nu}}^{4}\cos 2\pi\left(nt_j+\lfloor\tfrac{n}{2}\rfloor t_\nu\right).$$

Using (4.23), we see that, $\mathbb{G}_n^{2,2}=\{(2n,2n,-2n,-2n)\sigma : \sigma\in\mathcal{G}\}$ and, if n is even, then $\mathbb{G}_n^{1,3}=\{(n,n,n,-3n)\sigma : \sigma\in\mathcal{G}\}$ and $\mathbb{G}_n^{3,1}=\{(3n,-n,-n,-n)\sigma : \sigma\in\mathcal{G}\}$, whereas if n is odd, then $\mathbb{G}_n^{1.3}=\mathbb{G}_n^{3,1}=\emptyset$. As a result, it follows that

$$\sum_{\mathbf{k}\in\mathbb{G}_n^{2,2}}\phi_{\mathbf{k}}(\mathbf{t}) = \sum_{1\le\mu<\nu\le4} e^{2\pi i n(t_\mu+t_\nu)} = \sum_{1\le\mu<\nu\le4}\cos 2\pi n(t_\mu+t_\nu),$$

where we have used the fact that $t_1+t_2+t_3+t_4=0$, and

$$\sum_{\mathbf{k}\in\mathbb{G}_n^{1,3}\cup\mathbb{G}_n^{3,1}}\phi_k(\mathbf{t}) = \sum_{j=1}^{4}\left(e^{2\pi i n t_j}+e^{-2\pi i n t_j}\right) = 2\sum_{j=1}^{4}\cos 2\pi n t_j,$$

if n is even while it is equal to 0 if n is odd.

Putting all these into (4.31) completes the proof. □

Theorem 4.9. *Let $\|I_n^*\|_\infty$ denote the norm of the operator $I_n^* : C\left([-\frac{1}{2},\frac{1}{2}]^3\right)\mapsto C\left([-\frac{1}{2},\frac{1}{2}]^3\right)$. Then, there is a constant c, independent of n, such that*

$$\|I_n^*\|_\infty \le c(\log n)^3.$$

Proof. Following the standard procedure, we see that

$$\|I_n^*\|_\infty = \max_{x\in[-\frac12,\frac12]^3} \sum_{k\in X_n^*} \left|\Phi_n^*(x-\tfrac{k}{4n})\right|.$$

Using the formula of Φ_n^* in (4.29), it is easy to see that it suffices to prove that

$$\max_{x\in[-\frac12,\frac12]^3} \sum_{k\in X_n^*} \left|D_n^*(x-\tfrac{k}{2n})\right| \le c(\log n)^3, \quad n\ge 0.$$

Furthermore, using the explicit formula of $D_n^{\mathrm{odd}}(\mathbf{t})$ and (3.19) in [9], we see that our main task is to estimate the sums in the form of

$$I_{\{1,2,3\}} := \frac{1}{2n^3} \max_{\mathbf{t}\in Q} \sum_{k\in X_n^*} \left| \frac{\sin \pi n(t_1-\frac{k_2+k_3}{2n}) \sin \pi n(t_2-\frac{k_1+k_3}{2n}) \sin \pi n(t_3-\frac{k_1+k_2}{2n})}{\sin \pi(t_1-\frac{k_2+k_3}{2n}) \sin \pi(t_2-\frac{k_1+k_3}{2n}) \sin \pi(t_3-\frac{k_1+k_2}{2n})} \right|$$

and three other similar estimates $I_{\{1,2,4\}}$, $I_{\{1,3,4\}}$ and $I_{\{2,3,4\}}$, respectively, as well as similar sums in which the denominator becomes product of $\sin 2\pi(t_i-\frac{k_i}{2n})$ and n in the numerator is replace by $n+1$ or $n+2$. Here Q is the image of $[-1,1]^3$ under the mapping (4.11); that is,

$$Q = \left\{\mathbf{t}\in\mathbb{R}_H^4 : -\tfrac12 \le t_1+t_2, t_2+t_3, t_3+t_1 \le \tfrac12\right\}.$$

Changing the summation indices and enlarging the set X_n^*, we see that

$$I_{\{1,2,3\}} \le 4 \max_{t\in[-1,1]} \left(\frac{1}{2n} \sum_{k=0}^{2n} \left| \frac{\sin n\pi\left(t-\frac{k}{2n}\right)}{\sin \pi\left(t-\frac{k}{2n}\right)} \right| \right)^3 \le c(\log n)^3,$$

where the last step follows from the standard estimate of one variable (cf. [20, Vol. II, p. 19]). □

4.6. Interpolation by algebraic polynomials

The main outcome of Theorem 4.7 in the previous section is that we can derive a genuine interpolation by trigonometric polynomials based on the set of points in $\{\frac{k}{2n} : k\in \Xi_n\}$ defined at (4.6). The development below is similar to the case of $d=2$. We define

$$\mathscr{P} f(x) := \frac18 \sum_{\varepsilon\in\{-1,1\}^3} f(\varepsilon_1 x_1, \varepsilon_2 x_2, \varepsilon_3 x_3).$$

Theorem 4.10. *For $n\ge 0$ define*

$$\mathscr{L}_n f(x) = \sum_{k\in\Xi_n} f\left(\tfrac{k}{2n}\right)\ell_k(x), \quad \ell_k(x) := \lambda_k^{(n)} \mathscr{P}\left[\Phi_n^*(\cdot - \tfrac{k}{2n})\right](x),$$

with $\lambda_k^{(n)}$ given in (4.9). Then, $\mathscr{L}_n f \in \mathscr{T}_n$ is even in each of its variables and it satisfies

$$\mathscr{L}_n f\left(\tfrac{j}{2n}\right) = f\left(\tfrac{j}{2n}\right), \quad \forall j\in\Xi_n.$$

Proof. As shown in (4.30), $R_k(x) := \Phi_n^*\big(x - \frac{k}{2n}\big)$ satisfies $R_k\big(\frac{j}{2n}\big) = 1$ when $k \equiv j$ mod $2n\mathbb{Z}^3$ and 0 otherwise. Hence, if $j \in \Xi_n^\circ$, then

$$(\mathscr{P}R_k)\big(\tfrac{j}{2n}\big) = \frac{1}{8}R_k\big(\tfrac{j}{2n}\big) = [\lambda_k^{(n)}]^{-1}\delta_{k,j}.$$

If $j \in \Xi_n^* \backslash \Xi_n^\circ$, then we need to consider several cases, depending on how many components of $\mathbf{j}$ are zero, which determines how many distinct terms are in the sum $(\mathscr{P}R_k)\big(\frac{j}{2n}\big)$ and how many distinct k can be obtained from j by congruent in $\mathbb{Z}^3$. For example, if $j \in \Xi_n^f$ and none of the components of j are zero, then there are 2 elements in $\mathscr{S}_j$, j and the one in the opposite face, and the sum $\mathscr{P}R_k\big(\frac{j}{2n}\big)$ contains 8 terms, so that

$$(\mathscr{P}R_k)\big(\tfrac{j}{2n}\big) = \frac{1}{4}\delta_{j,k} = [\lambda_k^{(n)}]^{-1}\delta_{k,j}.$$

The other cases can be verified similarly, just as in the case of $d = 2$. We omit the details. $\square$

The above theorem yields immediately interpolation by algebraic polynomials upon applying the change of variables (4.5). Recall Γ_n defined in (4.7) and the polynomial subspace

$$\Pi_n^* = \mathrm{span}\{s_1^{k_1}s_2^{k_2}s_3^{k_3} : k_1, k_2, k_3 \ge 0, k_i + k_j \le n,\ \ 1 \le i,\ \ j \le 3\}.$$

Theorem 4.11. *For $n \ge 0$, let*

$$\mathscr{L}_n f(s) = \sum_{z_k \in \Gamma_n} f(z_k)\ell_k^*(s), \quad \ell_k^*(s) = \ell_k(x) \ \textit{with}\ s = \cos 2\pi x.$$

Then, $\mathscr{L}_n f \in \Pi_n^*$ and it satisfies $\mathscr{L}_n f(z_k) = f(z_k)$ for all $z_k \in \Gamma_n$.

This theorem follows immediately from the change of variables (4.5). The explicit compact formula of $\ell_k(x)$, thus $\ell_k^*(s)$, can be derived from Theorem 4.8.

The theorem states that the interpolation space for the point set Γ_n is exactly Π_n^*, which consists of monomials that have indices in the positive quadrant of the rhombic dodecahedron, as depicted in Fig. 3.

The set Γ_n consists of roughly $n^3/4(1 + \mathscr{O}(n^{-1})$ points. The interpolation polynomial $\mathscr{L}_n f \in \Pi_n^*$ is about a total degree of $3n/2$. The compact formula of the fundamental interpolation polynomial provides a convenient way of evaluating the interpolation polynomial. Furthermore, the Lebesgue constant of this interpolation process remains at the order of $(\log n)^3$, as the consequence of Theorem 4.9 and the change of variables.

Corollary 4.12. *Let $\|\mathscr{L}_n\|_\infty$ denote the operator norm of $\mathscr{L}_n : C([-1,1]^3) \mapsto C([-1,1]^3)$. Then, there is a constant c, independent of n, such that*

$$\|\mathscr{L}_n\|_\infty \le c(\log n)^3.$$

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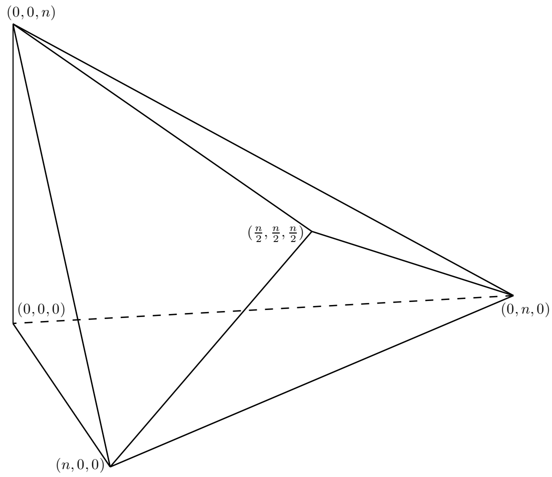

Figure 3: Index set of Π_n^*.

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