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Standard and Economical Cascadic Multigrid Methods for the Mortar Finite Element Methods

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Abstract. In this paper, standard and economical cascadic multigrid methods are considered for solving the algebraic systems resulting from the mortar finite element methods. Both cascadic multigrid methods do not need full elliptic regularity, so they can be used to tackle more general elliptic problems. Numerical experiments are reported to support our theory.

AMS subject classifications: 65F10, 65N30
Key words: Cascadic multigrid, mortar finite elements.

1. Introduction

Cascadic multigrid (CMG) [8, 9] is a type of multigrid methods which requires no coarse grid corrections at all that may be viewed as a "one way" multigrid. The main advantage of the cascadic multigrid method is its simplicity. Numerical experiments [9, 10] show that this method is very effective. Meanwhile, it has been proved [8,23] that the cascadic multigrid which uses the P1 conforming element for second-order elliptic problem in 3-D is accurate with optimal computational complexity for all one-step conventional iterative methods, like the weighted Jacobi, Gauss-Seidel and Richardson iteration as well as for the conjugate gradient method as a smoother. However, in 2-D case, the cascadic multigrid gives accurate solution with optimal computational complexity for the conjugate gradient method, but only nearly optimal complexity for other conventional iterative smoothers. In recent years, there have been several theoretical analysis and applications of these methods, cf. [24, 30] for nonconforming element methods and plate bending problems, [25] for parabolic problems, [21, 31] for nonlinear problems, [12] for Stokes problems, [14] for mortar element methods, [27] for the finite volume methods.

Recently, we proposed in [28] a new type of cascadic multigrid method. Compared with the standard cascadic multigrid method developed by Bornemann and Deuflhard

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[8,9], the new one requires less iterations on each level, especially on coarser grids. Many operations can be reduced in the new cascadic multigrid algorithms. So we call it an economical cascadic multigrid method (ECMG). It is proved that the new cascadic multigrid algorithm is still as optimal as the standard cascadic multigrid algorithm in both accuracy and computational complexity.

The mortar finite element method as a special domain decomposition methodology appears very attractive because it can handle situations where meshes on different subdomains need not align across interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. In [7], Bernardi, Maday and Patera first introduced basic concepts of general mortar element methods, including the coupling of spectral elements with finite elements. Then it was extensively used and analyzed by many authors. In [5], Belgacem studied the mortar element method under a primal hybrid finite element formulation. Meanwhile, some extensions and convergence results in three dimensions were considered in [4,6]. Recently, many works have been done in constructing efficient iterative solvers for the discrete system resulting from the mortar element method. The first approaches were based on iterative substructuring methods, see [1–3,18]. Multigrid methods for the mortar element methods have also been considered. Gopalakrishnan and Pasciak [20] presented a variable V-cycle preconditioner, while Braess, Dahmen and Wieners [13] and Wohlmuth [29] established a W-cycle multigrid based on a hybrid formulation which gives rise to a saddle point problem.

The objective of this paper is to design efficient cascadic multigrid (CMG) solvers for the mortar finite element methods. Note that Braess, Deuflhard and Lipnikov [14] constructed a subspace cascadic multigrid method for the mortar element method based on a saddle point formulation. Moreover, the authors only considered second-order elliptic problems with full regularity. In this paper, we will treat the mortar element method under the framework of nonconforming methods and assume that the Lagrange multiplier has been eliminated. We will construct the standard and economical cascadic multigrid methods for solving the algebraic system resulting from such kind of mortar finite element method and then give their convergence analysis for more general second elliptic problems without full regularity such as L-shape domain problems.

This paper is organized as follows: Section 2 introduces the mortar element method developed by Bernardi, Maday and Patera in [7]. In Section 3, we give the standard cascadic multigrid method and its convergence analysis. In Section 4, we introduce the economical cascadic multigrid method. Finally, numerical results that confirm our theory will be given in the Section 5.

2. The mortar element method

The mortar finite element method allows the coupling of different discretizations across subdomain boundaries. The idea of the mortar finite element method is to weakly impose the transmission conditions across the interface of difference subdomains by means of Lagrange multiplier. The key argument is to construct a suitable discrete Lagrange multiplier space in order to ensure the stability of the discrete problem. It is interesting to design some efficient iterative solvers for the arising systems from the mortar finite element method. In this paper, we will construct standard and economical cascadic multigrid methods for solving the algebraic systems. Cascadic multigrid is a new type of multigrid methods which requires no coarse correction at all that may be viewed as a "one way" multigrid, while the economical cascadic multigrid needs less iterations on each level, especially on coarser levels.

Throughout this paper, we adopt standard notations from Lebesgue and Sobolev space theory (cf., e.g., [17]). In particular, we refer to $(\cdot,\cdot)_0$ and $\|\cdot\|_0$ as the inner product and norm on $L^2(\Omega)$ and to $\|\cdot\|_\alpha$ as the norm on the Sobolev space $H^\alpha(\Omega)$, $\alpha\in[1,2]$. Let $\Omega\subset\mathbb{R}^d, d=2,3$ be a bounded polygonal domain. We consider the following elliptic Dirichlet problem:

$$\begin{cases} -\mathrm{div}(a(x)\nabla u)=f & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $a(x)$ is a sufficiently smooth and uniformly positive definite matrix in $\Omega\subset\mathbb{R}^d$.

The variational form of (2.1) is to find $u\in H_0^1(\Omega)$ such that

$$a(u,v)=(f,v)\quad \forall v\in H_0^1(\Omega), \tag{2.2}$$

where the bilinear form

$$a(u,v)=\int_\Omega a(x)\nabla u\cdot\nabla v dx \quad \forall u,v\in H^1(\Omega).$$

It is known that the problem (2.2) has a unique solution.

We assume some regularity for the solution of problem (2.1), i.e.,

(A1). There exists an $\alpha\in(\frac{1}{2},1]$ such that

$$\|u\|_{1+\alpha}\le C\|f\|_{\alpha-1}.$$

Remark 2.1. It is known that (A1) holds for a wide class of domains [19]. Note that we do not require the full regularity ($\alpha=1$) as in [14].

We now divide Ω into nonoverlapping polygonal subdomains such that

$$\overline{\Omega}=\bigcup_{i=1}^N\overline{\Omega}_i \quad \text{and} \quad \Omega_i\cap\Omega_j=, \quad i\neq j.$$

They are arranged so that the intersection of $\Omega_i\cap\Omega_j$ for $i\neq j$ is either an empty set, an edge or a vertex, i.e., the partition is geometrically conforming. The interface

$$\Gamma=\bigcup_{i=1}^N\partial\Omega_i\backslash\partial\Omega$$

is broken into a set of disjoint open straight segments γ_m $(1\le m\le M)$ (that are the edges of subdomains) called mortars, i.e.,

$$\Gamma=\bigcup_{m=1}^{M}\bar{\gamma}_m,\quad \gamma_m\cap\gamma_n=,\quad \text{if } m\neq n.$$

We denote the common open edge to Ω_i and Ω_j by γ_m. By $\gamma_{m(i)}$ we denote an edge of Ω_i called mortar and by $\delta_{m(j)}$ an edge of Ω_j that geometrically occupies the same place called nonmortar.

Let $\mathscr{T}_1^i$ be the coarsest triangulation of Ω_i with the mesh size h_1. The triangulation generally does not align at the subdomain interface. Denote the global mesh $\cup_i\mathscr{T}_1^i$ by $\mathscr{T}_1$. We refine the triangulation $\mathscr{T}_1$ to produce $\mathscr{T}_2$ by joining all mid-points of edges of triangles in $\mathscr{T}_1$. Obviously, the mesh size h_2 in $\mathscr{T}_2$ is $h_2=h_1/2$. Repeating this process, we get the l-time refined triangulation $\mathscr{T}_l$ with the mesh size $h_l=h_1 2^{-l+1}$ $(l=1,\cdots,L)$.

In the following, we define the mortar finite element space as in [7,20]. First, define the space Z as follows:

$$Z=\Big\{v|\ v|_{\Omega_i}\in H^1(\Omega_i),\ \ \forall i=1,\cdots,N,\ v=0 \text{ on } \partial\Omega\Big\}. \tag{2.3}$$

On each level l, the P1 linear continuous finite element space over the triangulation $\mathscr{T}_l^i$ is denoted by $V_{l,i}$, whose functions have zero trace on $\partial\Omega$. Let

$$\tilde{V}_l=\prod_{i=1}^{N}V_{l,i},\quad l=1,\cdots,L. \tag{2.4}$$

Obviously,

$$\tilde{V}_1\subseteq\cdots\subseteq\tilde{V}_L.$$

For any interface

$$\gamma_m=\gamma_{m(i)}=\delta_{m(j)},\quad 1\le m\le M,$$

there are two different and independent 1D triangulations $\mathscr{T}_l(\gamma_{m(i)})$ and $\mathscr{T}_l(\delta_{m(j)})$, which are the restriction of triangulations $\mathscr{T}_l^i$ and $\mathscr{T}_l^j$ on γ_m. Let $M_l(\gamma_{m(i)})$ and $M_l(\delta_{m(j)})$ be the piecewise continuous linear function space corresponding to the triangulation $\mathscr{T}_l(\gamma_{m(i)})$ and $\mathscr{T}_l(\delta_{m(j)})$, respectively. In addition, we define an auxiliary test space $S_l(\delta_{m(j)})$ as a subspace of the nonmortar space $M_l(\delta_{m(j)})$ such that its functions are constants on elements that intersect the ends of $\delta_{m(j)}$. Based on the above preparation, we can now define the following mortar finite element space

$$V_l=\Big\{v_l\subset\tilde{V}_l|\forall\delta_{m(j)}\subset\Gamma,\ \int_{\delta_{m(j)}}(v_{l,i}-v_{l,j})\varphi ds=0,\ \forall\varphi\in S_l(\delta_{m(j)})\Big\}. \tag{2.5}$$

Note that though the space $\{\tilde{V}_l\}$ is nested, the multilevel space $\{V_l\}$ is generally nonnested. Define

$$
\begin{aligned}
&|v|_{l,i} \hat{=} \sum_{K\in\mathcal{T}_l^i} \int_K \nabla v \cdot \nabla v dx, \quad \|v\|_l \hat{=} \sum_{i=1}^N |v|_{l,i}^2.\\
&a_{l,i}(u,v) \hat{=} \sum_{K\in\mathcal{T}_l^i} \int_K (a(x)\nabla u)\cdot \nabla v dx, \quad \forall u,v \in V_{l,i},\\
&a_l(u,v) \hat{=} \sum_{i=1}^N a_{l,i}(u,v).
\end{aligned}
$$

Then the mortar element approximation of the problem (2.2) is to find $u_l \in V_l$ such that

$$
a_l(u_l, v_l) = f(v_l), \quad \forall v_l \in V_l, \tag{2.6}
$$

where

$$
f(v_l) = \sum_{i=1}^N \int_{\Omega_i} f v_l ds.
$$

It is shown in [7] that (2.6) has a unique solution.

The following theorem can be found in [7].

Theorem 2.1. *Let* u, u_l *be the solution of* (2.2), (2.6) *respectively. Then*

$$
\|u - u_l\|_l \le C h_l^\alpha \|u\|_{1+\alpha}. \tag{2.7}
$$

3. Cascadic multigrid algorithm and its convergence analysis

In this section, we will give the convergence analysis of the cascadic multigrid method. It is shown that the cascadic multigrid method is an optimal methods, which means that we obtain both the accuracy

$$
\|u_L - u_L^*\|_L \approx |u - u_L|_L
$$

and the multigrid complexity

$$
\text{amount of work} = \mathcal{O}(n_L), \quad n_L = dimV_L.
$$

Due to the nonnestedness of the mesh spaces, we first introduce an intergrid transfer operator which was constructed in [20] for the standard multigrid method.

Define the space $W_l(\delta_{m(j)})$ by

$$
\begin{aligned}
W_l(\delta_{m(j)}) = \{v|\ v \text{ is a linear continuous function on } \delta_{m(j)}, \text{ and } v \text{ is}\\
\text{vanishing at endpoints of } \delta_{m(j)}\}.
\end{aligned}
$$

Accordingly, define a projection operator $\pi_{l,m} : L^2(\Omega) \to W_l(\delta_{m(j)})$ by [7, 20]:

$$\int_{\delta_{m(j)}} (\pi_{l,m} v)\chi ds = \int_{\delta_{m(j)}} u\chi ds, \quad \forall \chi \in S_l(\delta_{m(j)}). \tag{3.1}$$

This projection is known to be stable in $L^2(\gamma_m)$ and $H_{00}^{1/2}(\gamma_m)$ [7, 13], i.e.,

$$\|\pi_{l,m} v\|_{0,\gamma_m} \le C\|v\|_{0,\gamma_m},$$
$$\|\pi_{l,m} v\|_{H_{00}^{1/2}(\gamma_m)} \le C\|v\|_{H_{00}^{1/2}(\gamma_m)}.$$

Let $\{y_l^i\}$ denote the nodes of $\delta_{m(j)}$ and the operator $\Xi_{l,\delta_{m(j)}} : X \to \tilde{V}_l$ is defined by

$$(\Xi_{l,\delta_{m(j)}}(v))(y_l^i) = \begin{cases} (\pi_{l,m}(v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}}))(y_l^i) & y_l^i \in \delta_{m(j)}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we can give the intergrid transfer operator I_l for the nonnested space V_l ($l = 1, \cdots, L$) as follows [20]:

$$I_l v = v + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(v), \quad \forall v \in V_{l-1}, \tag{3.2}$$

Remark 3.1. In [15], Braess, Dryja and Hackbusch presented another intergrid transfer operator for the space V_l.

Remark 3.2. For 3D problems, we can also define corresponding mortar element spaces V_l and intergrid transfer operators, see [6, 11, 14] for details.

Lemma 3.1. *For the operator I_l defined in* (3.2)*, we have*

$$(1).\ \|I_l v\|_l \le C\|v\|_{l-1},$$
$$(2).\ \|v - I_l v\|_0 \le Ch_l\|v\|_{l-1}.$$

Proof. Please refer to [20] for a proof. $\square$

Moreover, based on the operator $\Xi_{l,\delta_{m(j)}}$, we give an approximation function for any $\xi \in H^2(\Omega) \cap H_0^1(\Omega)$ in the space V_l:

$$\Pi_l \xi = \tilde{C}_l \xi + \sum_{m=1}^{M} \Xi_{l,\delta_{m(j)}}(\tilde{C}_l \xi). \tag{3.3}$$

It is easy to see that $\Pi_l \xi \in V_l$.

Lemma 3.2. *For the operator Π_l, we have*

$$\|\xi - \Pi_l \xi\|_0 + h_l\|\xi - \Pi_l \xi\|_l \le Ch_l^{1+\alpha}\|\xi\|_{1+\alpha} \quad \forall \xi \in H_0^1(\Omega) \cap H^{1+\alpha}(\Omega).$$

Proof. The proof follows by a similar argument as in Lemma 4.2 in [32]. □

For the following convergence analysis, we define a mesh dependent norm over the space V_l, i.e.,

$$|||v|||_{s,l} \widehat{=} (A_l^s v, v) \quad \forall v \in V_l, \;\; s \in \mathbb{R}, \tag{3.4}$$

where the operator A_l is given by

$$(A_l u, v) = a_l(u, v) \quad \forall u, v \in V_l.$$

It is easy to check that $|||v|||_{1,l} = a_l^{\frac{1}{2}}(v, v)$, $|||v|||_{0,l} = \|v\|_0$, and

$$|||v|||_{s,l} \le C h^{t-s} |||v|||_{t,l} \quad (t < s), \tag{3.5}$$

$$(u, v) \le |||u|||_{s,l} |||v|||_{-s,l} \quad \forall u, v \in V_l, \; s \in \mathbb{R}. \tag{3.6}$$

Lemma 3.3. *Let $s \in [0, \frac{1}{2})$ or $(\frac{1}{2}, 1]$, for any $v_l \in V_l$. It holds that*

$$\|v_l\|_{-s} \le C |||v|||_{-s,l}.$$

Proof. Let Q_l be the L^2 projection operator from $L^2(\Omega)$ to V_l, i.e.,

$$(Q_l v, w) = (v, w) \quad \forall v \in L^2(\Omega), \;\; w \in V_l,$$

where $(\cdot, \cdot)$ denotes the usual inner product over the space $L^2(\Omega)$. It is easy to see that

$$\|Q_l v\|_0 \le \|v\|_0, \tag{3.7}$$

$$\|v - Q_l v\|_0 \le 2\|v\|_0, \quad \forall v \in V_l. \tag{3.8}$$

By Lemma 3.2 and the property of the projection Q_l, we can derive that

$$\|v - Q_l v\|_0 \le C h_l^{1+\alpha} \|v\|_{1+\alpha}, \quad \forall v \in H^{1+\alpha}(\Omega), \;\; \alpha \in (0.5, 1]. \tag{3.9}$$

Using the interpolation [16] between (3.8) and (3.9), we get

$$\|v - Q_l v\|_0 \le C h_l \|v\|_1, \quad \forall v \in H^1(\Omega). \tag{3.10}$$

For each K on Γ_l, define $Q_K : L^2(K) \to P_1(K)$ being the $L^2(K)$ orthogonal projection, here $P_1(K)$ is the piecewise polynomial space on K, then it is easy to show that

$$\sum_{i=0}^{1} h_l^i |v - Q_K v|_{i,K} \le C h_l |v|_{1,K}, \quad \forall v \in H^1(K). \tag{3.11}$$

Using (3.1), (3.11), then for any $v \in H_0^1(\Omega)$, we have

$$\begin{aligned}
\|Q_l v\|_{1,l}^2 &= C \sum_{K\in\Gamma_l} |Q_l v|_{1,K}^2 \\
&\le C \sum_{K\in\Gamma_l} \Big(|Q_l v - Q_K v|_{1,K}^2 + |Q_K v|_{1,K}^2 \Big) \\
&\le C \sum_{K\in\Gamma_l} \Big(C h_l^{-2} \|Q_l v - Q_K v\|_{0,K}^2 + |Q_K v|_{1,K}^2 \Big) \\
&\le C \sum_{K\in\Gamma_l} \Big[C h_l^{-2} (\|Q_l v - v\|_{0,K}^2 + \|v - Q_K v\|_{0,K}^2) + |Q_K v|_{1,K}^2 \Big] \\
&\le C \left[h_l^{-2} (\|Q_l v - v\|_0^2 + h_l^2 \sum_{K\in\Gamma_l} |v|_{1,K}^2) + \|v\|_1^2 \right] \\
&\le C \|v\|_1^2. \qquad (3.12)
\end{aligned}$$

An application of the interpolation between (3.7) and (3.12) yields

$$|||Q_l v|||_{s,l} \le C \|v\|_s \quad s \in [0, 0.5) \text{ or } (0.5, 1]. \qquad (3.13)$$

By the definition of negative norm in Sobolev space, we have

$$\|v_l\|_{-s} = \sup_{w\in H_0^1(\Omega)} \frac{(v_l, w)}{\|w\|_s}.$$

On the other hand, it follows from (3.6) and (3.13) that

$$\begin{aligned}
(v_l, w) = (v_l, Q_l w) &\le |||v_l|||_{-s,l} |||Q_l v|||_{s,l} \\
&\le C |||v_l|||_{-s,l} \|w\|_s.
\end{aligned}$$

Combining the above two inequalities gives the proof. □

Next, we choose an iterative operator $C_l : V_l \to V_l$ on the level l and assume that there exists a linear operator $T_l : V_l \to V_l$ such that

$$u_l - C_l^{m_l} u_l^0 = T_l^{m_l} (u_l - u_l^0),$$

and

(H1): the following assumptions:

$$\begin{aligned}
&(1).\ \ \|T_l^{m_l} v\|_l \le C \frac{h_l^{-\alpha}}{m_l^{\alpha\gamma}} |||v|||_{1-\alpha,l} \quad && \forall v \in V_l, \\
&(2).\ \ \|T_l^{m_l} v\|_l \le \|v\|_l && \forall v \in V_l,
\end{aligned}$$

where m_l is the number of iteration steps on the level l, and γ is a positive number depending on the given iteration.

Based on the smoothing operator and intergrid transfer operator I_l, we can write the cascadic algorithm for the mortar finite element method as follows:

Cascadic Multigrid Algorithm (CMG)

(1) Set $u_1^1 = u_1^* \widehat{=} u_1$ and let $(l = 2, \cdots, L)$

$$u_l^1 = I_l u_{l-1}^*. \tag{3.14}$$

(2) For $l = 2, \cdots, L$, do iterations:

$$u_l^{m_l} = C_l^{m_l} u_l^1. \tag{3.15}$$

(3) Set $u_l^* \widehat{=} u_l^{m_l}$.

Following [8], we call a cascadic multigrid optimal in the energy norm on the level L, if we obtain both the accuracy

$$\|u_L - u_L^*\|_L \approx \|u - u_L\|_L \tag{3.16}$$

which means that the iterative error is comparable to the approximation error, and the multigrid complexity

$$\text{amount of work} = \mathcal{O}(n_L), \quad n_L = dim V_L.$$

For obtaining the convergence rate of the above CMG method, we first prove the following lemma.

Lemma 3.4. *Let u, u_l be the solution of equations* (2.2), (2.6) *respectively. Then*

$$|||\Pi_l u - u_l|||_{1-\alpha,l} \leq C h_l^{2\alpha} \|u\|_{1+\alpha}.$$

Proof. Define

$$g_l = A_l^{1-\alpha}(\Pi_l u - u_l).$$

Consider the following auxiliary problems

$$a(\xi, v) = (g_l, v) \quad \forall v \in H_0^1(\Omega).$$
$$a_l(\xi_l, v) = (g_l, v) \quad \forall v \in V_l.$$

We have

$$\|\xi - \xi_l\|_l \leq C h_l^{\alpha} \|\xi\|_{1+\alpha} \quad \forall \xi \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega). \tag{3.17}$$

Using the regularity assumption (A1) and Lemma 3.3, we get

$$\|\xi\|_{1+\alpha} \leq C\|g_l\|_{\alpha-1} \leq C|||g_l|||_{\alpha-1,l}.$$

By the definition of the mesh-dependent norm $|||\cdot|||_{s,l}$, it follows that

$$\begin{aligned}|||g_l|||^2_{\alpha-1,l} &= (A_l^{\alpha-1} g_l, g_l)\\ &= (A_l^{\alpha-1} A_l^{1-\alpha}(\Pi_l u - u_l), A_l^{1-\alpha}(\Pi_l u - u_l))\\ &= (A_l^{1-\alpha}(\Pi_l u - u_l), \Pi_l u - u_l)\\ &= |||\Pi_l u - u_l|||_{1-\alpha,l}.\end{aligned}$$

Combining above two inequalities, we obtain

$$\|\xi\|_{1+\alpha} \le C|||\Pi_l u - u_l|||_{1-\alpha,l}. \tag{3.18}$$

On the other hand,

$$\begin{aligned}|||\Pi_l u - u_l|||^2_{1-\alpha,l} &= (A_l^{1-\alpha}(\Pi_l u - u_l), \Pi_l u - u_l)\\ &= (g_l, \Pi_l u - u_l) = a_l(\xi_l, \Pi_l u - u_l)\\ &= a_l(\xi_l, \Pi_l u - u) + a_l(\xi_l, u - u_l)\\ &= a_l(\xi_l - \xi, \Pi_l u - u) + a_l(\xi_l, u - u_l) + a_l(\xi, \Pi_l u - u)\\ &= \sum_{i=1}^{3} e_i.\end{aligned} \tag{3.19}$$

We now estimate the terms e_i one by one. For e_1, it is easy to check that

$$|e_1| \le C h_l^{2\alpha} \|\xi\|_{1+\alpha} \|u\|_{1+\alpha}. \tag{3.20}$$

For e_2, we have

$$\begin{aligned}|e_2| &\le \Big|a_l(\xi_l, u - u_l)\Big| = \Big|a_l(\xi_l, u) - a_l(\xi_l, u_l)\Big|\\ &= \Big|(-\triangle u, \xi_l) + \sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla u)\cdot n[\xi_l]ds - f(\xi_l)\Big|\\ &= \Big|\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla u)\cdot n[\xi_l]ds\Big|\\ &= \Big|\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla u)\cdot n[\xi_l - \xi]ds\Big|\\ &\le C h_l^{2\alpha}\|u\|_{1+\alpha}\|\xi\|_{1+\alpha},\end{aligned} \tag{3.21}$$

where $[\cdot]$ denotes the jump of a function across the interface Γ. Moreover, we have used the following fact (cf. [7] for details)

$$\Big|\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla u)\cdot nv ds\Big| \le C h_l^{\alpha}\|u\|_{1+\alpha}\|v\|_l \quad \forall u \in H^{1+\alpha}(\Omega),\ v \in V_l + H_0^1(\Omega). \tag{3.22}$$

For the last term e_3,

$$\begin{aligned}
&a(\xi,\Pi_l u-u)\\
&=(-\triangle\xi,\Pi_l u)-(-\triangle\xi,u)+\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla\xi)\cdot n[\Pi_l u-u]ds\\
&=(g_l,\Pi_l u-u)+\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla\xi)\cdot n[\Pi_l u-u]ds\\
&\,\hat{=}\, T_1+T_2.
\end{aligned}\tag{3.23}$$

We can estimate the term T_2 in the above inequality as follows:

$$\begin{aligned}
|T_2|&\le Ch_l^{\alpha}\|\xi\|_{1+\alpha}\|\Pi_l u-u\|_l\\
&\le Ch_l^{2\alpha}\|\xi\|_{1+\alpha}\|u\|_{1+\alpha}.
\end{aligned}\tag{3.24}$$

For the term T_1, we have

$$\begin{aligned}
|T_1|&=(g_l,Q_l(\Pi_l u-u))=(g_l,\Pi_l u-Q_l u)\\
&\le |||g_l|||_{\alpha-1,l}|||Q_l u-\Pi_l u|||_{1-\alpha,l}\\
&=|||\Pi_l u-u_l|||_{1-\alpha,l}|||Q_l u-\Pi_l u|||_{1-\alpha,l}.
\end{aligned}\tag{3.25}$$

On the other hand,

$$\begin{aligned}
|||Q_l u-\Pi_l u|||_{1-\alpha,l}&\le Ch_l^{\alpha-1}\|Q_l u-\Pi_l u\|_0\\
&\le Ch_l^{\alpha-1}(\|Q_l u-u\|_0+\|\Pi_l u-u\|_0)\\
&\le Ch_l^{\alpha-1}(h^{1+\alpha}\|u\|_{1+\alpha})\\
&\le Ch_l^{2\alpha}\|u\|_{1+\alpha}.
\end{aligned}\tag{3.26}$$

It follows from (3.23)-(3.26) that

$$|e_3|\le Ch_l^{2\alpha}|||\Pi_l u-u_l|||_{1-\alpha,l}\|u\|_{1+\alpha}.\tag{3.27}$$

Finally, combining (3.19)-(3.21), (3.27) gives the proof. □

Based on Lemma 3.4, we can prove the following important theorem.

Theorem 3.1. *Let u_l be the solution of* (2.6). *Then*

$$|||u_l-I_l u_{l-1}|||_{1-\alpha,l}\le Ch_l^{2\alpha}\|u\|_{1+\alpha}.$$

Proof. Using the same argument as in Lemma 4.3 in [32], we can prove that

$$||\Pi_l v-I_l\Pi_{l-1}v||_0\le Ch^{1+\alpha}\|v\|_{1+\alpha},\quad \forall v\in H^{1+\alpha}(\Omega).$$

Then by (3.5), we have

$$|||\Pi_l v-I_l\Pi_{l-1}v|||_{1-\alpha,l}\le Ch_l^{\alpha-1}h_l^{1+\alpha}\|v\|_{1+\alpha}=Ch_l^{2\alpha}\|v\|_{1+\alpha}.\tag{3.28}$$

Using the triangle inequality, (3.28) and Lemma 3.4, we get

$$\begin{aligned}
&|||u_l - I_l u_{l-1}|||_{1-\alpha,l} \\
\le\ & |||u_l - \Pi_l u|||_{1-\alpha,l} + |||\Pi_l u - I_l \Pi_{l-1} u|||_{1-\alpha,l} + |||I_l(\Pi_{l-1}u - u_{l-1})|||_{1-\alpha,l} \\
\le\ & C h_l^{2\alpha} \|u\|_{1+\alpha} + |||I_l(\Pi_{l-1}u - u_{l-1})|||_{1-\alpha,l}.
\end{aligned} \tag{3.29}$$

An application of Lemma 3.1 and the interpolation [16] yields

$$|||I_l v|||_{1-\alpha,l} \le C|||v|||_{1-\alpha,l-1}. \tag{3.30}$$

Then based on (3.30) and Lemma 3.4, we get

$$|||I_l(\Pi_{l-1}u - u_{l-1})|||_{1-\alpha,l} \le |||\Pi_{l-1}u - u_{l-1}|||_{1-\alpha,l-1} \le C h_l^{2\alpha}\|u\|_{1+\alpha}. \tag{3.31}$$

Combining (3.29), (3.31) gives Theorem 3.1. $\square$

Moreover, following the general framework of the convergence analysis for the nonnested cascadic multigrid method, we introduce a projection operator $P_l : V_{l-1} + V_l \to V_l$ defined by

$$a_l(P_l v, w) = a_l(v, w) \quad \forall w \in V_l. \tag{3.32a}$$

From the definition, it is easy to see that

$$\|P_l v\|_l \le \|v\|_{l-1} \quad \forall v \in V_{l-1}. \tag{3.32b}$$

Theorem 3.2. *For the projection operator P_l defined by* (3.32)*, we have*

$$|||(I_l - P_l)v|||_{1-\alpha,l} \le C h_l^{\alpha}\|v\|_{l-1} \quad \forall v \in V_{l-1}.$$

Proof. Similar to Lemma 3.4, we use a duality technique to prove this theorem. Define

$$g_l = A_l^{1-\alpha}(I_l v - P_l v).$$

Consider the following auxiliary problems

$$\begin{aligned}
a(\eta, v) &= (g_l, v) \quad && \forall v \in H_0^1(\Omega), \\
a_l(\eta_l, v) &= (g_l, v) \quad && \forall v \in V_l.
\end{aligned}$$

We have

$$\|\eta - \eta_l\|_l \le C h_l^{\alpha}\|\eta\|_{1+\alpha}. \tag{3.33}$$

Using the same argument as in the proof of Lemma 3.4, we can see that

$$|||g_l|||_{\alpha-1,l} = |||(I_l - P_l)v|||_{1-\alpha,l}, \tag{3.34}$$

$$\|\eta\|_{1+\alpha} \le C|||(I_l - P_l)v|||_{1-\alpha,l}. \tag{3.35}$$

Meanwhile,

$$
\begin{aligned}
&|||(I_l-P_l)v|||^2_{1-\alpha,l}\\
&=(g_l,(I_l-P_l)v)=a_l(\eta_l,(I_l-P_l)v)\\
&=a_l(\eta_l-\eta,(I_l-P_l)v)+a_l(\eta,(I_l-P_l)v).
\end{aligned}
\tag{3.36}
$$

For the first term on the right hand of (3.36), an application of Lemma 3.1 and (3.32), (3.33) yields

$$
|a_l(\eta_l-\eta,(I_l-P_l)v)|\le Ch_l^{\alpha}\|\eta\|_{1+\alpha}\|v\|_{l-1}. \tag{3.37}
$$

We now estimate the second term on the right hand of (3.36). In fact

$$
a_l(\eta,(I_l-P_l)v)=a_l(\eta,v-P_lv)+a_l(\eta,I_lv-v)\hat{=}F_1+F_2. \tag{3.38}
$$

By the definition of P_l, we obtain

$$
\begin{aligned}
|F_1|&=|a_l(\eta-\Pi_l\eta,v-P_lv)|\\
&\le Ch_l^{\alpha}\|\eta\|_{1+\alpha}\|v-P_lv\|_l\le Ch_l^{\alpha}\|\eta\|_{1+\alpha}\|v\|_{l-1}.
\end{aligned}
\tag{3.39}
$$

For the term F_2,

$$
\begin{aligned}
F_2&=(-\triangle\eta,I_lv)-(-\triangle\eta,v)+\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla\eta)\cdot nI_lvds+\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla\eta)\cdot nvds\\
&=(g_l,I_lv-v)+\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla\eta)\cdot nI_lvds+\sum_{\gamma_m\subset\Gamma}\int_{\gamma_m}(a(x)\nabla\eta)\cdot nvds\\
&\hat{=}\sum_1^3 G_i.
\end{aligned}
\tag{3.40}
$$

It is easy to see that

$$
|G_i|\le Ch_l^{\alpha}\|\eta\|_{1+\alpha}\|v\|_l,\quad i=2,3. \tag{3.41}
$$

For the term G_1, we have

$$
\begin{aligned}
|G_1|&=|(g_l,Q_l(I_lv-v)|\le|||g_l|||_{\alpha-1,l}|||Q_l(I_lv-v)|||_{1-\alpha,l}\\
&=|||(I_l-P_l)v|||_{1-\alpha,l}|||Q_l(I_lv-v)|||_{1-\alpha,l}\\
&\le|||(I_l-P_l)v|||_{1-\alpha,l}Ch^{\alpha-1}\|Q_l(I_lv-v)\|_0\\
&\le|||(I_l-P_l)v|||_{1-\alpha,l}Ch^{\alpha-1}\|I_lv-v\|_0\\
&\le Ch_l^{\alpha}|||(I_l-P_l)v|||_{1-\alpha,l}\|v\|_{l-1}.
\end{aligned}
\tag{3.42}
$$

Then by (3.40)-(3.42), we get

$$
|F_2|\le Ch_l^{\alpha}|||(I_l-P_l)v|||_{1-\alpha,l}\|v\|_{l-1}. \tag{3.43}
$$

Combining (3.35)-(3.39) and (3.43) yields the proof. □

Note that the mesh size on the level l is

$$h_l = h_L 2^{L-l}. \tag{3.44}$$

Let m_l $(1 \le l \le L)$ be the smallest integer satisfying

$$m_l \ge \beta^{L-l} m_L \tag{3.45}$$

for some fixed $\beta > 1$, where m_L is the number of iterations on the finest level L.

Lemma 3.5. *Let m_l satisfy* (3.45)*, and assume that the smoothing condition* (H1) *holds. Then*

$$\|u_L - u_L^*\|_L \le C_0 \sum_{l=1}^{L} \frac{h_l^\alpha}{m_l^{\alpha\gamma}} \|u\|_{1+\alpha},$$

where the constant C_0 is independent of the mesh size h_L and the level L.

Proof. The proof of this lemma is similar as the one of Lemma 2.1 in [24] where the cascadic multigrid method with full elliptic regularity for standard finite element approximations was considered. For completeness, we give an outline of this proof.

It is easy to check that

$$\begin{aligned}
\|u_l - u_l^*\|_l &= \|T_l^{m_l}(u_l - I_l u_{l-1}^*)\|_l \\
&\le \|T_l^{m_l}(u_l - I_l u_{l-1})\|_l + \|T_l^{m_l} I_l(u_{l-1} - u_{l-1}^*)\|_l \\
&\le \|T_l^{m_l}(u_l - I_l u_{l-1})\|_l + \|T_l^{m_l} P_l(u_{l-1} - u_{l-1}^*)\|_l \\
&\quad + \|T_l^{m_l}(I_l - P_l)(u_{l-1} - u_{l-1}^*)\|_l \\
&= J_1 + J_2 + J_3.
\end{aligned} \tag{3.46}$$

Then we estimate J_i one by one. For J_1, by Theorem 3.1 and (H1)-(1), we have

$$J_1 \le C \frac{h_l^{-\alpha}}{m_l^{\alpha\gamma}} |||u_l - I_l u_{l-1}|||_{1-\alpha,l} \le C \frac{h_l^\alpha}{m_l^{\alpha\gamma}} \|u\|_{1+\alpha}. \tag{3.47}$$

For J_2, an application of (H1)-(2) and (3.32) yields

$$J_2 \le \|P_l(u_{l-1} - u_{l-1}^*)\|_l \le \|u_{l-1} - u_{l-1}^*\|_{l-1}. \tag{3.48}$$

For the last term J_3, using (H1)-(1), Theorem 3.2, we get

$$\begin{aligned}
J_3 &\le C \frac{h_l^{-\alpha}}{m_l^{\alpha\gamma}} |||(I_l - P_l)(u_{l-1} - u_{l-1}^*)|||_{1-\alpha,l} \\
&\le \frac{C}{m_l^{\alpha\gamma}} \|u_{l-1} - u_{l-1}^*\|_{l-1}.
\end{aligned} \tag{3.49}$$

Combining (3.46)-(3.49) yields

$$\|u_l - u_l^*\|_l \le C\frac{h_l^\alpha}{m_l^{\alpha\gamma}}\|u\|_{1+\alpha} + \Big(1+\frac{C}{m_l^{\alpha\gamma}}\Big)\|u_{l-1}-u_{l-1}^*\|_{l-1}. \tag{3.50}$$

Recurrently, we get

$$\|u_L - u_L^*\|_L \le C\sum_{l=1}^{L}\prod_{i=0}^{l-1}\Big(1+\frac{C}{m_{L-i}^{\alpha\gamma}}\Big)\frac{h_{L-i}^\alpha}{m_{L-i}^{\alpha\gamma}}\|u\|_{1+\alpha}. \tag{3.51}$$

Noticing (3.45) and $m_L \ge 1$, we obtain

$$\begin{aligned}&\prod_{i=0}^{L-1}\Big(1+\frac{C}{m_{L-i}^{\alpha\gamma}}\Big) \le \exp\Big(\sum_{i=0}^{L-1}\frac{C}{m_{L-i}^{\alpha\gamma}}\Big)\\ \le& \exp\Big(\frac{C}{m_L^{\alpha\gamma}}\sum_{i=0}^{L-1}\beta^{-i\alpha\gamma}\Big) \le \exp\Big(\frac{C\beta^{\alpha\gamma}}{m_L^{\alpha\gamma}(\beta^{\alpha\gamma}-1)}\Big) \le C_0.\end{aligned} \tag{3.52}$$

Finally, inserting (3.52) into (3.51) proves Lemma 3.5. □

Based on Lemma 3.5 and a similar argument as in [8], we can prove that the following lemma is valid.

Lemma 3.6. *Under the assumption* (H1)*, if m_l, the number of iterations at level l, is given by* (3.45)*, then the accuracy of the cascadic multigrid is*

$$\|u_L - u_L^*\|_L \le \begin{cases} C\dfrac{1}{1-(\frac{2}{\beta^\gamma})^\alpha}\dfrac{h_L^\alpha}{m_L^{\alpha\gamma}}\|u\|_{1+\alpha} & \textit{for}\ \ \beta > 2^{\frac{1}{\gamma}},\\ CL\dfrac{h_L^\alpha}{m_L^{\alpha\gamma}}\|u\|_{1+\alpha} & \textit{for}\ \ \beta = 2^{\frac{1}{\gamma}}.\end{cases} \tag{3.53}$$

Lemma 3.7 ([8])**.** *The computational cost of the cascadic multigrid is proportional to*

$$\sum_{l=1}^{L} m_l n_l \le \begin{cases} C\dfrac{1}{1-\beta 2^{-d}}m_L n_L & \textit{for}\ \ \beta < 2^d,\\ CLm_L n_L & \textit{for}\ \ \beta = 2^d,\end{cases} \tag{3.54}$$

where d is the dimension of the domain Ω.

3.1. Basic iterative methods

We consider the Richardson iteration as a smoother. Using the same argument as in [4], it is shown that the Richardson iterative operator

$$T_l = I - \frac{1}{\lambda_l}A_l$$

satisfies the assumption (H1) with $\gamma = 0.5$.

Combining the methods of constructing smoothers in [16, 20], we can also show that the operator $T_l = I - R_l A_l$, where R_l denotes Jacobi or Gauss-Seidel iterations on the level l, satisfies (H1).

Theorem 3.3. *If a standard iteration like the Richardson, Jacobi, and Gauss-Seidel iterations is used as smoother, then*

(1). If d=3, then the cascadic multigrid method is optimal.

(2). If d=2, and the number of iterations on the level L is $m_L = [m_* L^2]$, then the error of the cascadic multigrid method is

$$\|u_L - u_L^*\|_L \le C h_L^\alpha m_*^{-\frac{1}{2}} \|u\|_{1+\alpha},$$

and the complexity of computation is

$$\sum_{l=1}^{L} m_l n_l \le C m_* n_L (1 + \log n_L)^{1+\frac{2}{\alpha}},$$

which means the CMG algorithm in this case is nearly optimal.

3.2. Conjugate gradient (CG) method

Assume that u_l^0 is the initial value of the CG method on the level l. Let $C_l^{m_l} u_l^0$ be the m_l steps of the CG iteration. Then the error of the CG method can be expressed by

$$\|u_l - C_l^{m_l} u_l^0\|_l = \min_{p \in P_{m_l}, p(0)=1} \|p(A_l)(u_l - u_l^0)\|_l, \tag{3.55}$$

where P_{m_l} denotes the set of polynomials p with degree $p \le m_l$, see [3].

Combining Theorem 2.2 in [8] and the fact the L^2 norm of basis function $\{\phi_l^i\}_{i=1}^{N_l} \in V_l$ is $\mathcal{O}(h_l^2)$ (see the proof of Lemma 4.3 in [20]), we can prove

Lemma 3.8. *There exists a linear operator $T_l \in P_{m_l} = \phi_{\lambda_l, m_l}(A_l)$, where $\phi_{\lambda,m} \in P_m$, $\phi_{\lambda,m}(0) = 1$ such that*

$$(1).\ \ \|T_l^{m_l} v_l\|_l \le C \frac{h_l^{-1}}{2m_l + 1} \|v_l\|_0 \quad \forall v_l \in V_l, \tag{3.56}$$

$$(2).\ \ \|T_l^{m_l} v_l\|_l \le \|v_l\|_l \quad \forall v_l \in V_l. \tag{3.57}$$

So (H1) holds for CG method with $\gamma = 1$. Finally, we have

Theorem 3.4. *If the conjugate gradient method is used as smoother, then the cascadic multigrid method is optimal for both 2D and 3D problems.*

Remark 3.3. We can extend the cascadic multigrid method developed in this paper to the mortar-type P1 nonconforming element method which was first introduced by Marcinkowski in [22] and the mortar-type Wilson element method [26].

4. An economical cascadic multigrid method

In this section, we will introduce the economical cascadic multigrid method for solving the system resulting from the mortar finite element method. It is shown that the ECMG method is also optimal both the accuracy and the computational complexity. In fact, compared with the standard CMG method, the ECMG method requires less work operations on each level.

For simplicity, we consider the 2-D case. Based on [28], we define a level parameter L_0 as the largest positive integer, which satisfies the following inequality:

$$L_0 \leq \min\Big\{\frac{L\log\beta + \log m_L + 2\log h_0}{\log\beta + 2\log 2}, \frac{L}{2}\Big\}. \tag{4.1}$$

First, we give a new criteria.

New criteria:

(i). If $l > L_0$, then

$$m_l = [m_L \beta^{L-l}].$$

(ii). If $l \leq L_0$, then

$$m_l = [m_*^{\frac{1}{2}}(L-(2-\varepsilon_0)l)\kappa_l],$$

where $0 < \varepsilon_0 \leq 1$ is a fixed positive number.

In practical implementation, because $\kappa_l \approx h_l^{-2}$, the above terms can be replaced by:

$$m_l = [m_*^{\frac{1}{2}}(L-(2-\varepsilon_0)l)h_l^{-2}], \quad m_l = [m_*^{\frac{1}{2}} h_l^{-2}].$$

Then our economical cascadic multigrid method can be written as follows:

Economical Cascadic Multigrid

(1) Set $u_0^0 = u_0^* \hat{=} u_0$ and let

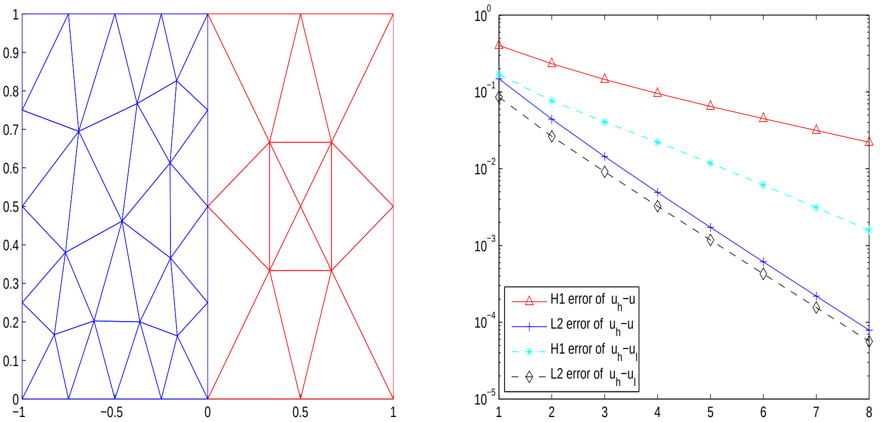
$$u_l^0 = I_l u_{l-1}^*.$$

(2) For $l = 1, \cdots, L$

$$u_l^{m_l} = C_l^{m_l} u_l^0.$$

(3) Set $u_l^* \hat{=} u_l^{m_l}$, where m_l is determined by the new criteria.

Similar as the standard CMG method, optimal error estimates and computational complexity can be also obtained for this economical cascadic multigrid method. The proof is just a simple combination of the theory of Section 3 in this paper and the theoretical results in [28]. We will omit the details here.



(a) The initial regular mesh (b) The error for $\alpha=\frac{1}{2}$

Figure 1: Left: the initial mesh of finite element method; Right: the solid lines represent the relative errors of e_{H_1} (red line) and e_{L_2} (blue line); The dashed lines represent $e^I_{H_1}$ (cyan line) and $e^I_{L_2}$ (black line).

5. Numerical experiments

In this section, we show the performance of the cascadic multigrid method and economical cascadic multigrid methods for the mortar element methods. For the experiments, we consider our model problem to be defined on a square domain and an L-shaped domain.

We use ECMG and usual CMG algorithms to solve the following problem:

$$\begin{cases} -\triangle u = f, & \text{in } \Omega = (-1,1)\times(-1,1), \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where f is chosen such that the exact solution of the problem is

$$u(x,y) = (1-x^2)(1-y^2)r^\alpha \sin(\alpha\theta),$$

where r and θ are the polar coordinates, i.e., $x = r\cos\theta, y = r\sin\theta$, and the parameter $\alpha \in (1,0.5]$. We assume that $0 \in \partial\Omega$. Then it is known that $u \in H^{1+\alpha-\epsilon}(\Omega) \setminus H^{1+\alpha}(\Omega)$ (see [19]).

Experiment 5.1. In this experiment, we assume that $\Omega = [-1,1]\times[0,1]$, and the domain Ω is divided into two subdomains $\Omega_1 = [-1,0]\times[0,1]$ and $\Omega_2 = [0,1]\times[0,1]$. The initial mesh ($l = 1$) is plotted in Fig. 1(a), and the mesh of level l is refined uniformly from the mesh of level $l-1$, that is to say, every element at level $l-1$ is cut into four new triangles by joining the midpoints of its edges. The mortar side is taken at the boundary of $\partial\Omega_1$. In our experiments, we compute the following terms to estimate the error:

$$e_{H_1} = \frac{\|\nabla u_h - \nabla u\|_{L^2(\Omega)}}{\|\nabla u\|_{L^2(\Omega)}}, \quad \text{and} \quad e_{L_2} = \frac{\|u_h - u\|_{L_2(\Omega)}}{\|u\|_{L^2(\Omega)}},$$

Table 1: Errors of the mortar finite element method (H^1 norm).

Level	$\alpha = \frac{9}{10}$		$\alpha = \frac{2}{3}$		$\alpha = \frac{1}{2}$	
	e_{H_1}	$\log_2 r_1$	e_{H_1}	$\log_2 r_1$	e_{H_1}	$\log_2 r_1$
1	3.9281e-1		3.8355e-1		4.0284e-1	
2	1.9085e-1	1.0414	1.9639e-1	9.6571e-1	2.3303e-1	7.897e-1
3	9.3990e-2	1.0219	1.0477e-1	9.0646e-1	1.4528e-1	6.8217e-1
4	4.6687e-2	1.0095	5.7855e-2	8.5674e-1	9.5510e-2	6.052e-1
5	2.3313e-2	1.0019	3.2953e-2	8.1202e-1	6.4926e-2	5.569e-1
6	1.1676e-2	9.9752e-1	1.9277e-2	7.7355e-1	4.4977e-2	5.296e-1
7	5.8592e-3	9.9484e-1	1.1523e-2	7.4231e-1	3.1472e-2	5.151e-1
8	2.9439e-3	9.9297e-1	7.0033e-3	7.1845e-1	2.2137e-2	5.076e-1

Table 2: Errors of the mortar finite element method (L^2 norm).

Level	$\alpha = \frac{9}{10}$		$\alpha = \frac{2}{3}$		$\alpha = \frac{1}{2}$	
	e_{L_2}	$\log_2 r_2$	e_{L_2}	$\log_2 r_2$	e_{L_2}	$\log_2 r_2$
1	1.5696e-1		1.4110e-1		1.4841e-1	
2	3.7308e-2	2.0728	3.6045e-2	1.9688	4.4356e-2	1.7424
3	9.0536e-3	2.0429	9.7462e-3	1.8869	1.4401e-2	1.623
4	2.2313e-3	2.0206	2.7546e-3	1.8230	4.9145e-3	1.5511
5	5.5511e-4	2.0071	8.0604e-4	1.7729	1.7238e-3	1.5114
6	1.3882e-4	1.9996	2.4214e-4	1.7350	6.1315e-4	1.4913
7	3.4819e-5	1.9953	7.4158e-5	1.7072	2.1959e-4	1.4814
8	8.7500e-6	1.9925	2.3022e-5	1.6876	7.8895e-5	1.4768

and we also compute the difference between the finite element solution with the interpolant of the original solution:

$$e^I_{H_1} = \frac{\|\nabla u_h - \nabla u^I\|_{L^2(\Omega)}}{\|\nabla u^I\|_{L^2(\Omega)}}, \quad \text{and} \quad e^I_{L_2} = \frac{\|u_h - u^I\|_{L_2(\Omega)}}{\|u^I\|_{L^2(\Omega)}},$$

where u^I is the P_1 conforming finite element interpolant of u. Fig. 1(b) shows the convergence behaviors of the mortar element method: For the H_1 norm, the mortar finite element u_h is closer to the interpolant u^I, which is typical called superconvergence.

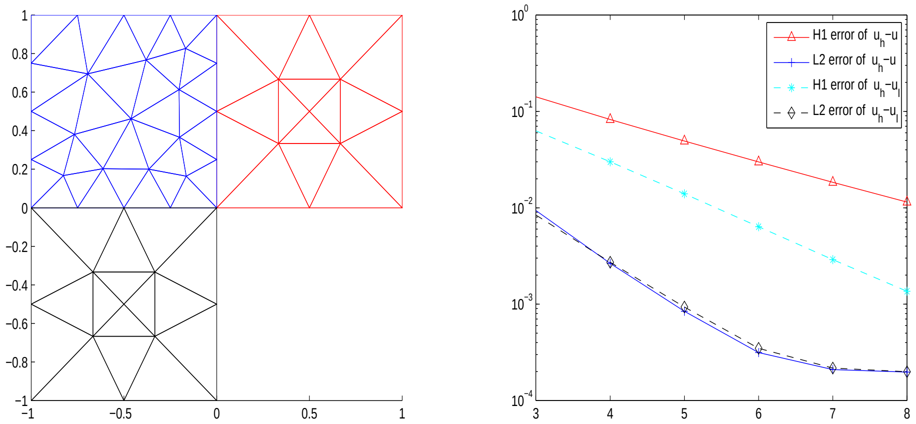
Now, following the convergence analysis of Theorem 2.1, we will compute the ratios of the errors at different level l:

$$r_1 = \frac{e_{H_1} \text{ on level } l-1}{e_{H_1} \text{ on level } l}, \quad \text{and} \quad r_2 = \frac{e_{L_2} \text{ on level } l-1}{e_{L_2} \text{ on level } l},$$

and note that the mesh size of the lth level is half of the $(l-1)$th one, then we expect $r_1 \approx 2^\alpha$, or $\log_2 r_1 \approx \alpha$. This expectation can be verified by the results in Table 1: For different α, the value $\log_2 r_1$ is a good approximation of α.

Here, we also display the error behaviors in the L^2 norm. Table 2 shows that the convergence of the mortar finite element behaves like

$$\|u - u_l\|_{L^2(\Omega)} \le c h_l^{1+\alpha} \|u\|_{1+\alpha}.$$



(a) The initial mesh (b) The error for $\alpha = \frac{2}{3}$

Figure 2: Left: the initial mesh on the L-shaped domain; Right: the solid lines represent the relative errors of e_{H_1} (red line) and e_{L_2} (blue line); The dashed lines represent $e^I_{H_1}$ (cyan line) and $e^I_{L_2}$ (black line).

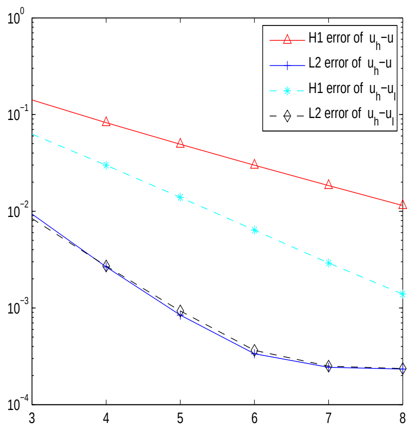


Figure 3: ECMG is used to solve the mortar finite element ($\alpha = \frac{2}{3}$). The solid lines represent the relative errors of e_{H_1} (red line) and e_{L_2} (blue line); The dashed lines represent $e^I_{H_1}$ (cyan line) and $e^I_{L_2}$ (black line).

Experiment 5.2. In Experiment 5.1, we have verified the convergence of the mortar finite element method. In this experiment, we want to check the convergence behavior of the cascadic multigrid algorithm (see (3.14) and (3.15)). Here we use the Jacobi iteration as the smoother, and we set $\beta = 5$.

Fig. 2(a) shows the initial mesh on the L-shaped domain. Here, we use the same refinement as in Experiment 5.1. In Fig. 2(b), the mortar finite element on level 3 is solved exactly, and the cascadic multigrid algorithm (3.14)-(3.15) is used to obtain the iterative solution. Here, the error behavior in the H^1 norm is the same as in Experiment 5.1, and the errors in the L^2 norm will stop at some error level. That is to say that the cascadic multigrid algorithm is optimal in the H^1 norm but not in the L^2 norm.

Experiment 5.3. In this experiment, the ECMG (Economical Cascadic Multigrid) algorithm is used. Compared with CMG algorithm, less smoothing steps are needed when $l \leq L_0$.

Table 3: Errors of CMG and ECMG for $\alpha = \frac{2}{3}$.

Level	CMG				ECMG			
	H^1	$\log_2 r_1$	L^2	$\log_2 r_2$	H^1	$\log_2 r_1$	L_2	$\log_2 r_2$
3	1.4154e-1		9.3528e-3		1.4154e-1		9.3528e-3	
4	8.2424e-2	0.7801	2.6458e-3	1.8217	8.2421e-2	0.7801	2.6554e-3	1.8165
5	4.9196e-2	0.7445	8.4412e-4	1.6482	4.9196e-2	0.7445	8.4326e-4	1.6549
6	2.9889e-2	0.7189	3.1492e-4	1.4225	2.9891e-2	0.7188	3.3561e-4	1.3292
7	1.8394e-2	0.7004	2.0963e-4	0.5871	1.8405e-2	0.6996	2.4340e-4	0.4635
8	1.1444e-2	0.6846	1.9715e-4	0.0886	1.1455e-2	0.6841	2.3303e-4	0.0628

Fig. 3 shows that the error behavior is almost the same as in Experiment 5.1. In Table 3, the relative errors in the H^1 and L_2 norms are listed. Here, the mortar finite element on level 3 is solved exactly, and when the CMG and ECMG algorithms are used, the errors are almost the same.

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