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A Fourier Companion Matrix (Multiplication Matrix) with Real-Valued Elements: Finding the Roots of a Trigonometric Polynomial by Matrix Eigensolving

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Abstract. We show that the zeros of a trigonometric polynomial of degree N with the usual $(2N+1)$ terms can be calculated by computing the eigenvalues of a matrix of dimension $2N$ with real-valued elements M_{jk}. This matrix $\vec{\vec{M}}$ is a multiplication matrix in the sense that, after first defining a vector $\vec{\phi}$ whose elements are the first $2N$ basis functions, $\vec{\vec{M}}\vec{\phi} = 2\cos(t)\vec{\phi}$. This relationship is the eigenproblem; the zeros t_k are the arccosine function of $\lambda_k/2$ where the λ_k are the eigenvalues of $\vec{\vec{M}}$. We dub this the "Fourier Division Companion Matrix", or FDCM for short, because it is derived using trigonometric polynomial division. We show through examples that the algorithm computes both real and complex-valued roots, even double roots, to near machine precision accuracy.

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Key words: Fourier series, trigonometric polynomial, rootfinding, secular, companion matrix.

1. Introduction

More than a century ago, Frobenius showed that the roots of a polynomial could be found as the eigenvalues of a so-called "companion matrix" whose elements are trivial functions of the coefficients of the polynomial in the monomial basis. Similar companion matrices are now known to find the zeros of truncated series of Chebyshev, Legendre, Gegenbauer, Hermite and Bernstein polynomials as reviewed in [2].

A trigonometric polynomial of degree N is a truncated Fourier series of the form $f_N(t) \equiv \sum_{j=0}^{N} a_j \cos(jt) + \sum_{j=1}^{N} b_j \sin(jt)$. It is known that such a trigonometric polynomial can always be converted by the change of coordinate $z = \exp(it)$ into an ordinary

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polynomial in z with complex-valued coefficients. The zeros can then be computed by finding eigenvalues z_k of the Frobenius companion matrix with complex-valued coefficients and applying $t_k = -i \log(z_k)$.

In this note, we show that it is possible to obtain a companion matrix for a truncated Fourier series directly. This provides a simple way to find the zeros of a function represented by its Fourier expansion. The determination of the maxima and minima and inflection points of a function are also problems in rootfinding because these points are the zeros of the first or second derivative of the function, and these derivatives can easily be found in Fourier form by term-by-term differentiation of the Fourier expansion for $f(t)$.

Trigonometric root finding problems arise in many applications. For example, computing the intersection of two curves is a common task in computer graphics. If one curve is specified implicitly as the zero set ("affine variety") of a bivariate algebraic polynomial $P(x, y)$ and the other is a closed curve, parameterized by a pair of trigonometric polynomials, the intersection problem may be reduced to finding zeros of a trigonometric polynomial. If the parameterized curve is specified by some functions $(x(t), y(t))$, then the univariate trigonometric polynomial whose roots are needed is $f(t) \equiv P(x(t), y(t))$. Later, we thus compute the intersection of an algebraic curve (a trifolium) with a parameterized ellipse.

2. Previous work on computing the zeros of trigonometric polynomials

Three transformations have been used to convert trigonometric polynomials into algebraic polynomials so that the standard rootfinders for the latter can be deployed. Weidner set $z = \exp(it)$, which yields a polynomial with complex coefficients and maps the real zeros in t to roots on the unit circle in z [18]. Schweikard avoided complex coefficients by the substitutions

$$t = 2\arctan(s) \qquad \longleftrightarrow \qquad s = \tan(t/2), \tag{2.1}$$

which convert a trigonometric polynomial to a rational function and then, after clearing denominators, to a polynomial in s via the identities [16, 17]

$$\cos(t) = \frac{1-s^2}{1+s^2}, \qquad \sin(t) = \frac{2s}{1+s^2}. \tag{2.2}$$

Lastly, one may write $\cos(t) = c, \sin(t) = s$ and add the constraint $Q(c, s) \equiv c^2 + s^2 - 1 = 0$ which yields a system of two algebraic equations in the unknowns (c, s). This option is very popular in robotics and yields the ECM companion matrix [4].

Other authors have applied interval arithmetic [7–9, 17] and the Durand-Kerner iteration for finding all roots simultaneously [1, 7–9, 13, 14]. Earlier and complementary companion matrix studies by the author include [2–4, 6].

We omit a full-scale review because we have already provided one in [2].

Table 1: A bibliography of trigonometric rootfinding.

Angelova& Semerdzhiev [1]	trigonometric generalization of Durand-Kerner
Boyd [3]	symmetry-exploiting companion matrices for cosine polynomials, sine polynomials and polynomials with double parity
Boyd [2]	comprehensive review of orthogonal polynomial and Fourier rootfinding
Boyd [4]	Chebyshev companion matrix (ECM) derived from the $c=\cos(t)$, $s=\sin(t)$ substitution
Boyd & Sadiq [6]	Zeros of Fourier series with a linear, secular term
Carstensen & Petkovic, Carstensen & Reinders, Carstensen [7–9]	Durand-Kerner method, higher order generalizations and interval arithmetic
Frommer [13]	simplified Durand-Kerner iteration
Makrelov & Semerdzhiev [14]	Durand-Kerner simultaneous rootfinding
Makrelov & Semerdzhiev [15]	primarily polynomials of exponentials
Schweikard [16]	$x=\tan(t/2)$ transformation, square-free method
Schweikard [17]	interval arithmetic
Weidner [18]	$z=\exp(it)$ transformation to algebraic polynomial in z

3. Multiplication matrices

When discretizing a differential equation using a set of basis functions $\phi_n(t)$, a common task is to calculate the product of a function $\lambda(t)$ with the basis functions. For trigonometric polynomials, it is convenient to define a vector $\vec{\phi}$ whose elements are the first $2N$ basis functions:

$$\phi_n(x) \equiv \begin{cases} \cos([n-1]t), & n=1,\cdots,N, \\ \sin(nt), & n=N+1,\cdots,2N. \end{cases} \tag{3.1}$$

(Note that one of the basis functions in the trigonometric polynomial of degree N, $\cos(Nt)$, is not included.) The product of the multiplier with each basis function can then be expanded as a series in the basis:

$$\lambda(t)\phi_n(t) = \sum_{k=1}^{2N} H_{n,k}\phi_k(t) + \rho_n(t), \tag{3.2}$$

where the "residuals" ρ_j include all terms which cannot be represented using the first $2N$ basis functions. In the usual matrix notation, collecting the residuals as a vector $\vec{\rho}$, we can write

$$\lambda(t)\vec{\phi} = \vec{\vec{H}}\vec{\phi} + \vec{\rho}. \tag{3.3}$$

To discretize a term in a differential equation, the residual vector is simply ignored. The resulting relationship has the form of an eigenproblem, $\vec{\vec{H}}\vec{\phi} = \lambda\vec{\phi}$, but is only an approximation.

Suppose, however, that we can write each residual in the form

$$\rho_n(t) = f(t)q_n(t) + \tilde{r}_n(t), \tag{3.4}$$

where $f(t)$ is the trigonometric polynomial whose roots we seek and each remainder $\tilde{r}_n$ is a trigonometric polynomial that can be exactly represented by sum of the first $2N$ basis functions. When t is one of the roots of the polynomial, each residual collapses into the corresponding remainder, and each remainder $\tilde{r}_n$ can be absorbed into the square matrix by defining a new square matrix $\vec{\vec{M}}$ whose elements are equal to those of $\vec{\vec{H}}$ minus the expansion coefficients of the remainders $\tilde{r}_n(t)$. The eigenproblem $\vec{\vec{M}}\vec{\phi} = \lambda\vec{\phi}$ is then *exact*, but only when $t = t_k$ where t_k is one of the roots of $f(t)$. Therefore, each of the eigenvalues is the multiplier $\lambda(t)$, evaluated at one of the roots, and the root can be obtained by inverting $\lambda(t)$.

For polynomial companion matrices, the multiplier function λ is simply t itself, no version of the multiplier is necessary, all the residuals are identically zero except for that in the last row, and the factorization of the residual as $\rho_n = f(t)q_n(t) + r_n(t)$ is the result of synthetic division, also known as "polynomial division with the remainder".

In our case, the product of t with a trigonometric function is an infinite series. If instead we choose the multiplier as

$$\lambda = 2\cos(t), \tag{3.5}$$

then the trigonometric identities below show that all the residuals are zero except for ρ_N, generated by $2\cos(t)\cos([N-1]t) = \cos([N-2]t)+\cos(Nt)$ where $\cos(Nt)$ is outside the vector $\vec{\phi}$, and ρ_{2N}, generated by $2\cos(t)\sin(Nt) = \sin([N-1]t) + \sin([N+1]t)$ where $\sin([N+1]t)$ is also outside the vector $\vec{\phi}$. Both of these can be factored by a process analogous to polynomial division.

3.1. Fourier synthetic division

Synthetic division of an ordinary polynomial $p(x)$ by another polynomial $d(x)$ yields $p(x)/d(x) = q(x) + r(x)/d(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. This is equivalent to factorizing the polynomial as $p(x) = q(x)d(x)+r(x)$ where a classical theorem asserts that the degree of the remainder is less than the degree of the divisor. Here, we compute similar factorizations for trigonometric polynomials using $f(t)$, the polynomial whose roots are sought, as the divisor. We make heavy use of the trigonometric identities

$$2\cos(t)\cos(jt) = \cos([j-1]t) + \cos([j+1]t), \tag{3.6a}$$
$$2\cos(t)\sin(jt) = \sin([j-1]t) + \sin([j+1]t). \tag{3.6b}$$

The factorization of $2\cos(t)\cos([N-1]t)$ is easy because the quotient is a constant:

$$2\cos(t)\cos([N-1]t) = q_1 f(t) + r_N \tag{3.7a}$$
$$\Updownarrow$$
$$\cos([N-2]t) + \cos(Nt) = q_1\left\{a_N\cos(Nt) + b_N\sin(Nt) + \text{lower degree}\right\} + r_N(t), \tag{3.7b}$$

where we used a trigonometric identity to go from the upper left to the bottom left. (Note that it is now convenient to factorize $\lambda\phi_n$ instead of just the residuals, and we have dropped the tildes over the remainders to indicate that these are now the remainders of the entire product of the multiplier function with the basis function.) By choosing the quotient as

$$q_1 = 1/a_N, \tag{3.8}$$

we can match the terms in $\cos(Nt)$ on both sides of the equation, thus ensuring that the remainder $r_N(t) \equiv \cos([N-2]t) + \cos(Nt) - f(t)/a_N$ contains only $-(b_N/a_N)\sin(Nt)$ plus terms of lower degree than $\cos(Nt)$. Because $\sin(Nt)$ is included in $\vec{\phi}$, $r_N(t)$ is a trigonometric polynomial in only the $2N$ basis functions included in $\vec{\phi}$; its coefficients can therefore be completely absorbed into the matrix and in fact are the elements of the N-th row of the multiplication matrix $\vec{\vec{M}}$ as catalogued below in (3.15e).

The factorization of $2\cos(t)\sin(Nt) = \sin([N-1]t) + \sin([N+1]t)$ is more complicated because $\sin([N+1]t)$ is one degree higher than $f(t)$. Therefore, the quotient of $q_2(t)$ must be a *linear* trigonometric polynomial of the form

$$q_2(t) \equiv w_1 + w_2\cos(t) + w_3\sin(t). \tag{3.9}$$

The factorization is

$$2\cos(t)\sin(Nt) = (w_1 + w_2\cos(t) + w_3\sin(t))f(t) + r_2(t) \tag{3.10a}$$

$$\Updownarrow$$

$$\begin{aligned} &\sin([N-1]t) + \sin([N+1]t) \\ &\quad = \{w_3 b_N - w_2 a_N\}\cos([N+1]t) + \{-w_2 b_N - w_3 a_N + 1\}\sin([N+1]t) \\ &\qquad + \{-2w_1 a_N + w_3 b_{N-1} - w_2 a_{N-1}\}\cos(Nt) + \text{lower degree} + r_2(t). \end{aligned} \tag{3.10b}$$

The condition that the three terms in braces all vanish, removing high degree terms from the remainder, yields

$$w_1 = -1/2\,\frac{-a_N b_{N-1} + b_N a_{N-1}}{\left({a_N}^2 + {b_N}^2\right) a_N}, \tag{3.11a}$$

$$w_2 = \frac{b_N}{{a_N}^2 + {b_N}^2}, \tag{3.11b}$$

$$w_3 = \frac{a_N}{{a_N}^2 + {b_N}^2}. \tag{3.11c}$$

The elements of the multiplication matrix can be written in symbolic form as

$$M_{j,k} =< \overline{2\cos(t)\phi_j(t)}, \phi_k(t) > / < \phi_k, \phi_k >, \tag{3.12}$$

where the inner product here is, for arbitrary functions $p(t)$ and $q(t)$,

$$< p(t), q(t) > \equiv \int_{-\pi}^{\pi} dt\, p(t) q(t), \tag{3.13}$$

and we define

$$\overline{2\cos(t)\phi_j(t)} = \begin{cases} r_N(t), & j = N, \\ r_{2N}(t), & j = 2N, \\ 2\cos(t)\phi_j(t), & \text{otherwise.} \end{cases} \tag{3.14}$$

Evaluating the inner products gives explicitly

$$M_{1,j} = 2\,\delta_{j,2}, \tag{3.15a}$$
$$M_{n,j} = \delta_{j,n-1} + \delta_{j,n+1}, \qquad n = 2,\cdots,(N-1), \tag{3.15b}$$
$$M_{N+1,j} = \delta_{j,N+2}, \tag{3.15c}$$
$$M_{n,j} = \delta_{j,n-1} + \delta_{j,n+1}, \qquad n = (N+2),\cdots,(2N-1), \tag{3.15d}$$
$$M_{N,j} = \delta_{j,N-1} - \frac{a_{j-1}}{a_N}, \qquad j = 1,\cdots,N, \tag{3.15e}$$
$$M_{N,j} = -\frac{b_{j-N}}{a_N}, \qquad j = N+1,\cdots 2N, \tag{3.15f}$$
$$M_{2N,1} = -2w_1a_0 - w_2a_1 - w_3b_1, \tag{3.15g}$$
$$M_{2N,2} = -2w_1a_1 - w_2(2a_0 + a_2) + w_3(-b_2), \tag{3.15h}$$
$$M_{2N,N+1} = -2w_1b_1 - w_2(b_2) + w_3(a_2 - 2a_0), \tag{3.15i}$$
$$M_{2N,2N-1} = -2w_1b_{N-1} - w_2(b_{N-2} + b_N) + w_3(a_N - a_{N-2}) + 1, \tag{3.15j}$$
$$M_{2N,2N} = -2w_1b_N - w_2(b_{N-1}) + w_3(-a_{N-1}), \tag{3.15k}$$
$$M_{2N,n} = -2w_1a_{n-1} - w_2(a_{n-2} + a_n) + w_3(b_{n-2} - b_n), \qquad n = 3,\cdots,N, \tag{3.15l}$$
$$M_{2N,n+N} = -2w_1b_n - w_2(b_{n-1} + b_{n+1}) + w_3(a_{n+1} - a_{n-1}), \qquad n = 2,\cdots,(N-2). \tag{3.15m}$$

4. Processing the eigenvalues

For each eigenvalue, we calculate the two branches of the arccosine function in the strip $\Re(t) \in (-\pi,\pi]$

$$t_j^{\pm} = \pm\text{arccos}\left(\lambda_j/2\right). \tag{4.1}$$

Since there are $4N$ of these numbers and a theorem [2] shows that a trigonometric polynomial of degree N has precisely $2N$ roots, counted according to their multiplicity, with $\Re(t) \in (-\pi,\pi]$, it follows that half of these $t_j^{\pm}$ are spurious. The simplest procedure to resolve this ambiguity is to evaluate $|f(t)|$ at each of the candidates. The genuine roots have tiny residuals whereas $|f(t)|$ is not small for the spurious zeros as illustrated for our numerical example in Table 2.

However, our goal is not to minimize the residuals, but rather to approximate the zeros. We therefore recommend refining the arccosines of the eigenvalues by applying Newton's iteration to all the candidates. For each pair, the Newton corrections are

$$v_j^{\pm} = \frac{f(t_j^{\pm})}{df/dt(t_j^{\pm})}. \tag{4.2}$$

Of these $4N$ Newton corrections, half will be tiny and half will be $\mathcal{O}(1)$. We select the $2N$ corrections of tiny magnitude, subtract these from the corresponding $t_j^{\pm}$, and return the corrected values as the final approximations to the roots.

5. Removing difficulties by translation

The companion matrix fails if:

1. $a_N = 0$.
2. One or more pairs of roots are symmetric about the origin, that is, $\pm t_*$ are *both* zeros of $f(t)$ for some t_*.

If the highest *cosine* coefficient a_N in $f(t)$ is zero, then the constant w_1 in the quotient $q_2(t)$ is infinite. It is then impossible for $w_1 f(t)$ to remove $\cos(Nt)$ from the remainder $r_{2N}(t)$. Except for the special case of a pure sine polynomial ($f(-t) = -f(t)$ for all t, and all $a_n = 0$), which is best handled by special algorithms [3], this situation is very unlikely in practice where typically the sine and cosine coefficients are similar in magnitude. We shall nevertheless describe a remedy in a moment.

When there is a pair of symmetric roots, postprocessing will yield two valid roots for a given eigenvalue. This can fool Newton's iteration or the small $|f(t)|$ criterion into endorsing more than $2N$ roots unless the postprocessing is very careful.

The remedy for both difficulties is to make a change of coordinate

$$t \equiv \overline{t} + s \quad \Longleftrightarrow \quad \overline{t} = t - s, \tag{5.1}$$

where s is a positive constant. Then

$$\begin{aligned} f_N(\overline{t} + s) &\equiv g_N(\overline{t}) \\ &= a_0 + \sum_{j=1}^{N} \left\{ a_j \cos(js) + b_j \sin(js) \right\} \cos(j\overline{t}) \\ &\quad + \sum_{j=1}^{N} \left\{ b_j \cos(js) - a_j \sin(js) \right\} \sin(j\overline{t}). \end{aligned} \tag{5.2}$$

Once the roots of $g_N(\overline{t})$ have been found as the set $\{\overline{t}_k\}$, the zeros of the original, unshifted problem are

$$t_r \equiv \overline{t}_r + s. \tag{5.3}$$

It is easy to embed a companion matrix procedure in a loop and repeat the computation with a different translation s whenever either of these singularity conditions is detected. A fuller discussion is given in [4].

6. Numerical examples

When

$$a_0 = 1/13; \quad a_1 = 3/14; \quad a_2 = -11/37; \tag{6.1a}$$

$$a_3 = -1/3; \quad a_4 = 3/14; \quad a_5 = -161/71, \tag{6.1b}$$

$$b_1 = -2; \quad b_2 = -9/4; \quad b_3 = -17/11; \quad b_4 = 1/11; \quad b_5 = -51/23, \tag{6.1c}$$

the multiplication matrix is

$$\vec{\vec{M}} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.35897 & -1 & 2.3874 & 1.5556 & 9.3333 & 10.500 & 7.2121 & -0.42424 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 9.5435 & 14.692 & -9.3018 & -17.756 & -49.835 & -54.521 & -33.402 & 6.2670 \end{bmatrix}. \tag{6.2}$$

Note that all but two of the rows of the multiplication matrix are completely trivial in the sense that there are at most two nonzero elements in the trivial rows, and these elements are all ones except for $M_{1,2} = 2$.

The eigenvalues of the multiplication matrix are

$$\vec{\lambda} = \begin{bmatrix} 5.35451999999999994 + 5.66558999999999990\, i \\ 5.35451999999999994 - 5.66558999999999990\, i \\ 1.99987000000000004 + 0\, i \\ -1.99811000000000006 + 0\, i \\ 0.0681942999999999994 + 0\, i \\ -0.403386000000000022 + 0\, i \\ -1.56857000000000002 + 0\, i \\ -0.984462999999999977 + 0\, i \end{bmatrix}. \tag{6.3}$$

The arccosine function is multivalued, and thus the arccosine of $\lambda_k/2$ for each eigenvalue generates two numbers which are negatives of each other. To condense these $4N$ "root-candidates" into $2N$ genuine roots and $2N$ numerical artifacts, the easiest procedure is to evaluate the residuals, that is, the absolute value of $f(t)$ at each of the candidates, as illustrated in Table 2.

Residuals are not the same as errors; as noted above, it is a good precaution to apply a Newton iteration to the approximate zeros. The eight candidates with tiny Newton

Table 2: Candidate roots and their residuals.

k	$t_k^+ \equiv +\arccos(\lambda_k/2)$	$\lvert f(t_k^+)\rvert$	$t_k^- \equiv -\arccos(\lambda_k/2)$	$\lvert f(t_k^-)\rvert$
1	$.8299872013010460 - 2.054875306957421\, i$	701.6	$\mathbf{-.8299872013010460 + 2.054\cdots}\, i$	$0.77e-11$
2	$.8299872013010460 + 2.054875306957421\, i$	701.6	$\mathbf{-.8299872013010460 - 2.054\cdots}\, i$	$0.77e-11$
3	$0.1161394466375658e-1$	.2502	$\mathbf{-0.1161394466375658e-1}$	$0.40e-11$
4	**3.098130435942467**	$0.38e-12$	-3.098130435942467	.22
5	**1.536692578178622**	$0.94e-14$	-1.536692578178622	1.25
6	1.773882520239653	.53	**−1.773882520239653**	$0.71e-14$
7	2.472342091902223	.98	**−2.472342091902223**	$0.14e-14$
8	**2.085447543296132**	$0.20e-14$	-2.085447543296132	.62

The genuine roots are in bold face and are listed next to their tiny residuals; the spurious zeros are in ordinary type next to their $\mathcal{O}(1)$ residuals. The imaginary parts were truncated in the fourth column to fit the table to the width of the page; the missing digits are given in the second column.

corrections were the same eight that had small residuals: the eight zeros of $f(t)$ are thus

$$\begin{bmatrix} -2.472342091902224 \\ -1.773882520239652 \\ -0.01161394466338633 \\ 1.536692578178624 \\ 2.085447543296132 \\ 3.098130435942322 \\ -0.8299872013010614 - 2.054875306957421\, i \\ -0.8299872013010614 + 2.054875306957421\, i \end{bmatrix}. \tag{6.4}$$

The largest Newton correction for any of these eight roots has a magnitude of only 0.37×10^{-12}.

Fig. 1 shows the exemplary $f(t)$, its graph visually confirming the six real roots of the eigenvalue calculation.

As a second example, consider the intersection of the algebraic curve known as the "trifolium" with an ellipse as illustrated in Fig. 2. The trifolium is specified implicitly as

$$P(x,y) \equiv (x^2+y^2)^2 - c(x^3 - 3xy^2) = 0, \tag{6.5}$$

where we shall arbitrarily choose the parameter $c=1$. The ellipse is centered at the origin with a horizontal semimajor axis, represented in parametric form as

$$x = A\cos(t-\pi/3), \qquad y = B\sin(t-\pi/3), \tag{6.6}$$

where the $\pi/3$ was inserted merely so that our Fourier polynomial will be a mixture of sines and cosines; this shifts the origin of t without altering the shape of the curve. We shall choose the axes to be $A=1$ and $B=1/2$. To find the intersection of the two curves,

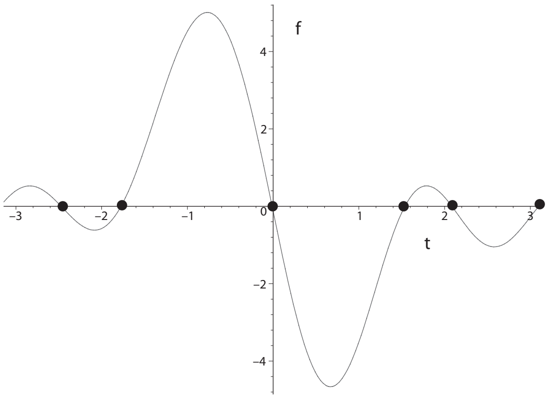

Figure 1: The trigonometric polynomial $f(t)$ for the first numerical example. This polynomial of degree four has six real roots as shown (black disks).

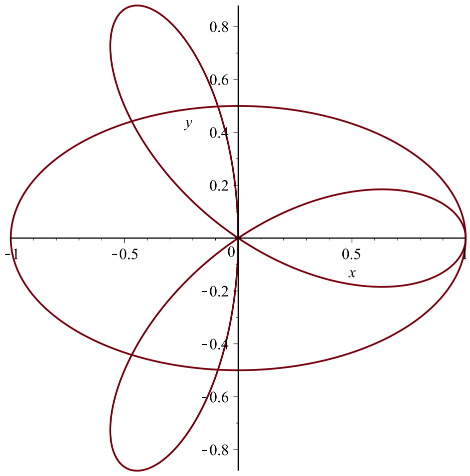

Figure 2: The trifolium, defined implicitly as the zero set of $(x^2+y^2)^2-(x^3-3xy^2)$, and the ellipse, parameterized by $x=\cos(t), y=(1/2)\sin(t)$, intersect at four simple roots and one double root. These intersections were computed by finding the zeros of a trigonometric polynomial of degree four.

the first step is to substitute for x and y in the algebraic polynomial of the trifolium using the parameterization of the ellipse, yielding

$$\begin{aligned} f(t) &\equiv P\left(x=A\cos(t-\pi/3), y=B\sin(t-\pi/3)\right) \\ &= \frac{59}{128}-\frac{15}{64}\cos(2t)+\frac{15}{64}\sin(2t)\sqrt{3}-\frac{9}{256}\sin(4t)\sqrt{3} \\ &\quad -\frac{9}{256}\cos(4t)\frac{7}{16}\cos(3t)-\frac{9}{32}\cos(t)-\frac{9}{32}\sin(t)\sqrt{3}. \end{aligned} \tag{6.7}$$

The intersection points are now the zeros of the polynomial $f(t)$; these roots t_j can be converted into the corresponding values in Cartesian coordinates by again invoking the parameterization of the ellipse: $(x_j=A\cos(t_j-\pi/3), y_j=\sin(t_j-\pi/3))$.

Table 3: Computed zeros and the distances between the computed (x_j^{app}, y_j^{app}) and exact intersection points $(x_j^{ex}, y_j^{ex})$for the intersection of a trifolium and an ellipse.

j	x_j	y_j	$\sqrt{(x_j^{app}-x_j^{ex})^2+(y_j^{app}-y_j^{ex})^2}$
1	$-.4680013939455222$	$-.4418638634424205$	$0.1e-15$
2	$-0.8898338250960578e-1$	$-.4980165553566340$	$0.2400e-15$
3	2.668095887566240 $+0.9418723816663841e-15\,i$	$0.5079636723542840e-15$ $-1.236803911827566\,i$	$0.15e-14$
4	$-0.8898338250960517e-1$	$.4980165553566340$	$0.3700e-15$
5	$-.4680013939455190$	$.4418638634424214$	$0.34e-14$
6	2.668095887566240 $-0.9418723816663841e-15\,i$	$0.5079636723542840e-15$ $+1.236803911827566\,i$	$0.15e-14$
7	1.000000000000000	$0.1903843391541801e-15$	$0.19e-15$
8	1.000000000000000	$0.1903843391541801e-15$	$0.19e-15$

Fig. 2 shows that the two curves intersect at five points. Table 3 shows that, in 16 decimal place arithmetic, the Fourier Division Companion Matrix computes intersection points that lie at most a distance of 0.34×10^{-14} from the true zeros! The double root at $(0,1)$ causes no problems; the corresponding eigenvalue appears twice in the output from the multiplication matrix and therefore the same zero appears twice after postprocessing, but its accuracy is actually the best of any of the roots. Although complex-valued intersections are irrelevant in compute graphics, it is noteworthy that the Fourier companion matrix finds them with extreme accuracy anyway.

More examples and detailed comparisons with other Fourier companion matrix strategies are given in the companion paper [4].

7. Explicit solutions and multiplication matrices for $N=1$ and $N=2$

For the special case $N=1$, the companion matrix is unnecessary because the roots of $f_1(t) \equiv a_0 + a_1\cos(t) + b_1\sin(t)$ are explicitly

$$S \equiv \sqrt{{b_1}^2\left({a_1}^2-{a_0}^2+{b_1}^2\right)}; \qquad R \equiv {a_1}^2+{b_1}^2, \tag{7.1a}$$

$$t^+ = -\arctan\left(\frac{a_0{b_1}^2+a_1S}{Rb_1}, \frac{-a_0a_1+S}{R}\right), \tag{7.1b}$$

$$t^- = -\arctan\left(\frac{a_0{b_1}^2-a_1S}{Rb_1}, \frac{-a_0a_1-S}{R}\right). \tag{7.1c}$$

The solutions for $N=2$ are

$$t = -\arctan\left(\frac{a_0-a_2+a_1Z+2\,a_2Z^2}{b_1+2\,b_2Z}, Z\right), \tag{7.2}$$

where Z is one of the four roots of the quartic

$$\left(4\,{b_2}^2+4\,{a_2}^2\right)Z^4+\left(4\,b_1b_2+4\,a_1a_2\right)Z^3+\left({a_1}^2-4\,{b_2}^2-4\,{a_2}^2+4\,a_0a_2+{b_1}^2\right)Z^2 \\ +\left(-2\,a_1a_2-4\,b_1b_2+2\,a_0a_1\right)Z+{a_0}^2+{a_2}^2-2\,a_0a_2-{b_1}^2=0. \tag{7.3}$$

This is no real improvement over the companion matrix method because the default strategy for calculating the zeros of an algebraic polynomial in Matlab, for example, is to find the eigenvalues of a Frobenius companion matrix which is the same size as the Fourier companion matrix. The explicit solution requires further processing with an arc tangent function; the multiplication matrix strategy requires postprocessing of eigenvalues with the arccosine function. We therefore give the $N=2$ Fourier multiplication matrix:

$$\begin{bmatrix} 0 & 2 & 0 & 0 \\ -\frac{a_0}{a_2}+1 & -\frac{a_1}{a_2} & -\frac{b_1}{a_2} & -\frac{b_2}{a_2} \\ 0 & 0 & 0 & 1 \\ M_{4,1} & M_{4,2} & M_{4,3} & M_{4,4} \end{bmatrix}, \tag{7.4}$$

where

$$M_{4,1}=\frac{-a_0a_2b_1-b_2a_1a_2+a_0b_2a_1-{a_2}^2b_1}{\left({b_2}^2+{a_2}^2\right)a_2}, \tag{7.5a}$$

$$M_{4,2}=-\frac{a_1a_2b_1+2\,b_2{a_2}^2+2\,b_2a_0a_2-{a_1}^2b_2}{\left({b_2}^2+{a_2}^2\right)a_2}, \tag{7.5b}$$

$$M_{4,3}=-\frac{2\,{a_2}^2a_0+{b_1}^2a_2-b_1b_2a_1-2\,{a_2}^3}{\left({b_2}^2+{a_2}^2\right)a_2}, \tag{7.5c}$$

$$M_{4,4}=-\frac{-{b_2}^2a_1+2\,b_2b_1a_2+{a_2}^2a_1}{\left({b_2}^2+{a_2}^2\right)a_2}. \tag{7.5d}$$

The explicit solutions were computed by the "solve" function in Maple followed by considerable manual simplification and numerical checking. The Maple algorithm employs the $s=\tan(t/2)$ transformation to convert the trigonometric polynomial to an algebraic polynomial in s, solves that, and then takes the arctan function.

8. Summary and generalizations

The new Fourier Division Companion Matrix (FDCM) described here is compared to two older alternatives in [4]. The new method gives very accurate (near-machine-precision) approximations to both real and complex valued zeros of a trigonometric polynomial for all of the additional examples described there.

One of the alternatives, the Elimination/Chebyshev (ECM) method, is useful, alas, only for real valued roots. The Complex Companion Matrix (CCM) [2, 4] has elements which are, as the name indicates, *complex valued* even when applied to *real* polynomials. There is, however, little to choose, for *real* roots, between the three companion matrix algorithms for either practical accuracy or speed.

Companion/multiplication matrices have been developed for many species of polynomials [2,5,10–12]. To extend the multiplication matrix method beyond Fourier series, one needs an easily-invertible $\lambda(x)$ such that the expansion of $\lambda(x)\phi_n(x)$ for $n = 1, \cdots, N$ has the following properties:

1. It *terminates*, including only a finite number of basis functions of degree $n > N$.
2. The residuals $\rho_n(x)$, which are defined as the sum of the terms in the expansion of $\lambda(x)\phi_n(x)$ which involve ϕ_j with $j > N$, must be factorizable as $\rho_n = q_n(x)f(x) + r_n(x)$ where $f(x)$ is the generalized polynomial whose roots are sought and $r_n(x)$ is a weighted sum of the first N basis functions.

Beyond those already known, no obvious basis sets fulfill these stiff conditions. Perhaps the reader will be more inventive!

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References

[1] E. D. Angelova and K. I. Semerdzhiev, *Methods for the simultaneous approximate derivation of the roots of algebraic, trigonometric and exponential equations*, U.S.S.R. Comput. Maths. Math. Phys., 22 (1982), pp. 226–232.

[2] J. P. Boyd, *Computing the zeros, maxima and inflection points of Chebyshev, Legendre and Fourier series: Solving transcendental equations by spectral interpolation and polynomial rootfinding*, J. Eng. Math., 56 (2006), pp. 203–219.

[3] J. P. Boyd, *Computing the zeros of a Fourier series or a Chebyshev series or general orthogonal polynomial series with parity symmetries*, Comput. Math. Appl., 54 (2007), pp. 336–349.

[4] J. P. Boyd, *A comparison of companion matrix methods to find roots of a trigonometric polynomial*, J. Comput. Phys., 246 (2013), pp. 96–112.

[5] J. P. Boyd, *Finding the zeros of a univariate equation: Proxy rootfinders, Chebyshev interpolation and the companion matrix*, SIAM Rev., 55 (2013), in proof.

[6] J. P. Boyd and B. A. Sadiq, *Computing the real roots of a Fourier series-plus-linear-polynomial: A Chebyshev companion matrix approach*, Appl. Math. Comput., 219 (2012), pp. 819–826.

[7] C. Carstensen, *A note on simultaneous rootfinding for algebraic, exponential, and trigonometric polynomials*, Comput. Math. Appl., 27 (1994), pp. 7–14.

[8] C. Carstensen and M. S. Petkovi'c, *On some interval methods for algebraic, exponential and trigonometric polynomials*, Computing, 51 (1993), pp. 313–326.

[9] C. Carstensen and M. Reinders, *On a class of higher order methods for simultaneous rootfinding of generalized polynomials*, Numer. Math., 64 (1993), pp. 69–84.

[10] R. M. Corless, *Generalized companion matrices in the Lagrange basis*, in Proceedings EACA, L. Gonzalez-Vega and T. Recio, eds., 2004, pp. 317–322.

[11] R. M. Corless, *On a generalized companion matrix pencil for matrix polynomials expressed in the Lagrange basis*, in Symbolic-Numeric Computation, D. Wang and L.-H. Zhi, eds., Trends in Mathematics, Natl Nat Sci Fdn China, Birkhäuser Basel, 2007, pp. 1–15.

[12] R. M. Corless and G. Litt, *Generalized companion matrices for polynomials not expressed in monomial bases*, unpublished, University of Western Ontario, 2000. Available on Corless' web site.

[13] A. Frommer, *A unified approach to methods for the simultaneous computation of all zeros of generalized polynomials*, Num. Math., 54 (1988), pp. 105–116.

[14] I. V. Makrelov and H. I. Semerdzhiev, *Methods for the simultaneous determination of all zeros algebraic, trigonometric and exponential equations*, U. S. S. R. Comput. Maths. Math. Phys., 24 (1984), pp. 99–105.

[15] I. V. Makrelov and K. I. Semerdzhiev, *On the convergence of two methods for the simultaneous find of all roots of exponential equations*, IMA J. Numer. Anal., 5 (1985), pp. 191–200.

[16] A. Schweikard, *Trigonometric polynomials with simple roots*, Information Processing Lett., 39 (1991), pp. 231–236.

[17] A. Schweikard, *Real zero isolation for trigonometric polynomials*, ACM Trans. Math. Software, 18 (1992), pp. 350–359.

[18] P. Weidner, *The Durand-Kerner method for trigonometric and exponential polynomials*, Comput., 40 (1988), pp. 175–179.