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Quasi-Optimal Convergence Rate of an AFEM for Quasi-Linear Problems of Monotone Type

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Abstract. We prove the quasi-optimal convergence of a standard adaptive finite element method (AFEM) for a class of nonlinear elliptic second-order equations of monotone type. The adaptive algorithm is based on residual-type a posteriori error estimators and Dörfler's strategy is assumed for marking. We first prove a contraction property for a suitable definition of total error, analogous to the one used by Diening and Kreuzer (2008) and equivalent to the total error defined by Cascón et. al. (2008). This contraction implies linear convergence of the discrete solutions to the exact solution in the usual H^1 Sobolev norm. Secondly, we use this contraction to derive the optimal complexity of the AFEM. The results are based on ideas from Diening and Kreuzer and extend the theory from Cascón et. al. to a class of nonlinear problems which stem from strongly monotone and Lipschitz operators.

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1. Introduction

The main goal of this article is the study of convergence and optimality properties of an adaptive finite element method (AFEM) for quasi-linear elliptic partial differential equations over a polygonal/polyhedral domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ having the form

$$\begin{cases} Au := -\nabla \cdot \left[\alpha(\,\cdot\,, |\nabla u|^2)\nabla u\right] = f & \text{in}\,\Omega \\ \qquad\qquad\qquad\qquad\qquad\quad u = 0 & \text{on}\,\partial\Omega, \end{cases} \tag{1.1}$$

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where $\alpha : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded positive function whose precise properties will be stated in Section 2 below, and $f \in L^2(\Omega)$ is given. The assumptions on α guarantee that the nonlinear operator A is Lipschitz and strongly monotone; see (2.6)–(2.7). This kind of problems arises in many practical situations; see Section 2.2 below.

AFEMs are an effective tool for making an efficient use of the computational resources, and for certain problems, it is even indispensable to their numerical resolvability. The ultimate goal of AFEMs is to equidistribute the error and the computational effort obtaining a sequence of meshes with optimal complexity. Adaptive methods are based on a posteriori error estimators, that are computable quantities depending on the discrete solution and data, and indicate a distribution of the error. A quite popular, natural adaptive version of classical finite element methods consists of the loop

$$\text{SOLVE} \to \text{ESTIMATE} \to \text{MARK} \to \text{REFINE}, \tag{1.2}$$

that is: solve for the finite element solution on the current grid, compute the a posteriori error estimator, mark with its help elements to be subdivided, and refine the current grid into a new, finer one.

A general result of convergence for linear problems has been obtained by Morin, Siebert and Veeser [16], where very general conditions on the linear problems and the adaptive methods that guarantee convergence are stated. Following these ideas a (plain) convergence result for elliptic eigenvalue problems has been proved in [8]. On the other hand, optimality of adaptive methods using *Dörfler's marking strategy* [7] for linear elliptic problems has been stated by Stevenson [22] and Cascón, Kreuzer, Nochetto and Siebert [2]. Linear convergence of an AFEM for elliptic eigenvalue problems has been proved in [13], and optimality results can be found in [5,9]. For a summary of convergence and optimality results of AFEM we refer the reader to the survey [18] and the references therein. We restrict ourselves to those references strictly related to our work.

Well-posedness and a priori finite element error estimates for problem (1.1) have been stated in [4]. A posteriori error estimators for nonconforming approximations have been developed in [19]. Linear convergence of an AFEM for the φ-Laplacian problem in a context of Sobolev-Orlicz spaces has been established in [6]. Recently, the (plain) convergence of an adaptive *inexact* FEM for problem (1.1) has been proved in [10], where only a discrete *linear* system is solved before each adaptive refinement; albeit with stronger assumptions on α.

In this article we consider a standard adaptive loop of the form (1.2) based on classical residual-type a posteriori error estimators, where the Galerkin discretization for problem (1.1) is considered. We use the Dörfler's strategy for marking and assume a minimal bisection refinement. The goal of this paper is to prove the optimal complexity of this AFEM by stating two main results. The first one establishes the linear convergence of the adaptive loop through a contraction property. More precisely, we will prove the following

Theorem 1.1 (Contraction property). *Let u be the weak solution of problem* (1.1) *and let $\{U_k\}_{k\in\mathbb{N}_0}$ be the sequence of discrete solutions computed through the adaptive algorithm de-*

scribed in Section 4. Then, there exist constants $0 < \rho < 1$ *and* $\mu > 0$ *such that*

$$[\mathscr{F}(U_{k+1}) - \mathscr{F}(u)] + \mu\,\eta_{k+1}^2 \le \rho^2\big([\mathscr{F}(U_k) - \mathscr{F}(u)] + \mu\,\eta_k^2\big), \qquad \forall\, k \in \mathbb{N}_0, \tag{1.3}$$

where $[\mathscr{F}(U_k) - \mathscr{F}(u)]$ *is a notion equivalent to the energy error and* η_k *denotes the global a posteriori error estimator in the mesh corresponding to the step* k *of the iterative process.*

The second main result shows that, if the solution of the nonlinear problem (1.1) can be ideally approximated with adaptive meshes at a rate $(DOFs)^{-s}$, then the adaptive algorithm generates a sequence of meshes and discrete solutions which converge with this rate. Specifically, we will prove the following

Theorem 1.2 (Quasi-optimal convergence rate)**.** *Assume that the solution* u *of problem* (1.1) *belongs to* $\mathbb{A}_s$.† *Let* $\{\mathscr{T}_k\}_{k\in\mathbb{N}_0}$ *and* $\{U_k\}_{k\in\mathbb{N}_0}$ *denote the sequence of meshes and discrete solutions computed through the adaptive algorithm described in Section 4, respectively. If the marking parameter* θ *in Dörfler's criterion is small enough (cf.* (4.1) *and* (5.1)*), then*

$$\big[\|\nabla(U_k - u)\|_\Omega^2 + \operatorname{osc}_{\mathscr{T}_k}^2(U_k)\big]^{\frac{1}{2}} = \mathscr{O}\left((\#\mathscr{T}_k - \#\mathscr{T}_0)^{-s}\right), \qquad \forall\, k \in \mathbb{N}. \tag{1.4}$$

The left-hand side is called total error *and consists of the energy error plus an oscillation term.*

Basically, we follow the steps presented in [2] for linear elliptic problems. However, due to the *nonlinearity* of problem (1.1) the generalization of the mentioned results is not obvious. In particular, for linear elliptic problems the *Galerkin orthogonality property (Pythagoras)*

$$\|\nabla(U - u)\|_\Omega^2 + \|\nabla(U - V)\|_\Omega^2 = \|\nabla(V - u)\|_\Omega^2, \tag{1.5}$$

where U is a discrete solution and V is a discrete test function, is used to prove the contraction property and a generalized Cea's Lemma (the quasi-optimality of the total error). This orthogonality property does not hold when we consider problem (1.1) though. To overcome this difficulty we resort to ideas from [6], replacing (1.5) by the trivial equality

$$[\mathscr{F}(U) - \mathscr{F}(u)] + [\mathscr{F}(V) - \mathscr{F}(U)] = [\mathscr{F}(V) - \mathscr{F}(u)],$$

where each term in brackets is equivalent to the corresponding term in (1.5) (cf. Theorem 4.1 below), and $\mathscr{F}$ is the energy functional of (1.1). We thus establish some kind of *quasi-orthogonality relationship* for the energy error (cf. Lemma 5.1) which is sufficient to prove the quasi-optimality of the total error (cf. Lemma 5.2).

Additionally, it is necessary to study the behavior of the error estimators and oscillation terms when refining. In order to do that, we need to show that a certain quantity, which measures the difference of error estimators and oscillation terms between two discrete functions (cf. (3.13)), is bounded by the energy of the difference between these functions (see Lemma 3.2 in Section 3.3). This result can be proved with usual techniques for linear elliptic problems using inverse inequalities and trace theorems, but the generalization of

†Roughly speaking, $u \in \mathbb{A}_s$ if u can be approximated with adaptive meshes with a rate $(DOFs)^{-s}$ (cf. (6.1) in Section 6).

this result to nonlinear problems requires some new technical results. We establish suitable hypotheses on the main coefficient α of problem (1.1) to be able to prove the mentioned estimation for the nonlinear problems that we study in this article (see (2.8)).

It is worth mentioning that even though we exploit ideas from [6], our results neither contain, nor are contained in those from [6]. They prove linear convergence of a φ-Laplacian problem in a context of Sobolev-Orlicz spaces through a contraction property analogous to (1.3). On the one hand, we prove the contraction property (1.3) for a class of nonlinear problems arising from Lipschitz and strongly monotone operators, which excludes the p-Laplacian, but allows for a spatial dependence of the nonlinearity α, and uses only the more familiar Sobolev norms, without resorting to Orlicz-Sobolev norms. Even though the use of these norms has been a breakthrough in the numerical investigation of p-Laplacian-like problems, being able to leave these norms aside allows for a simpler presentation, with more familiar and easily computable norms. On the other hand, we also study the complexity of the AFEM in terms of degrees of freedom, and establish the quasi-optimality bound (1.4). We thus conclude that the theory developed for linear problems in [2] can be generalized to quasi-linear problems arising from differential operators being Lipschitz continuous and strongly monotone, and believe that this is a step forward towards a more general optimality analysis of AFEMs for nonlinear problems.

This paper is organized as follows. In Section 2 we present specifically the class of problems that we study and some of its properties, together with some applications that fall into our theory. In Section 3, we present a posteriori error estimations. In Section 4 we state the adaptive loop that we use for the approximation of problem (1.1) and we prove its linear convergence through a contraction property. Finally, the last two sections of the article are devoted to prove that the AFEM converges with quasi-optimal rate.

2. Setting and applications

2.1. Setting

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal ($d = 2$) or polyhedral ($d = 3$) domain with Lipschitz boundary. A weak formulation of (1.1) consists in finding $u \in H_0^1(\Omega)$ such that

$$a(u;u,v) = L(v), \qquad \forall\, v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(w;u,v) = \int_\Omega \alpha(\,\cdot\,,|\nabla w|^2)\nabla u \cdot \nabla v, \qquad \forall\, w,u,v \in H_0^1(\Omega),$$

and

$$L(v) = \int_\Omega f\,v, \qquad \forall\, v \in H_0^1(\Omega).$$

In order to make this presentation clearer, we define $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\beta(x,t) := \frac{1}{2}\int_0^{t^2} \alpha(x,r)\,dr,$$

and note that from Leibniz's rule the derivative of β as a function of its second variable satisfies

$$D_2\beta(x,t) := \frac{\partial \beta}{\partial t}(x,t) = t\alpha(x,t^2).$$

We require that α is $\mathscr{C}^1$ as a function of its second variable and there exist positive constants c_a and C_a such that

$$c_a \leq \frac{\partial^2 \beta}{\partial t^2}(x,t) = \alpha(x,t^2) + 2t^2 D_2\alpha(x,t^2) \leq C_a, \qquad \forall\, x \in \Omega,\ t > 0. \tag{2.2}$$

Since

$$\alpha(x,t^2) = \frac{D_2\beta(x,t) - D_2\beta(x,0)}{t} = \frac{\partial^2 \beta}{\partial t^2}(x,r),$$

for some $0 < r < t$ the last assumption yields

$$c_a \leq \alpha(x,t) \leq C_a, \qquad \forall\, x \in \Omega,\ t > 0. \tag{2.3}$$

It is easy to check that the form a is linear and symmetric in its second and third variable. Additionally, from (2.3) it follows that a is bounded,

$$|a(w;u,v)| \leq C_a \|\nabla u\|_\Omega \|\nabla v\|_\Omega, \qquad \forall\, w,u,v \in H_0^1(\Omega), \tag{2.4}$$

and coercive,

$$c_a \|\nabla u\|_\Omega^2 \leq a(w;u,u), \qquad \forall\, w,u \in H_0^1(\Omega).$$

Now, we sketch the proof that (2.2) is sufficient to guarantee the well-posedness of problem (2.1). Let $\gamma : \Omega \times \mathbb{R}^d \to \mathbb{R}_+$ be given by

$$\gamma(x,\xi) := \beta(x,|\xi|) = \frac{1}{2}\int_0^{|\xi|^2} \alpha(x,r)\, dr,$$

and note that if $\nabla_2\gamma$ denotes the gradient of γ as a function of its second variable, then

$$\nabla_2\gamma(x,\xi) = \alpha(x,|\xi|^2)\xi, \qquad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^d. \tag{2.5}$$

Condition (2.2) means that $D_2\beta$ is Lipschitz and strongly monotone as a function of its second variable and it can be seen that $\nabla_2\gamma$ so is [25].

If $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the operator given by

$$\langle Au, v\rangle := a(u;u,v), \qquad \forall\, u,v \in H_0^1(\Omega),$$

problem (2.1) is equivalent to the equation

$$Au = L,$$

where $L \in H^{-1}(\Omega)$ is given. It is easy to check that the properties of $\nabla_2 \gamma$ are inherited by A, i.e., A is Lipschitz and strongly monotone. More precisely, there exist positive constants C_A and c_A such that

$$\|Au - Av\|_{H^{-1}(\Omega)} \le C_A \|\nabla(u-v)\|_\Omega, \qquad \forall\, u,v \in H_0^1(\Omega), \tag{2.6}$$

and

$$\langle Au - Av, u - v\rangle \ge c_A \|\nabla(u-v)\|_\Omega^2, \qquad \forall\, u,v \in H_0^1(\Omega). \tag{2.7}$$

As a consequence of (2.6) and (2.7), problem (2.1) has a unique stable solution [24, 25], which will be denoted throughout this article by u.

In order to have the behavior of the error estimator and oscillation terms under control when refining, we need some additional assumptions on $\alpha(x,t)$ and $D_2\alpha(x,t)t$ with respect to the space variable $x \in \Omega$. From now on we assume that $\alpha(\cdot,t)$ and $D_2\alpha(\cdot,t)t$ are piecewise Lipschitz over an initial triangulation $\mathscr{T}_0$ of Ω uniformly in $t>0$. More precisely, there exists a constant $C_\alpha > 0$ such that

$$\begin{aligned} |\alpha(x,t) - \alpha(y,t)| + |D_2\alpha(x,t)t - D_2\alpha(y,t)t| &\le C_\alpha |x-y|, \\ \text{for all } x,y \in T, \text{ all } T \in \mathscr{T}_0 \text{ and all } t>0&, \end{aligned} \tag{2.8}$$

where $\mathscr{T}_0$ is the initial triangulation of the domain Ω.

2.2. Applications

As seen in the last section, condition (2.2) guarantees the existence and uniqueness of the solutions of problem (2.1), and it is a standard assumption allowing a unified theory [25] in a framework of the familiar Sobolev norms. In this section we show that there exist several applications in which (2.2) is reasonable.

Example 2.1 Problems like (2.1) arise in electromagnetism; see the presentation from nonlinear Maxwell equations in [14] and for nonlinear magnetostatic field in [3]. Concrete formulas such as

$$\alpha(t) = \frac{1}{\mu_0}\left(a + (1-a)\frac{t^8}{t^8+b}\right), \tag{2.9}$$

appear in [14], and characterize the reluctance of stator sheets in the cross-sections of an electrical motor [14] (μ_0 is the vacuum permeability and $a, b > 0$ are characteristic constants). Also, x-dependent nonlinearities arise where typically the function α is independent of x in some subdomain $\Omega_1 \subset \Omega$ and constant on the complement, where these subdomains correspond to ferromagnetic and other media, respectively. In the case of the nonlinearity (2.9) on Ω_1, we have

$$\alpha(x,t) = \begin{cases} \dfrac{1}{\mu_0}\left(a + (1-a)\dfrac{t^8}{t^8+b}\right) & \text{if } x \in \Omega_1,\ t>0 \\ a & \text{if } x \in \Omega \setminus \Omega_1, \end{cases}$$

where $a > 0$ is a constant magnetic reluctance.

The formula

$$\alpha(t) = \left(1 - (c-d)\frac{1}{t^2 + c}\right),$$

is stated in [3] and describes the magnetostatic field ($c > d > 0$ are constants).

It is easy to check that the functions α just described satisfy assumption (2.2) for all $t > 0$.

In the examples that follow, α does not fulfill (2.2) for all $t > 0$ but it does for t in any interval of the form $(0, T)$ with $T > 0$. Therefore, under the assumption that an upper bound for the gradient of the solution $|\nabla u|$ is known, the function α could be replaced by one satisfying (2.2) without changing the solution. This replacement of α is not needed in practice, but is rather a theoretical tool for proving that this assumption holds. We note that in several applications, an upper bound for the gradient of the solution $|\nabla u|$ is known or can be computed.

Example 2.2 For the equation of prescribed mean curvature, the unknown u defines the graph of the surface whose curvature is prescribed by f and

$$\alpha(t) = \frac{1}{(1+t)^{\frac{1}{2}}}.$$

This function α satisfies (as can be easily checked) assumption (2.2) on any interval of the form $(0, T)$ with $T > 0$. Therefore, this example falls into our theory when we are computing a solution with $|\nabla u|^2$ uniformly bounded. This assumption is made in [15] and can be proved for several domains and right-hand side functions f.

Example 2.3 In [20], a problem like (2.1) arises from Forchheimer flow in porous media and Ergun's law for incompressible fluid flow. In the case of Forchheimer's law the unknown u denotes the pressure and

$$\alpha(t) = \frac{2}{c + \sqrt{c^2 + d t^{\frac{1}{2}}}},$$

in the absence of gravity, where $c = \frac{\mu}{k}$ (μ is the viscosity of the fluid, $k = k(x)$ is the permeability of the medium) and $d = 4b\rho$ (b is a dynamic viscosity and ρ is the fluid density), all taken to be uniformly positive. Again, it is easy to check that this function α satisfies (2.2) on any interval of the form $(0, T)$ with $T > 0$. Under the constraint that $|\nabla u|^2$ is uniformly bounded from above, as is done in [20], this example falls within our theory.

Example 2.4 The concept of fictitious gas has been introduced to regularize the transonic flow problem for shock free airfoil design (see [4] and the references therein). The velocity potential u for the fictitious gas is governed by an equation of the form (2.1) with

$$\alpha(t) = \left(1 - \frac{\gamma - 1}{2} t\right)^{\frac{1}{\gamma - 1}}.$$

The flow remains subsonic when $\gamma \leq -1$, and in this case α satisfies assumption (2.2) on any interval of the form $(0,T)$ with $T > 0$; notice that the case $\gamma = -1$ coincides with Example 2.2.

3. Discrete solutions and a posteriori error analysis

3.1. Discretization

In order to define discrete approximations to problem (2.1) we will consider *triangulations* of the domain Ω. Let $\mathscr{T}_0$ be an initial conforming triangulation of Ω, that is, a partition of Ω into d-simplices such that if two elements intersect, they do so at a full vertex/edge/face of both elements. Let us also assume that the initial mesh $\mathscr{T}_0$ is labeled satisfying condition (b) of Section 4 in Ref. [23]. Let $\mathbb{T}$ denote the set of all conforming triangulations of Ω obtained from $\mathscr{T}_0$ by refinement using the bisection procedure described by Stevenson [23], which coincides, (after some re-labeling) with the *newest vertex* bisection procedure in two dimensions and the Kossaczký's procedure in three dimensions [21].

Due to the processes of refinement used, the family $\mathbb{T}$ is shape regular, i.e.,

$$\sup_{\mathscr{T}\in\mathbb{T}} \sup_{T\in\mathscr{T}} \frac{\text{diam}(T)}{\rho_T} =: \kappa_{\mathbb{T}} < \infty,$$

where $\text{diam}(T)$ is the diameter of T, and ρ_T is the radius of the largest ball contained in it. Throughout this article, we only consider meshes $\mathscr{T}$ that belong to the family $\mathbb{T}$, so the shape regularity of all of them is bounded by the uniform constant $\kappa_{\mathbb{T}}$ which only depends on the initial triangulation $\mathscr{T}_0$ [21]. Also, the diameter of any element $T \in \mathscr{T}$ is equivalent to the local mesh-size $H_T := |T|^{1/d}$, which in turn defines the global mesh-size $H_{\mathscr{T}} := \max_{T\in\mathscr{T}} H_T$. Also, the complexity of the refinement can be controlled, as described in Lemma 6.3 below.

Hereafter, we denote the subset of $\mathscr{T}$ consisting of neighbors of T by $\mathscr{N}_{\mathscr{T}}(T)$ and the union of T and its neighbors in $\mathscr{T}$ by $\omega_{\mathscr{T}}(T)$. More precisely,

$$\mathscr{N}_{\mathscr{T}}(T) := \{T' \in \mathscr{T} \mid T' \cap T \neq \emptyset\}, \qquad \omega_{\mathscr{T}}(T) := \bigcup_{T'\in\mathscr{N}_{\mathscr{T}}(T)} T'.$$

For the discretization we consider the Lagrange finite element spaces consisting of continuous functions vanishing on $\partial\Omega$ which are piecewise linear over a mesh $\mathscr{T} \in \mathbb{T}$, i.e.,

$$\mathbb{V}_{\mathscr{T}} := \{V \in H_0^1(\Omega) \mid \quad V_{|_T} \in \mathscr{P}_1(T), \quad \forall\, T \in \mathscr{T}\}. \tag{3.1}$$

The discrete problem associated to (2.1) consists in finding $U \in \mathbb{V}_{\mathscr{T}}$ such that

$$a(U; U, V) = L(V), \qquad \forall\, V \in \mathbb{V}_{\mathscr{T}}. \tag{3.2}$$

Note that the discrete problem (3.2) has a unique solution because $A_{|_{\mathbb{V}_{\mathscr{T}}}}$ is Lipschitz and strongly monotone (cf. (2.6)–(2.7)).

At this point, it is important to remark that the discrete problem (3.2) is also *nonlinear*, and for our analysis we will assume that it can be solved exactly in every mesh $\mathcal{T} \in \mathbb{T}$. However, this assumption is usual even though in practice, even for discrete *linear* problems, we *compute* only approximations to the solution of discrete problems. The optimality of inexact methods has been studied for linear problems in [17,22], and a generalization to nonlinear problems is subject of future work.

3.2. A posteriori error estimators

In this section we present the *a posteriori error estimators* for the discrete approximation (3.2) of problem (2.1) and state results showing their *reliability* and *efficiency*. These estimations will be useful in order to prove the optimality of the AFEM in Section 6.

The *residual* of $V \in \mathbb{V}_{\mathcal{T}}$ is given by

$$\langle \mathbf{R}(V), v\rangle := a(V;V,v) - L(v), \qquad \forall\, v \in H_0^1(\Omega).$$

Integrating by parts on each $T \in \mathcal{T}$ we have that

$$\langle \mathbf{R}(V), v\rangle = \sum_{T\in\mathcal{T}} \left(\int_T R_{\mathcal{T}}(V)v + \int_{\partial T} J_{\mathcal{T}}(V)v \right), \qquad \forall\, v \in H_0^1(\Omega), \tag{3.3}$$

where $R_{\mathcal{T}}(V)$ denotes the *element residual* given by

$$R_{\mathcal{T}}(V)_{|_T} := -\nabla\cdot[\alpha(\cdot,|\nabla V|^2)\nabla V] - f, \qquad \forall\, T \in \mathcal{T}, \tag{3.4}$$

and $J_{\mathcal{T}}(V)$ the *jump residual* given by

$$J_{\mathcal{T}}(V)_{|_S} := \frac{1}{2}\left[(\alpha(\cdot,|\nabla V|^2)\nabla V)_{|_{T_1}}\cdot \vec{n}_1 + (\alpha(\cdot,|\nabla V|^2)\nabla V)_{|_{T_2}}\cdot \vec{n}_2 \right], \tag{3.5}$$

for each interior side S, and $J_{\mathcal{T}}(V)_{|_S} := 0$, if S is a side lying on the boundary of Ω. Here, T_1 and T_2 denote the elements of $\mathcal{T}$ sharing S, and $\vec{n}_1$ and $\vec{n}_1$ are the outward unit normals of T_1 and T_2 on S, respectively.

We define the *local a posteriori error estimator* $\eta_{\mathcal{T}}(V;T)$ of $V \in \mathbb{V}_{\mathcal{T}}$ by

$$\eta^2_{\mathcal{T}}(V;T) := H_T^2 \left\|R_{\mathcal{T}}(V)\right\|_T^2 + H_T \left\|J_{\mathcal{T}}(V)\right\|_{\partial T}^2, \quad \forall\, T \in \mathcal{T}, \tag{3.6}$$

and the *global error estimator* $\eta_{\mathcal{T}}(V)$ by

$$\eta^2_{\mathcal{T}}(V) := \sum_{T\in\mathcal{T}} \eta^2_{\mathcal{T}}(V;T).$$

In general, if $\Xi \subset \mathcal{T}$ we denote $\left(\sum_{T\in\Xi} \eta^2_{\mathcal{T}}(V;T)\right)^{\frac{1}{2}}$ by $\eta_{\mathcal{T}}(V;\Xi)$.

Recall that if $V \in \mathbb{V}_{\mathcal{T}}$ is the Scott-Zhang interpolant of $v \in H_0^1(\Omega)$ then

$$\|v - V\|_T + H_T^{1/2}\|v - V\|_{\partial T} \lesssim H_T \|\nabla v\|_{\omega_{\mathcal{T}}(T)}, \qquad \forall\, T \in \mathcal{T}.$$

Notice that $\langle \mathbf{R}(U), V\rangle = 0$ and therefore $\langle \mathbf{R}(U), v\rangle = \langle \mathbf{R}(U), v - V\rangle$ because $V \in \mathbb{V}_{\mathcal{T}}$ (cf. (3.2)). Using (3.3), Hölder's and Cauchy-Schwartz's inequalities and the definition (3.6) we obtain:

$$|\langle \mathbf{R}(U), v\rangle| \lesssim \sum_{T\in\mathcal{T}} \eta_{\mathcal{T}}(U;T)\, \|\nabla v\|_{\omega_{\mathcal{T}}(T)}, \qquad \forall\, v \in H_0^1(\Omega). \tag{3.7}$$

The next lemma establishes a local lower bound for the error. Its proof follows the usual techniques taking into account that if u denotes the solution of problem (2.1),

$$|\langle \mathbf{R}(V), v\rangle| = |a(V;V,v) - L(v)| = |a(V;V,v) - a(u;u,v)| \le C_A \|\nabla(V-u)\|_{\omega} \|\nabla v\|_{\omega},$$

for $V \in \mathbb{V}_{\mathcal{T}}$, whenever $v \in H_0^1(\Omega)$ vanishes outside of ω, for any $\omega \subset \overline{\Omega}$.

Lemma 3.1 (Local lower bound). *Let $u \in H_0^1(\Omega)$ be the solution of problem* (2.1). *Let $\mathcal{T} \in \mathbb{T}$ and $T \in \mathcal{T}$ be fixed. If $V \in \mathbb{V}_{\mathcal{T}}$,*[‡]

$$\begin{aligned}\eta_{\mathcal{T}}(V;T) \lesssim \|\nabla(V-u)\|_{\omega_{\mathcal{T}}(T)} + H_T \left\| R_{\mathcal{T}}(V) - \overline{R_{\mathcal{T}}(V)} \right\|_{\omega_{\mathcal{T}}(T)} \\ + H_T^{\frac{1}{2}} \left\| J_{\mathcal{T}}(V) - \overline{J_{\mathcal{T}}(V)} \right\|_{\partial T},\end{aligned} \tag{3.8}$$

where $\overline{R_{\mathcal{T}}(V)}_{|_{T'}}$ denotes the mean value of $R_{\mathcal{T}}(V)$ on T', for all $T' \in \mathcal{N}_{\mathcal{T}}(T)$, and for each side $S \subset \partial T$, $\overline{J_{\mathcal{T}}(V)}_{|_S}$ denotes the mean value of $J_{\mathcal{T}}(V)$ on S.

The last result is known as *local efficiency of the error estimator*. According to the lemma, if a local estimator is large, then so is the corresponding local error, provided the last two terms in the right-hand side of (3.8) are relatively small.

We define the *local oscillation* corresponding to $V \in \mathbb{V}_{\mathcal{T}}$ by

$$\operatorname{osc}^2_{\mathcal{T}}(V;T) := H_T^2 \left\| R_{\mathcal{T}}(V) - \overline{R_{\mathcal{T}}(V)} \right\|_T^2 + H_T \left\| J_{\mathcal{T}}(V) - \overline{J_{\mathcal{T}}(V)} \right\|_{\partial T}^2, \quad \forall\, T \in \mathcal{T},$$

and the *global oscillation* by

$$\operatorname{osc}^2_{\mathcal{T}}(V) := \sum_{T\in\mathcal{T}} \operatorname{osc}^2_{\mathcal{T}}(V;T).$$

In general, if $\Xi \subset \mathcal{T}$ we denote $\left(\sum_{T\in\Xi} \operatorname{osc}^2_{\mathcal{T}}(V;T)\right)^{\frac{1}{2}}$ by $\operatorname{osc}_{\mathcal{T}}(V;\Xi)$.

As an immediate consequence of the last lemma, adding over all elements in the mesh we obtain the following

Theorem 3.1 (Global lower bound). *Let $u \in H_0^1(\Omega)$ denote the solution of problem* (2.1). *Then, there exists a constant $C_L = C_L(d, \kappa_{\mathbb{T}}, C_A) > 0$ such that*

$$C_L \eta^2_{\mathcal{T}}(V) \le \|\nabla(V-u)\|^2_{\Omega} + \operatorname{osc}^2_{\mathcal{T}}(V), \qquad \forall\, V \in \mathbb{V}_{\mathcal{T}}, \quad \forall\, \mathcal{T} \in \mathbb{T}.$$

[‡]From now on, we will write $a \lesssim b$ to indicate that $a \le Cb$ with $C > 0$ a constant depending on the data of the problem and possibly on shape regularity $\kappa_{\mathbb{T}}$ of the meshes. Also $a \simeq b$ will indicate that $a \lesssim b$ and $b \lesssim a$.

We conclude this section with two upper estimations for the error.

Theorem 3.2 (Global upper bound). *Let $u \in H_0^1(\Omega)$ be the solution of problem* (2.1). *Let $\mathscr{T} \in \mathbb{T}$ and let $U \in \mathbb{V}_{\mathscr{T}}$ be the solution of the discrete problem* (3.2). *Then, there exists $C_U = C_U(d, \kappa_{\mathbb{T}}, c_A) > 0$ such that*

$$\|\nabla(U-u)\|_\Omega^2 \le C_U \eta_{\mathscr{T}}^2(U). \tag{3.9}$$

Proof. Let $u \in H_0^1(\Omega)$ be the solution of problem (2.1). Let $\mathscr{T} \in \mathbb{T}$ and let $U \in \mathbb{V}_{\mathscr{T}}$ be the solution of the discrete problem (3.2). Since A is strongly monotone (cf. (2.7)), and u is the solution of problem (2.1) we have

$$c_A \|\nabla(U-u)\|_\Omega^2 \le \langle AU - Au, U-u\rangle = a(U;U,U-u) - L(U-u) = \langle \mathbf{R}(U), U-u\rangle.$$

Using (3.7) with $v = U-u$ the assertion (3.9) follows with $C_U = C_U(d, \kappa_{\mathbb{T}}, c_A) > 0$. □

Theorem 3.3 (Localized upper bound). *Let $\mathscr{T} \in \mathbb{T}$ and let $\mathscr{T}_* \in \mathbb{T}$ be a refinement of $\mathscr{T}$. Let $\mathscr{R}$ denote the subset of $\mathscr{T}$ consisting of the elements which are refined to obtain $\mathscr{T}_*$, that is, $\mathscr{R} := \{T \in \mathscr{T} \mid T \notin \mathscr{T}_*\}$. Let $U \in \mathbb{V}_{\mathscr{T}}$ and $U_* \in \mathbb{V}_{\mathscr{T}_*}$ be the solutions of the discrete problem* (3.2) *in $\mathbb{V}_{\mathscr{T}}$ and $\mathbb{V}_{\mathscr{T}_*}$, respectively. Then, there exists a constant $C_{LU} = C_{LU}(d, \kappa_{\mathbb{T}}, c_A) > 0$ such that*

$$\|\nabla(U-U_*)\|_\Omega^2 \le C_{LU} \eta_{\mathscr{T}}^2(U;\mathscr{R}). \tag{3.10}$$

Proof. Let $\mathscr{T}$, $\mathscr{T}_*$, $\mathscr{R}$, U and U_* be as in the assumptions of the theorem. Analogously to the last proof, using that A is strongly monotone and that U_* is the solution of problem (3.2) in $\mathbb{V}_{\mathscr{T}_*}$ we have that

$$\begin{aligned} c_A \|\nabla(U-U_*)\|_\Omega^2 &\le \langle AU - AU_*, U-U_*\rangle \\ &= a(U;U,U-U_*) - L(U-U_*) = \langle \mathbf{R}(U), U-U_*\rangle. \end{aligned} \tag{3.11}$$

Now, we build, using the Scott-Zhang operator, an approximation $V \in \mathbb{V}_{\mathscr{T}}$ of $U - U_*$ that coincides with $U-U_*$ over all unrefined elements $T \in \mathscr{T} \setminus \mathscr{R}$, and satisfies (see [2] for details)

$$\|(U-U_*) - V\|_T + H_T^{1/2} \|(U-U_*) - V\|_{\partial T} \lesssim \begin{cases} H_T \|\nabla(U-U_*)\|_{\omega_{\mathscr{T}}(T)} & \text{if } T \in \mathscr{R}, \\ 0 & \text{if } T \in \mathscr{T} \setminus \mathscr{R}. \end{cases}$$

Since $V \in \mathbb{V}_{\mathscr{T}}$, $\langle \mathbf{R}(U), U-U_*\rangle = \langle \mathbf{R}(U), (U-U_*) - V\rangle$ (cf. (3.2)). Using (3.3), Hölder's and Cauchy-Schwartz's inequalities and the definition (3.6) we obtain:

$$|\langle \mathbf{R}(U), U-U_*\rangle| \lesssim \sum_{T \in \mathscr{R}} \eta_{\mathscr{T}}(U;T) \|\nabla(U-U_*)\|_{\omega_{\mathscr{T}}(T)}. \tag{3.12}$$

Finally, from (3.11) and (3.12) the assertion (3.10) follows with $C_{LU} = C_{LU}(d, \kappa_{\mathbb{T}}, c_A) > 0$. □

3.3. Estimator reduction and perturbation of oscillation

In order to prove the contraction property it is necessary to study the effects that refinement has upon the error estimators and oscillation terms. We thus present two main results in this section. The first one is related to the error estimator and it will be used in Theorem 4.2.

Proposition 3.1 (Estimator reduction). *Let $\mathscr{T} \in \mathbb{T}$ and let $\mathscr{M}$ be any subset of $\mathscr{T}$. Let $\mathscr{T}_* \in \mathbb{T}$ be obtained from $\mathscr{T}$ by bisecting at least $n \geq 1$ times each element in $\mathscr{M}$. If $V \in \mathbb{V}_{\mathscr{T}}$ and $V_* \in \mathbb{V}_{\mathscr{T}_*}$, then*

$$\eta^2_{\mathscr{T}_*}(V_*) \leq (1+\delta)\left\{\eta^2_{\mathscr{T}}(V) - (1-2^{-\frac{n}{d}})\eta^2_{\mathscr{T}}(V;\mathscr{M})\right\} + (1+\delta^{-1})C_E\|\nabla(V_* - V)\|^2_{\Omega},$$

for all $\delta > 0$, where $C_E > 1$ is a constant (cf. Lemma 3.2 below).

The second result is related to the oscillation terms. It will be used to establish the *quasi-optimality for the error* (see Lemma 5.2) and to prove Lemma 5.3 in the next section.

Proposition 3.2 (Oscillation perturbation). *Let $\mathscr{T} \in \mathbb{T}$ and let $\mathscr{T}_* \in \mathbb{T}$ be a refinement of $\mathscr{T}$. If $V \in \mathbb{V}_{\mathscr{T}}$ and $V_* \in \mathbb{V}_{\mathscr{T}_*}$, then*

$$\operatorname{osc}^2_{\mathscr{T}}(V;\mathscr{T}\cap\mathscr{T}_*) \leq 2\operatorname{osc}^2_{\mathscr{T}_*}(V_*;\mathscr{T}\cap\mathscr{T}_*) + 2C_E\|\nabla(V_* - V)\|^2_{\Omega},$$

where $C_E > 1$ is a constant (cf. Lemma 3.2 below).

In order to prove Propositions 3.1 and 3.2 we observe that if we define for $\mathscr{T} \in \mathbb{T}$ and $V, W \in \mathbb{V}_{\mathscr{T}}$

$$g_{\mathscr{T}}(V,W;T) := H_T \left\|R_{\mathscr{T}}(V) - R_{\mathscr{T}}(W)\right\|_T + H_T^{\frac{1}{2}} \left\|J_{\mathscr{T}}(V) - J_{\mathscr{T}}(W)\right\|_{\partial T}, \tag{3.13}$$

then from the definition of the local error estimators (3.6) and the triangle inequality it follows that

$$\eta_{\mathscr{T}}(W;T) \leq \eta_{\mathscr{T}}(V;T) + g_{\mathscr{T}}(V,W;T), \qquad \forall\, T \in \mathscr{T}, \tag{3.14}$$

and analogously

$$\operatorname{osc}_{\mathscr{T}}(W;T) \leq \operatorname{osc}_{\mathscr{T}}(V;T) + g_{\mathscr{T}}(V,W;T), \qquad \forall\, T \in \mathscr{T}. \tag{3.15}$$

After proving that $g_{\mathscr{T}}(V,W;T)$ is bounded by $\|\nabla(V-W)\|_{\omega_{\mathscr{T}}(T)}$, the first terms on the right-hand sides of (3.14) and (3.15) may be treated as in [2, Corollary 3.4 and Corollary 3.5] for linear elliptic problems, respectively, and the assertions of Propositions 3.1 and 3.2 follow. On the other hand, while proving that $g_{\mathscr{T}}(V,W;T) \lesssim \|\nabla(V-W)\|_{\omega_{\mathscr{T}}(T)}$ is easy for linear problems by using inverse inequalities and trace theorems, it is not so obvious for nonlinear problems. Therefore, we omit the details of the proofs of the last two propositions, but we prove the following lemma, which is the main difference with linear problems [2].

Lemma 3.2. *Let $\mathscr{T} \in \mathbb{T}$ and let $g_{\mathscr{T}}$ be given by* (3.13). *Then, there holds that*

$$g_{\mathscr{T}}(V, W; T) \lesssim \|\nabla(V - W)\|_{\omega_{\mathscr{T}}(T)}, \qquad \forall\, V, W \in \mathbb{V}_{\mathscr{T}}, \quad \forall\, T \in \mathscr{T}. \tag{3.16}$$

Consequently, there exists a constant $C_E > 1$ which depends on d, $\kappa_{\mathbb{T}}$ and the problem data, such that

$$\sum_{T \in \mathscr{T}} g_{\mathscr{T}}^2(V, W; T) \le C_E \|\nabla(V - W)\|_{\Omega}^2, \qquad \forall\, V, W \in \mathbb{V}_{\mathscr{T}}. \tag{3.17}$$

In order to prove Lemma 3.2, we define

$$\Gamma_V(x) := \nabla_2 \gamma(x, \nabla V(x)) = \alpha(x, |\nabla V(x)|^2) \nabla V(x), \qquad \forall\, x \in \Omega, \tag{3.18}$$

and prove first the following auxiliary result.

Lemma 3.3. *Let $T \in \mathscr{T}$. Let $D_2^2 \gamma$ be the Hessian matrix of γ as a function of its second variable. If*

$$\|D_2^2 \gamma(x, \xi) - D_2^2 \gamma(y, \xi)\|_2 \le C_\gamma |x - y|, \qquad \forall\, x, y \in T,\ \xi \in \mathbb{R}^d,$$

for some constant $C_\gamma > 0$, then for all $V, W \in \mathscr{P}_1(T)$, there holds that

$$|\Gamma_V(x) - \Gamma_W(x) - \Gamma_V(y) + \Gamma_W(y)| \le C_\gamma \|\nabla(V - W)\|_{L^\infty(T)} |x - y|, \qquad \forall\, x, y \in T.$$

Proof. Let $T \in \mathscr{T}$. Let $V, W \in \mathscr{P}_1(T)$ and $x, y \in T$. Taking into account that V and W are linear over T, we denote $\mathbf{v} := \nabla V(x) = \nabla V(y)$ and $\mathbf{w} := \nabla W(x) = \nabla W(y)$. Thus, we have that

$$\begin{aligned}
&|\Gamma_V(x) - \Gamma_W(x) - \Gamma_V(y) + \Gamma_W(y)| \\
&= |\nabla_2 \gamma(x, \mathbf{v}) - \nabla_2 \gamma(x, \mathbf{w}) - \nabla_2 \gamma(y, \mathbf{v}) + \nabla_2 \gamma(y, \mathbf{w})| \\
&= \left| \int_0^1 \left[D_2^2 \gamma(x, \mathbf{w} + r(\mathbf{v} - \mathbf{w})) - D_2^2 \gamma(y, \mathbf{w} + r(\mathbf{v} - \mathbf{w})) \right] (\mathbf{v} - \mathbf{w})\, dr \right| \\
&\le C_\gamma |x - y| |\mathbf{v} - \mathbf{w}|,
\end{aligned}$$

which completes the proof of the lemma. □

We conclude this section with the proof of Lemma 3.2, where we use that

$$R_{\mathscr{T}}(V)_{|_T} = -\nabla \cdot \Gamma_V - f, \qquad \text{and} \qquad J_{\mathscr{T}}(V)_{|_S} = \frac{1}{2} \left(\Gamma_{V|_{T_1}} \cdot \vec{n}_1 + \Gamma_{V|_{T_2}} \cdot \vec{n}_2 \right), \quad S \subset \Omega,$$

which is an immediate consequence of (3.18) and the definitions of the element residual (3.4) and the jump residual (3.5).

Proof. [Proof of Lemma 3.2] [1] Taking into account (2.5), we have that, if $x \in \Omega$ and $\xi \in \mathbb{R}^d$,

$$(D_2^2 \gamma(x, \xi))_{ij} = 2 D_2 \alpha(x, |\xi|^2) \xi_i \xi_j + \alpha(x, |\xi|^2) \delta_{ij},$$

for $1 \le i, j \le d$, where δ_{ij} denotes the Kronecker's delta. Assumption (2.8) then implies that $D_2^2\gamma(x,\xi)$ is piecewise Lipschitz as a function of its first variable, i.e., there exists a constant $C_\gamma > 0$ such that

$$\|D_2^2\gamma(x,\xi) - D_2^2\gamma(y,\xi)\|_2 \le C_\gamma |x-y|, \qquad \forall\, x, y \in T,\ \xi \in \mathbb{R}^d,$$

for all $T \in \mathcal{T}_0$. In particular this holds for any $T \in \mathcal{T}$, $\mathcal{T} \in \mathbb{T}$, and the assumptions of Lemma 3.3 hold.

[2] Let $\mathcal{T} \in \mathbb{T}$, let $V, W \in \mathbb{V}_{\mathcal{T}}$ and let $T \in \mathcal{T}$ be fixed. By Lemma 3.3, for the element residual we have that

$$\begin{aligned}\left\|R_{\mathcal{T}}(V) - R_{\mathcal{T}}(W)\right\|_T = \left\|\nabla\cdot(\Gamma_V - \Gamma_W)\right\|_T &\le H_T^{\frac{d}{2}} \|\nabla\cdot(\Gamma_V - \Gamma_W)\|_{L^\infty(T)} \\ &\lesssim H_T^{\frac{d}{2}} \sup_{\substack{x,y\in T\\ x\ne y}} \frac{|\Gamma_V(x) - \Gamma_W(x) - \Gamma_V(y) + \Gamma_W(y)|}{|x-y|} \\ &\lesssim H_T^{\frac{d}{2}} \|\nabla(V-W)\|_{L^\infty(T)} = \|\nabla(V-W)\|_T,\end{aligned}$$

and thus,

$$H_T \left\|R_{\mathcal{T}}(V) - R_{\mathcal{T}}(W)\right\|_T \lesssim \|\nabla(V-W)\|_T. \tag{3.19}$$

[3] Consider now the term corresponding to the jump residual. If S is a side of T which is interior to Ω and if T_1 and T_2 are the elements sharing S, we have that

$$\begin{aligned}\left\|J_{\mathcal{T}}(V) - J_{\mathcal{T}}(W)\right\|_S = \left\|\frac{1}{2}\sum_{i=1,2} (\Gamma_V - \Gamma_W)_{|T_i} \cdot \vec{n}_i\right\|_S &\le \sum_{i=1,2} \left\|(\Gamma_V - \Gamma_W)_{|T_i}\right\|_S \\ &\lesssim \sum_{i=1,2} \left(H_T^{-\frac{1}{2}} \|\Gamma_V - \Gamma_W\|_{T_i} + H_T^{\frac{1}{2}} \|\nabla(\Gamma_V - \Gamma_W)\|_{T_i}\right),\end{aligned}$$

where we have used a scaled trace theorem. Since $\nabla_2\gamma$ is Lipschitz as a function of its second variable, we have that

$$|\Gamma_V(x) - \Gamma_W(x)| = |\nabla_2\gamma(x, \nabla V(x)) - \nabla_2\gamma(x, \nabla W(x))| \lesssim |\nabla V(x) - \nabla W(x)|,$$

for $x \in T_i$ $(i = 1, 2)$, and therefore,

$$\|\Gamma_V - \Gamma_W\|_{T_i} \lesssim \|\nabla(V-W)\|_{T_i}, \qquad i = 1, 2.$$

Using the same argument as in [2], we have that

$$\|\nabla(\Gamma_V - \Gamma_W)\|_{T_i} \lesssim \|\nabla(V-W)\|_{T_i}, \quad \text{for } i = 1, 2,$$

and in consequence,

$$H_T^{\frac{1}{2}} \left\|J_{\mathcal{T}}(V) - J_{\mathcal{T}}(W)\right\|_{\partial T} \lesssim \|\nabla(V-W)\|_{\omega_{\mathcal{T}}(T)}. \tag{3.20}$$

Finally, (3.16) follows from (3.19) and (3.20), taking into account (3.13). $\square$

4. Linear convergence of an adaptive FEM

In this section we present the adaptive FEM and establish one of the main results of this article (Theorem 4.2 below) which guarantees the convergence of the adaptive sequence.

4.1. The adaptive loop

We consider the following adaptive loop to approximate the solution u of problem (2.1).

Adaptive Algorithm. Let $\mathcal{T}_0$ be an initial conforming mesh of Ω and let θ be a parameter satisfying $0 < \theta < 1$. Let $k = 0$.

1. $U_k := \mathsf{SOLVE}(\mathcal{T}_k)$.
2. $\{\eta_k(T)\}_{T\in\mathcal{T}_k} := \mathsf{ESTIMATE}(U_k, \mathcal{T}_k)$.
3. $\mathcal{M}_k := \mathsf{MARK}(\{\eta_k(T)\}_{T\in\mathcal{T}_k}, \mathcal{T}_k, \theta)$.
4. $\mathcal{T}_{k+1} := \mathsf{REFINE}(\mathcal{T}_k, \mathcal{M}_k, n)$.
5. Increment k and go back to step 1.

Now we explain each module in the last algorithm.

- **The module SOLVE.** This module takes a conforming triangulation $\mathcal{T}_k$ of Ω as input argument and outputs the solution U_k of the discrete problem (3.2) in $\mathcal{T}_k$; i.e., $U_k \in \mathbb{V}_k := \mathbb{V}_{\mathcal{T}_k}$ satisfies

$$a(U_k; U_k, V) = L(V), \qquad \forall\, V \in \mathbb{V}_k.$$

- **The module ESTIMATE.** This module computes the a posteriori local error estimators $\eta_k(T)$ of U_k over $\mathcal{T}_k$ given by $\eta_k(T) := \eta_{\mathcal{T}_k}(U_k; T)$, for all $T \in \mathcal{T}_k$, (see (3.6)).
- **The module MARK.** Based on the local error estimators, the module MARK selects a subset $\mathcal{M}_k$ of $\mathcal{T}_k$, using an *efficient Dörfler's strategy*. More precisely, given the marking parameter $\theta \in (0,1)$, the module MARK selects a *minimal* subset $\mathcal{M}_k$ of $\mathcal{T}_k$ such that

$$\eta_k(\mathcal{M}_k) \geq \theta\, \eta_k(\mathcal{T}_k), \tag{4.1}$$

  where $\eta_k(\mathcal{M}_k) = \left(\sum_{T\in\mathcal{M}_k} \eta_k^2(T)\right)^{\frac{1}{2}}$ and $\eta_k(\mathcal{T}_k) = \left(\sum_{T\in\mathcal{T}_k} \eta_k^2(T)\right)^{\frac{1}{2}}$.
- **The module REFINE.** Finally, the module REFINE takes the mesh $\mathcal{T}_k$ and the subset $\mathcal{M}_k \subset \mathcal{T}_k$ as inputs. By using the bisection rule described by Stevenson in [23], this module refines (bisects) n times (where $n \geq 1$ is fixed) each element in $\mathcal{M}_k$.

After that, with the goal of keeping conformity of the mesh, possibly some further bisections are performed leading to a new conforming triangulation $\mathscr{T}_{k+1} \in \mathbb{T}$ of Ω, which is a refinement of $\mathscr{T}_k$ and the output of this module.

From now on, U_k, $\{\eta_k(T)\}_{T\in\mathscr{T}_k}$, $\mathscr{M}_k$, $\mathscr{T}_k$ will denote the outputs of the corresponding modules SOLVE, ESTIMATE, MARK and REFINE, when iterated after starting with a given initial mesh $\mathscr{T}_0$.

4.2. An equivalent notion for the error

In order to prove a contraction property for the error of a similar AFEM for linear elliptic problems the well-known *Galerkin orthogonality relationship* is used(see [2]). In this case, due to the nonlinearity of our problem, this property does not hold. We present an equivalent notion of error so that it is possible to establish a property analogous to the orthogonality (cf. (4.9) below).

It is easy to check that $\mathscr{J} : H_0^1(\Omega) \to \mathbb{R}$ given by

$$\mathscr{J}(v) := \int_0^1 \langle A(rv), v\rangle \, dr = \int_\Omega \gamma(\cdot, \nabla v)\, dx, \quad \forall\, v \in H_0^1(\Omega),$$

is a potential for the operator A. More precisely, if $\mathbb{W}$ is a closed subspace of $H_0^1(\Omega)$, the following claims are equivalent

- $w \in \mathbb{W}$ is solution of
$$a(w; w, v) = L(v), \qquad \forall\, v \in \mathbb{W}, \tag{4.2}$$
where $L(v) = \int_\Omega f v$, for $v \in H_0^1(\Omega)$.
- $w \in \mathbb{W}$ minimizes the functional $\mathscr{F} : H_0^1(\Omega) \to \mathbb{R}$ over $\mathbb{W}$, where $\mathscr{F}$ is given by
$$\mathscr{F}(v) := \mathscr{J}(v) - L(v) = \int_\Omega \gamma(\cdot, \nabla v) - f v \, dx, \quad v \in H_0^1(\Omega). \tag{4.3}$$

The following theorem states a notion equivalent to the $H_0^1(\Omega)$-error. The proof follows the ideas used in [6] and uses that the Hessian matrix of γ, denoted by $D_2^2\gamma$, is uniformly elliptic, i.e.,

$$c_A|\zeta|^2 \le D_2^2\gamma(x,\xi)\zeta \cdot \zeta \le C_A|\zeta|^2, \qquad \forall\, x \in \Omega,\ \xi, \zeta \in \mathbb{R}^d. \tag{4.4}$$

This fact holds because $\nabla_2\gamma$ is Lipschitz and strongly monotone as a function of its second variable.

Theorem 4.1. *Let $\mathbb{W}$ be a closed subspace of $H_0^1(\Omega)$ and let $\mathscr{F}$ be given by* (4.3). *If $w \in \mathbb{W}$ satisfies* (4.2)*, then*

$$\frac{c_A}{2}\|\nabla(v-w)\|_\Omega^2 \le \mathscr{F}(v) - \mathscr{F}(w) \le \frac{C_A}{2}\|\nabla(v-w)\|_\Omega^2, \quad \forall\, v \in \mathbb{W}.$$

Proof. Let $\mathbb{W}$ be a closed subspace of $H_0^1(\Omega)$ and let $w \in \mathbb{W}$ be the solution of (4.2). Let $v \in \mathbb{W}$ be fixed and arbitrary. For $z \in \mathbb{R}$, we define $\phi(z) := (1-z)w + zv$, and note that

$$\phi'(z) = v - w \qquad \text{and} \qquad \nabla\phi(z) = (1-z)\nabla w + z\nabla v.$$

If we define $\psi(z) := \mathscr{F}(\phi(z))$, integration by parts yields

$$\mathscr{F}(v) - \mathscr{F}(w) = \psi(1) - \psi(0) = \psi'(0) + \int_0^1 \psi''(z)(1-z)\,dz. \tag{4.5}$$

From (4.3) it follows that

$$\psi(z) = \mathscr{F}(\phi(z)) = \int_\Omega \gamma(x, \nabla\phi(z))\,dx - \int_\Omega f\phi(z)\,dx, \tag{4.6}$$

and therefore, in order to obtain the derivatives of ψ we first compute $\frac{\partial}{\partial z}(\gamma(x, \nabla\phi(z)))$, for each $x \in \Omega$ fixed. On the one hand, we have that

$$\frac{\partial}{\partial z}\gamma(\cdot, \nabla\phi(z)) = \nabla_2\gamma(\cdot, \nabla\phi(z)) \cdot \frac{\partial}{\partial z}\nabla\phi(z) = \nabla_2\gamma(\cdot, \nabla\phi(z)) \cdot \nabla(v-w),$$

and then

$$\frac{\partial^2}{\partial z^2}\gamma(\cdot, \nabla\phi(z)) = D_2^2\gamma(\cdot, \nabla\phi(z))\nabla(v-w)\cdot\nabla(v-w),$$

where $D_2^2\gamma$ is the Hessian matrix of γ as a function of its second variable. Thus, taking into account that $\phi''(z) = 0$ for all $z \in \mathbb{R}$, from (4.6) it follows that

$$\psi''(z) = \int_\Omega D_2^2\gamma(x, \nabla\phi(z))\nabla(v-w)\cdot\nabla(v-w)\,dx. \tag{4.7}$$

Since w minimizes $\mathscr{F}$ over $\mathbb{W}$, we have that $\psi'(0) = 0$; and using (4.7), from (4.5) we obtain that

$$\mathscr{F}(v) - \mathscr{F}(w) = \int_0^1 \int_\Omega D_2^2\gamma(x, \nabla\phi(z))\nabla(v-w)\cdot\nabla(v-w)(1-z)\,dx\,dz.$$

Finally, since $D_2^2\gamma$ is uniformly elliptic (cf. (4.4)) we have that

$$\begin{aligned}\frac{c_A}{2}\|\nabla(v-w)\|_\Omega^2 &\le \int_0^1 \int_\Omega D_2^2\gamma(x, \nabla\phi(z))\nabla(v-w)\cdot\nabla(v-w)(1-z)\,dx\,dz \\ &\le \frac{C_A}{2}\|\nabla(v-w)\|_\Omega^2,\end{aligned}$$

which concludes the proof. □

As an immediate consequence of the last theorem,

$$\frac{c_A}{2}\|\nabla(U_k - U_p)\|_\Omega^2 \le \mathscr{F}(U_k) - \mathscr{F}(U_p) \le \frac{C_A}{2}\|\nabla(U_k - U_p)\|_\Omega^2, \qquad \forall\, k, p \in \mathbb{N}_0,\ k < p, \tag{4.8}$$

and the same estimation holds replacing U_p by u, the exact weak solution of problem (2.1).

4.3. Convergence of the adaptive FEM

Recall that u denotes the exact weak solution of problem (2.1), and U_k, $\{\eta_k(T)\}_{T\in\mathscr{T}_k}$, $\mathscr{M}_k$, $\mathscr{T}_k$ will denote the outputs of the corresponding modules SOLVE, ESTIMATE, MARK and REFINE of the Adaptive Algorithm when iterated after starting with a given initial mesh $\mathscr{T}_0$.

Taking into account the estimator reduction (Proposition 3.1), the global upper bound (Theorem 3.2) and (4.8), we now prove the following result which establish the convergence of the Adaptive Algorithm.

Theorem 4.2 (Contraction property). *There exist constants* $0<\rho<1$ *and* $\mu>0$ *which depend on* d, $\kappa_{\mathbb{T}}$, *of problem data, of number of refinements* n *performed on each marked element and the marking parameter* θ *such that*

$$[\mathscr{F}(U_{k+1})-\mathscr{F}(u)]+\mu\eta_{k+1}^2\le\rho^2([\mathscr{F}(U_k)-\mathscr{F}(u)]+\mu\eta_k^2),\qquad\forall\,k\in\mathbb{N}_0,$$

where $\eta_k:=\left(\sum_{T\in\mathscr{T}_k}\eta_k^2(T)\right)^{\frac{1}{2}}$ *denotes the global error estimator in* $\mathscr{T}_k$.

Proof. Let $k\in\mathbb{N}_0$, using that

$$\mathscr{F}(U_k)-\mathscr{F}(u)=\mathscr{F}(U_k)-\mathscr{F}(U_{k+1})+\mathscr{F}(U_{k+1})-\mathscr{F}(u),\tag{4.9}$$

and the estimator reduction given by Proposition 3.1 with $\mathscr{T}=\mathscr{T}_k$ and $\mathscr{T}_*=\mathscr{T}_{k+1}$ we have that

$$\begin{aligned}&[\mathscr{F}(U_{k+1})-\mathscr{F}(u)]+\mu\eta_{k+1}^2\\ \le&[\mathscr{F}(U_k)-\mathscr{F}(u)]-[\mathscr{F}(U_k)-\mathscr{F}(U_{k+1})]\\ &+(1+\delta)\mu\left\{\eta_k^2-\xi\eta_k^2(\mathscr{M}_k)\right\}+(1+\delta^{-1})C_E\mu\|\nabla(U_k-U_{k+1})\|_\Omega^2,\end{aligned}$$

for all $\delta,\mu>0$, where $\xi:=1-2^{-\frac{n}{d}}$ and $\eta_k^2(\mathscr{M}_k):=\sum_{T\in\mathscr{M}_k}\eta_k^2(T)$. By choosing $\mu:=\frac{c_A}{2(1+\delta^{-1})C_E}$, and using (4.8) it follows that

$$[\mathscr{F}(U_{k+1})-\mathscr{F}(u)]+\mu\eta_{k+1}^2\le[\mathscr{F}(U_k)-\mathscr{F}(u)]+(1+\delta)\mu\left\{\eta_k^2-\xi\eta_k^2(\mathscr{M}_k)\right\}.$$

Dörfler's strategy yields $\eta_k(\mathscr{M}_k)\ge\theta\eta_k$ and thus

$$\begin{aligned}&[\mathscr{F}(U_{k+1})-\mathscr{F}(u)]+\mu\eta_{k+1}^2\\ \le&[\mathscr{F}(U_k)-\mathscr{F}(u)]+(1+\delta)\mu\eta_k^2-(1+\delta)\mu\xi\theta^2\eta_k^2\\ =&[\mathscr{F}(U_k)-\mathscr{F}(u)]+(1+\delta)\mu\left(1-\frac{\xi\theta^2}{2}\right)\eta_k^2-(1+\delta)\mu\frac{\xi\theta^2}{2}\eta_k^2.\end{aligned}$$

Using (4.8), the global upper bound (Theorem 3.2) and that $(1+\delta)\mu=\frac{c_A\delta}{2C_E}$ it follows that

$$\begin{aligned}&[\mathscr{F}(U_{k+1})-\mathscr{F}(u)]+\mu\eta_{k+1}^2\\ \le&[\mathscr{F}(U_k)-\mathscr{F}(u)]+(1+\delta)\mu\left(1-\frac{\xi\theta^2}{2}\right)\eta_k^2-\frac{c_A\delta\xi\theta^2}{2C_UC_EC_A}[\mathscr{F}(U_k)-\mathscr{F}(u)].\end{aligned}$$

If we define

$$\rho_1^2(\delta) := \left(1 - \frac{c_A \delta \xi \theta^2}{2C_U C_E C_A}\right), \qquad \rho_2^2(\delta) := \left(1 - \frac{\xi\theta^2}{2}\right)(1+\delta),$$

we thus have that

$$[\mathscr{F}(U_{k+1}) - \mathscr{F}(u)] + \mu\eta_{k+1}^2 \le \rho_1^2(\delta)[\mathscr{F}(U_k) - \mathscr{F}(u)] + \mu\rho_2^2(\delta)\eta_k^2.$$

The proof concludes choosing $\delta > 0$ small enough to satisfy

$$0 < \rho := \max\{\rho_1(\delta), \rho_2(\delta)\} < 1. \qquad \square$$

The last result, coupled with (4.8) allows us to conclude that the sequence $\{U_k\}_{k\in\mathbb{N}_0}$ of discrete solutions obtained through the Adaptive Algorithm converges to the weak solution u of the nonlinear problem (2.1), and moreover, there exists $\rho \in (0,1)$ such that

$$\|\nabla(U_k - u)\|_\Omega \le C\rho^k, \qquad \forall\, k \in \mathbb{N}_0,$$

for some constant $C > 0$. Also, the global error estimators $\{\eta_k\}_{k\in\mathbb{N}_0}$ tend to zero, and in particular,

$$\eta_k \le C\rho^k, \qquad \forall\, k \in \mathbb{N}_0,$$

for some constant $C > 0$.

5. Optimality of the total error and optimal marking

In this section we introduce the notion of *total error*, we show an analogous of Cea's lemma for this new notion (see Lemma 5.2) and a result about *optimal marking* (see Lemma 5.3). Both of them will be very important to establish a control of marked elements in each step of the adaptive procedure (cf. Lemma 6.2 in Section 6).

We first present an auxiliary result that will allow us to show the analogous of Cea's lemma for the *total error*. Its proof is an immediate consequence of Theorem 4.1 and will thus be omitted.

Lemma 5.1 (Quasi-orthogonality property in a mesh). *If $U \in \mathbb{V}_{\mathscr{T}}$ denotes the solution of the discrete problem* (3.2) *for some $\mathscr{T} \in \mathbb{T}$, then*

$$\|\nabla(U-u)\|_\Omega^2 + \|\nabla(U-V)\|_\Omega^2 \le \frac{C_A}{c_A}\|\nabla(V-u)\|_\Omega^2, \qquad \forall\, V \in \mathbb{V}_{\mathscr{T}},$$

where C_A and c_A are the constants appearing in (2.6) *and* (2.7).

Since the global oscillation term is smaller than the global error estimator, that is, $\operatorname{osc}_{\mathscr{T}}(U) \le \eta_{\mathscr{T}}(U)$, using the global upper bound (Theorem 3.2), we have that

$$\|\nabla(U-u)\|_\Omega^2 + \operatorname{osc}_{\mathscr{T}}^2(U) \le (C_U + 1)\eta_{\mathscr{T}}^2(U),$$

whenever u is the solution of problem (2.1) and $U \in \mathbb{V}_{\mathscr{T}}$ is the solution of the discrete problem (3.2). Taking into account the global lower bound (Theorem 3.1) we obtain that

$$\eta_{\mathscr{T}}(U) \approx \left(\|\nabla(U-u)\|_{\Omega}^2 + \operatorname{osc}_{\mathscr{T}}^2(U)\right)^{\frac{1}{2}}.$$

The quantity on the right-hand side is called *total error*, and since adaptive methods are based on the *a posteriori error estimators*, the convergence rate is characterized through properties of the *total error*.

Remark 5.1. *(Cea's Lemma) Taking into account that A is Lipschitz and strongly monotone, it is easy to check that*

$$\|\nabla(U-u)\|_{\Omega} \le \frac{C_A}{c_A} \inf_{V \in \mathbb{V}_{\mathscr{T}}} \|\nabla(V-u)\|_{\Omega}.$$

This estimation is known as Cea's Lemma and shows that the approximation U is optimal (up to a constant) of the solution u from $\mathbb{V}_{\mathscr{T}}$.

A generalization of Cea's Lemma for the total error is given in the following

Lemma 5.2 (Cea's Lemma for the total error)**.** *If* $U \in \mathbb{V}_{\mathscr{T}}$ *denotes the solution of the discrete problem* (3.2) *for some* $\mathscr{T} \in \mathbb{T}$, *then*

$$\|\nabla(U-u)\|_{\Omega}^2 + \operatorname{osc}_{\mathscr{T}}^2(U) \le \frac{2C_E C_A}{c_A} \inf_{V \in \mathbb{V}_{\mathscr{T}}} \left(\|\nabla(V-u)\|_{\Omega}^2 + \operatorname{osc}_{\mathscr{T}}^2(V)\right),$$

where $C_E > 1$ *is the constant given in* (3.17).

Proof. Let $\mathscr{T} \in \mathbb{T}$ and let $U \in \mathbb{V}_{\mathscr{T}}$ be the solution of the discrete problem (3.2). If $V \in \mathbb{V}_{\mathscr{T}}$, using Proposition 3.2 with $\mathscr{T}_* = \mathscr{T}$ and Lemma 5.1 we have that

$$\begin{aligned}
\|\nabla(U-u)\|_{\Omega}^2 + \operatorname{osc}_{\mathscr{T}}^2(U) &\le \|\nabla(U-u)\|_{\Omega}^2 + 2\operatorname{osc}_{\mathscr{T}}^2(V) + 2C_E\|\nabla(V-U)\|_{\Omega}^2 \\
&\le 2C_E\frac{C_A}{c_A}\|\nabla(V-u)\|_{\Omega}^2 + 2\operatorname{osc}_{\mathscr{T}}^2(V) \\
&\le \frac{2C_E C_A}{c_A}\left(\|\nabla(V-u)\|_{\Omega}^2 + \operatorname{osc}_{\mathscr{T}}^2(V)\right).
\end{aligned}$$

Since $V \in \mathbb{V}_{\mathscr{T}}$ is arbitrary, the claim of this lemma follows. □

The following result establishes a link between nonlinear approximation theory and AFEM through Dörfler's marking strategy. Roughly speaking, it is a reciprocal to the contraction property (Theorem 4.2). More precisely, we prove that if there exists a suitable total error reduction from $\mathscr{T}$ to a refinement $\mathscr{T}_*$, then the error indicators of the refined elements from $\mathscr{T}$ must satisfy a Dörfler's property. In other words, Dörfler's marking and total error reduction are intimately connected. This result is known as *optimal marking* and was first proved for linear elliptic problems by Stevenson [22]. The notion of total error presented above was first introduced by Cascón et al. [2] for linear problems, together with the appropriate optimal marking result, which we mimic here.

In order to prove the *optimal marking* result we assume that the marking parameter θ satisfies

$$0 < \theta < \theta_0 := \left[\frac{C_L}{1 + 2C_{LU}(1 + C_E)} \right]^{1/2}, \tag{5.1}$$

where C_L, C_{LU} are the constants appearing in the global lower bound (Theorem 3.1) and in the localized upper bound (Theorem 3.3), respectively, and C_E is the constant appearing in (3.17).

Lemma 5.3 (Optimal marking). *Let $\mathcal{T} \in \mathbb{T}$ and let $\mathcal{T}_* \in \mathbb{T}$ be a refinement of $\mathcal{T}$. Let $\mathcal{R}$ denote the subset of $\mathcal{T}$ consisting of the elements which were refined to obtain $\mathcal{T}_*$, i.e., $\mathcal{R} = \mathcal{T} \setminus \mathcal{T}_*$. Assume that the marking parameter θ satisfies $0 < \theta < \theta_0$ and define $\nu := \frac{1}{2}\big(1 - \frac{\theta^2}{\theta_0^2}\big) > 0$. Let U and U_* be the solutions of the discrete problem* (3.2) *in $\mathbb{V}_{\mathcal{T}}$ and $\mathbb{V}_{\mathcal{T}_*}$, respectively. If*

$$\|\nabla(U_* - u)\|_\Omega^2 + \mathrm{osc}_{\mathcal{T}_*}^2(U_*) \leq \nu \left(\|\nabla(U - u)\|_\Omega^2 + \mathrm{osc}_{\mathcal{T}}^2(U) \right), \tag{5.2}$$

then

$$\eta_{\mathcal{T}}(U; \mathcal{R}) \geq \theta \eta_{\mathcal{T}}(U).$$

Proof. Let $\mathcal{T}$, $\mathcal{T}_*$, $\mathcal{R}$, U, U_*, θ and ν be as in the assumptions. Using (5.2) and the global lower bound (Theorem 3.1) we obtain that

$$\begin{aligned} (1 - 2\nu) C_L \eta_{\mathcal{T}}^2(U) &\leq (1 - 2\nu) \left(\|\nabla(U - u)\|_\Omega^2 + \mathrm{osc}_{\mathcal{T}}^2(U) \right) \\ &\leq \|\nabla(U - u)\|_\Omega^2 - 2\|\nabla(U_* - u)\|_\Omega^2 + \mathrm{osc}_{\mathcal{T}}^2(U) - 2\,\mathrm{osc}_{\mathcal{T}_*}^2(U_*). \end{aligned} \tag{5.3}$$

Since $\|\nabla(U - u)\|_\Omega \leq \|\nabla(U_* - u)\|_\Omega + \|\nabla(U_* - U)\|_\Omega$, we have that

$$\|\nabla(U - u)\|_\Omega^2 - 2\|\nabla(U_* - u)\|_\Omega^2 \leq 2\|\nabla(U_* - U)\|_\Omega^2. \tag{5.4}$$

Using Proposition 3.2 and that $\mathrm{osc}_{\mathcal{T}}^2(U; T) \leq \eta_{\mathcal{T}}^2(U; T)$ if $T \in \mathcal{R} = \mathcal{T} \setminus \mathcal{T}_*$, for the oscillation terms we obtain that

$$\mathrm{osc}_{\mathcal{T}}^2(U) - 2\,\mathrm{osc}_{\mathcal{T}_*}^2(U_*) \leq 2C_E \|\nabla(U_* - U)\|_\Omega^2 + \eta_{\mathcal{T}}^2(U; \mathcal{R}).$$

Taking into account (5.4) and the last inequality, from (5.3) it follows that

$$(1 - 2\nu) C_L \eta_{\mathcal{T}}^2(U) \leq 2\|\nabla(U - U_*)\|_\Omega^2 + 2C_E \|\nabla(U - U_*)\|_\Omega^2 + \eta_{\mathcal{T}}^2(U; \mathcal{R}),$$

and using the localized upper bound (Theorem 3.3) we have that

$$\begin{aligned} (1 - 2\nu) C_L \eta_{\mathcal{T}}^2(U) &\leq 2(1 + C_E) C_{LU} \eta_{\mathcal{T}}^2(U; \mathcal{R}) + \eta_{\mathcal{T}}^2(U; \mathcal{R}) \\ &= (1 + 2C_{LU}(1 + C_E)) \eta_{\mathcal{T}}^2(U; \mathcal{R}). \end{aligned}$$

Finally,

$$\frac{(1 - 2\nu) C_L}{1 + 2C_{LU}(1 + C_E)} \eta_{\mathcal{T}}^2(U) \leq \eta_{\mathcal{T}}^2(U; \mathcal{R}),$$

which completes the proof since $\frac{(1-2\nu)C_L}{1+2C_{LU}(1+C_E)} = (1 - 2\nu)\theta_0^2 = \theta^2$ by the definition of ν. □

6. Quasi-optimality of the adaptive FEM

In this section we state the second main result of this article, that is, the adaptive sequence computed through the Adaptive Algorithm converges with optimal rate to the weak solution of the nonlinear problem (2.1). For $N \in \mathbb{N}_0$, let $\mathbb{T}_N$ be the set of all possible conforming triangulations generated by refinement from $\mathscr{T}_0$ with at most N elements more than $\mathscr{T}_0$, i.e.,

$$\mathbb{T}_N := \{\mathscr{T} \in \mathbb{T} \mid \quad \#\mathscr{T} - \#\mathscr{T}_0 \leq N\}.$$

The quality of the best approximation in $\mathbb{T}_N$ is given by

$$\sigma_N(u) := \inf_{\mathscr{T}\in\mathbb{T}_N} \inf_{V\in\mathbb{V}_{\mathscr{T}}} \left[\|\nabla(V-u)\|_\Omega^2 + \operatorname{osc}_{\mathscr{T}}^2(V)\right]^{\frac{1}{2}}.$$

For $s > 0$, we say that $u \in \mathbb{A}_s$ if

$$|u|_s := \sup_{N\in\mathbb{N}_0} \left\{(N+1)^s \sigma_N(u)\right\} < \infty. \tag{6.1}$$

In other words, u belongs to the class $\mathbb{A}_s$ if it can be *ideally* approximated with adaptive meshes at a rate $(DOFs)^{-s}$. From another perspective, if $u \in \mathbb{A}_s$, then for each $\varepsilon > 0$ there exist a mesh $\mathscr{T}_\varepsilon \in \mathbb{T}$ and a function $V_\varepsilon \in \mathbb{V}_{\mathscr{T}_\varepsilon}$ such that

$$\#\mathscr{T}_\varepsilon - \#\mathscr{T}_0 \leq |u|_s^{\frac{1}{s}} \varepsilon^{-\frac{1}{s}} \quad \text{and} \quad \|\nabla(V_\varepsilon - u)\|_\Omega^2 + \operatorname{osc}_{\mathscr{T}_\varepsilon}^2(V_\varepsilon) \leq \varepsilon^2.$$

The study of classes of functions that will yield such rates is beyond the scope of this article. Some results along this direction can be found in [1, 11, 12].

The following result proved in [2, 22], provides a bound for the complexity of the overlay of two triangulations $\mathscr{T}^1$ and $\mathscr{T}^2$ obtained as refinements of $\mathscr{T}_0$.

Lemma 6.1 (Overlay of triangulations). *For $\mathscr{T}^1, \mathscr{T}^2 \in \mathbb{T}$ the overlay $\mathscr{T} := \mathscr{T}^1 \oplus \mathscr{T}^2 \in \mathbb{T}$, defined as the smallest admissible triangulation which is a refinement of $\mathscr{T}^1$ and $\mathscr{T}^2$, satisfies*

$$\#\mathscr{T} \leq \#\mathscr{T}^1 + \#\mathscr{T}^2 - \#\mathscr{T}_0.$$

The next lemma is essential for proving the main result below (see Theorem 6.1).

Lemma 6.2 (Cardinality of $\mathscr{M}_k$). *Let us assume that the weak solution u of problem* (2.1) *belongs to $\mathbb{A}_s$. If the marking parameter θ satisfies $0 < \theta < \theta_0$ (cf. (5.1)), then*

$$\#\mathscr{M}_k \leq \left(\frac{2C_E C_A}{\nu c_A}\right)^{\frac{1}{2s}} |u|_s^{\frac{1}{s}} \left[\|\nabla(U_k - u)\|_\Omega^2 + \operatorname{osc}_{\mathscr{T}_k}^2(U_k)\right]^{-\frac{1}{2s}}, \qquad \forall\, k \in \mathbb{N}_0,$$

where $\nu = \frac{1}{2}\left(1 - \frac{\theta^2}{\theta_0^2}\right)$ as in Lemma 5.3.

Proof. Let $k\in\mathbb{N}_0$ be fixed. Let $\varepsilon=\varepsilon(k)>0$ be a tolerance to be fixed later. Since $u\in\mathbb{A}_s$, there exist a mesh $\mathcal{T}_\varepsilon\in\mathbb{T}$ and a function $V_\varepsilon\in\mathbb{V}_{\mathcal{T}_\varepsilon}$ such that

$$\#\mathcal{T}_\varepsilon-\#\mathcal{T}_0\le|u|_s^{\frac{1}{s}}\varepsilon^{-\frac{1}{s}}\quad\text{and}\quad\|\nabla(V_\varepsilon-u)\|_\Omega^2+\operatorname{osc}^2_{\mathcal{T}_\varepsilon}(V_\varepsilon)\le\varepsilon^2.$$

Let $\mathcal{T}_*:=\mathcal{T}_\varepsilon\oplus\mathcal{T}_k$ the overlay of $\mathcal{T}_\varepsilon$ and $\mathcal{T}_k$ (cf. Lemma 6.1). Since $V_\varepsilon\in\mathbb{V}_{\mathcal{T}_*}$, we have that $\operatorname{osc}_{\mathcal{T}_\varepsilon}(V_\varepsilon)\ge\operatorname{osc}_{\mathcal{T}_*}(V_\varepsilon)$, and from Lemma 5.2, if $U_*\in\mathbb{V}_{\mathcal{T}_*}$ denotes the solution of the discrete problem (3.2) in $\mathbb{V}_{\mathcal{T}_*}$, we obtain that

$$\|\nabla(U_*-u)\|_\Omega^2+\operatorname{osc}^2_{\mathcal{T}_*}(U_*)\le2C_E\frac{C_A}{c_A}\left(\|\nabla(V_\varepsilon-u)\|_\Omega^2+\operatorname{osc}^2_{\mathcal{T}_\varepsilon}(V_\varepsilon)\right)\le2C_E\frac{C_A}{c_A}\varepsilon^2.$$

Let ε be such that

$$\|\nabla(U_*-u)\|_\Omega^2+\operatorname{osc}^2_{\mathcal{T}_*}(U_*)\le\nu\left(\|\nabla(U_k-u)\|_\Omega^2+\operatorname{osc}^2_{\mathcal{T}_k}(U_k)\right)=2C_E\frac{C_A}{c_A}\varepsilon^2,$$

where ν is the constant given by Lemma 5.3. Thus, this lemma yields

$$\eta_{\mathcal{T}_k}(U_k;\mathcal{R}_k)\ge\theta\eta_{\mathcal{T}_k}(U_k),$$

if $\mathcal{R}_k$ denotes the subset of $\mathcal{T}_k$ consisting of elements which were refined to get $\mathcal{T}_*$. Taking into account that $\mathcal{M}_k$ is a *minimal* subset of $\mathcal{T}_k$ satisfying the Dörfler's criterion, using Lemma 6.1 and recalling the choice of ε we conclude that

$$\begin{aligned}\#\mathcal{M}_k&\le\#\mathcal{R}_k\le\#\mathcal{T}_*-\#\mathcal{T}_k\le\#\mathcal{T}_\varepsilon-\#\mathcal{T}_0\le|u|_s^{\frac{1}{s}}\varepsilon^{-\frac{1}{s}}\\&=\left(\frac{2C_EC_A}{\nu c_A}\right)^{\frac{1}{2s}}|u|_s^{\frac{1}{s}}\left(\|\nabla(U_k-u)\|_\Omega^2+\operatorname{osc}^2_{\mathcal{T}_k}(U_k)\right)^{-\frac{1}{2s}}.\end{aligned}$$ □

The next result bounds the complexity of a mesh $\mathcal{T}_k$ in terms of the number of elements that were marked from the beginning of the iterative process, assuming that all the meshes were obtained by the bisection algorithm of [23], and that the initial mesh was properly labeled (satisfying condition (b) of Section 4 in [23]).

Lemma 6.3 (Complexity of REFINE). *Let us assume that $\mathcal{T}_0$ satisfies the labeling condition (b) of Section 4 in Ref.* [23], *and consider the sequence $\{\mathcal{T}_k\}_{k\in\mathbb{N}_0}$ of refinements of $\mathcal{T}_0$ where $\mathcal{T}_{k+1}:=$ REFINE$(\mathcal{T}_k,\mathcal{M}_k,n)$ with $\mathcal{M}_k\subset\mathcal{T}_k$. Then, there exists a constant $C_S>0$ solely depending on $\mathcal{T}_0$ and the number of refinements n performed by REFINE to marked elements, such that*

$$\#\mathcal{T}_k-\#\mathcal{T}_0\le C_S\sum_{i=0}^{k-1}\#\mathcal{M}_i,\qquad\text{for all }k\in\mathbb{N}.$$

The next result will use Lemma 6.3 and is a consequence of the global lower bound (Theorem 3.1), the bound for the cardinality of $\mathcal{M}_k$ given by Lemma 6.2 and the contraction property of Theorem 4.2. This is the second main result of the paper.

Theorem 6.1 (Quasi-optimal convergence rate)**.** *Let us assume that $\mathscr{T}_0$ satisfies the labeling condition (b) of Section 4 in Ref.* [23]. *Let us assume that the weak solution u of problem* (2.1) *belongs to $\mathbb{A}_s$. If $\{U_k\}_{k\in\mathbb{N}_0}$ denotes the sequence computed through the Adaptive Algorithm, and the marking parameter θ satisfies $0<\theta<\theta_0$ (cf.* (5.1)*), then*

$$\left[\|\nabla(U_k-u)\|_\Omega^2+\operatorname{osc}_{\mathscr{T}_k}^2(U_k)\right]^{\frac{1}{2}}\le C|u|_s(\#\mathscr{T}_k-\#\mathscr{T}_0)^{-s},\qquad \forall\, k\in\mathbb{N}, \tag{6.2}$$

where $C>0$ depends on d, $\kappa_{\mathbb{T}}$, problem data, the number of refinements n performed over each marked element, the marking parameter θ, and the regularity index s.

Proof. Let $k\in\mathbb{N}$ be fixed. The global lower bound (Theorem 3.1) yields

$$\|\nabla(U_i-u)\|_\Omega^2+\mu\eta_{\mathscr{T}_i}^2(U_i)\le\left(1+\mu C_L^{-1}\right)\left[\|\nabla(U_i-u)\|_\Omega^2+\operatorname{osc}_{\mathscr{T}_i}^2(U_i)\right],\qquad 0\le i\le k-1,$$

where μ is the constant appearing in Theorem 4.2. Using Lemmas 6.3 and 6.2 it follows that

$$\begin{aligned}\#\mathscr{T}_k-\#\mathscr{T}_0&\le C_S\sum_{i=0}^{k-1}\#\mathscr{M}_i\le C_S\left(\frac{2C_EC_A}{\nu c_A}\right)^{\frac{1}{2s}}|u|_s^{\frac{1}{s}}\sum_{i=0}^{k-1}\left[\|\nabla(U_i-u)\|_\Omega^2+\operatorname{osc}_{\mathscr{T}_i}^2(U_i)\right]^{-\frac{1}{2s}}\\&\le C_S\left(\frac{2C_EC_A}{\nu c_A}\right)^{\frac{1}{2s}}|u|_s^{\frac{1}{s}}\left(1+\mu C_L^{-1}\right)^{\frac{1}{2s}}\sum_{i=0}^{k-1}\left[\|\nabla(U_i-u)\|_\Omega^2+\mu\eta_{\mathscr{T}_i}^2(U_i)\right]^{-\frac{1}{2s}}.\end{aligned}\tag{6.3}$$

Since we do not have a contraction for the quantity $\left[\|\nabla(U_i-u)\|_\Omega^2+\mu\eta_{\mathscr{T}_i}^2(U_i)\right]$ as happens in the linear problem case, we now proceed as follows. We define

$$z_i^2:=[\mathscr{F}(U_i)-\mathscr{F}(u)]+\mu\eta_{\mathscr{T}_i}^2(U_i),$$

the contraction property (Theorem 4.2) yields $z_{i+1}\le\rho z_i$ and thus, $z_i^{-\frac{1}{s}}\le\rho^{\frac{1}{s}}z_{i+1}^{-\frac{1}{s}}$. Since $\rho<1$, taking into account (4.8), we obtain that[§]

$$\begin{aligned}&\sum_{i=0}^{k-1}\left(\|\nabla(U_i-u)\|_\Omega^2+\mu\eta_{\mathscr{T}_i}^2(U_i)\right)^{-\frac{1}{2s}}\\&\le(C_A/2)^{\frac{1}{2s}}\sum_{i=0}^{k-1}z_i^{-\frac{1}{s}}\le(C_A/2)^{\frac{1}{2s}}\sum_{i=1}^{\infty}(\rho^{\frac{1}{s}})^iz_k^{-\frac{1}{s}}=(C_A/2)^{\frac{1}{2s}}\frac{\rho^{\frac{1}{s}}}{1-\rho^{\frac{1}{s}}}z_k^{-\frac{1}{s}}\\&\le(C_Ac_A^{-1})^{\frac{1}{2s}}\frac{\rho^{\frac{1}{s}}}{1-\rho^{\frac{1}{s}}}\left(\|\nabla(U_k-u)\|_\Omega^2+\mu\eta_{\mathscr{T}_k}^2(U_k)\right)^{-\frac{1}{2s}}.\end{aligned}$$

Using the last estimation in (6.3), it follows that

$$\begin{aligned}&\#\mathscr{T}_k-\#\mathscr{T}_0\\&\le C_S\left(\frac{2C_EC_A}{\nu c_A}\right)^{\frac{1}{2s}}|u|_s^{\frac{1}{s}}\left(1+\mu C_L^{-1}\right)^{\frac{1}{2s}}(C_Ac_A^{-1})^{\frac{1}{2s}}\frac{\rho^{\frac{1}{s}}}{1-\rho^{\frac{1}{s}}}\left(\|\nabla(U_k-u)\|_\Omega^2+\mu\eta_{\mathscr{T}_k}^2(U_k)\right)^{-\frac{1}{2s}},\end{aligned}$$

[§]In this estimation we assume for simplicity that c_A and C_A are chosen so that $c_A\le 2\le C_A$.

and using that $\operatorname{osc}_{\mathscr{T}_k}(U_k) \le \eta_{\mathscr{T}_k}(U_k)$ and raising to the s-power we have that

$$
\begin{aligned}
&(\#\mathscr{T}_k - \#\mathscr{T}_0)^s \\
\le &\frac{C_S^s C_A}{c_A}\left(\frac{2C_E}{\nu}\right)^{\frac{1}{2}}\left(1+\mu C_L^{-1}\right)^{\frac{1}{2}}\frac{\rho}{(1-\rho^{\frac{1}{s}})^s}|u|_s\left(\|\nabla(U_k-u)\|_\Omega^2+\mu \operatorname{osc}^2_{\mathscr{T}_k}(U_k)\right)^{-\frac{1}{2}}.
\end{aligned}
$$

Finally, from the last estimate the assertion (6.2) follows, and the proof is concluded. □

We conclude this article with a few remarks.

Remark 6.1 The problem given by (1.1) is a particular case of the more general problem

$$
\begin{cases}
-\nabla \cdot \left[\alpha(\,\cdot\,, |\nabla u|^2_{\mathscr{A}})\mathscr{A}\nabla u\right] = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where $\alpha : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and $f \in L^2(\Omega)$ satisfy the properties assumed in the previous sections, and $\mathscr{A} : \Omega \to \mathbb{R}^{d\times d}$ is such that $\mathscr{A}(x)$ is a symmetric matrix, for all $x \in \Omega$, and uniformly elliptic, i.e., there exist constants $\underline{a}, \overline{a} > 0$ such that

$$
\underline{a}|\xi|^2 \le \mathscr{A}(x)\xi\cdot\xi \le \overline{a}|\xi|^2, \qquad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^d.
$$

If $\mathscr{A}$ is piecewise constant over an initial conforming mesh $\mathscr{T}_0$ of Ω, then the convergence and optimality results previously presented also hold for this problem.

Remark 6.2 We have assumed the use of *linear* finite elements for the discretization (see (3.1)), which is customary in nonlinear problems. It is important to notice that the only place where we used this is for proving (3.17), which is one of the key issues of our argument. The rest of the steps of the proof hold regardless of the degree of the finite element space.

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